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FREE ABELIAN TOPOLOGICAL GROUPS AND ADJUNCTION SPACES

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Dedicated to Bernard Banaschewski on his 60th birthday

1. Introduction

In recent years the authors have investigated the question of which free abelian topological groups can be embedded as subgroups of the free abelian topological group on the closed unit interval I and, more generally, the closed ball B^n, for positive integers n. For example, it was shown in [4] that the free abelian topological group $F((0,1))$ on the open interval $(0,1)$ can be embedded in F(I), and this result was extended in [6] to show that $F((0,1))^n) \leq F(B^n)$. In [5], it is shown that $F(S^n) \leq F(B^n)$, and that $F((S^1)^n) \leq F(B^n)$, where S^n denotes the n-sphere.

In this paper, we prove that if $F(X) \leq F(B^n)$, then $F(X \bigsqcup_f B^n) \leq F(B^n)$ also, where $X \bigsqcup_f B^n$ is any adjunction to X of B^n along its boundary S^{n-1}. A special case of this is the previously stated result that $F(S^n) \leq F(B^n)$. (For a discussion of adjunction spaces, see [2].)

It should be noted that, by contrast, much more is known about subgroups of free (non-abelian) topological groups. For example, the free topological group on a space Y is contained in the free topological group F on B^n if and only if Y is homeomorphic to a closed subspace of F.

2. Preliminaries

We record here the necessary definitions and background results.

A Hausdorff topological space X is said to be a k_ω-space with &-decomposition

$X=\bigcup_n X_n$ if X_n is compact, $X_n \subseteq X_{n+1}$ for $n=1,2,3,\ldots$ and X has the weak topology with respect to the sets X,,.

Definition. If X is a topological space with distinguished point e, the abelian topological group F(X) is said to be the (Graev) *free abelian topological group on X* if

(a) X is a subspace of F(X), and

(b) any continuous map ϕ from X into any abelian topological group H, sending e to the identity of H, extends uniquely to a continuous homomorphism Φ : F(X) $\rightarrow H$.

If X is any completely regular space, then F(X) exists, is unique, and is independent of the choice of e in X. Further, F(X) is algebraically the free abelian group on X \ (e}. If X is also Hausdorff, then F(X) is Hausdorff and has X as closed subspace [8]. For k_ω-spaces, one can say rather more.

Theorem A (Mack et al. [7]). *Let X= $\bigcup X_n$ be any k_ω-space with distinguished point e. Then F(X) is a k_ω-space and F(X) has k_ω-decomposition F(X)= $\bigcup_n$ $\mathrm{gp}_n(X_n)$, where $\mathrm{gp}_n(X_n)$ is the set of words of length not exceeding n in the subgroup generated by X_n.*

Definition. Let X= $\bigcup X_n$ be a k_ω-space, and let Y= $\bigcup Y_n$ be a closed k_ω-subspace of *F(X)*. Then Y is said to be *regularly situated* with respect to X if for each natural number n there is an integer m such that gp(Y) $\cap\, \mathrm{gp}_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$.

Theorem B (Mack et al. [7]). *Zf X is a k_ω-space, and Y is a closed subset of F(X) containing e such that Y \ {e} is a free algebraic basis for* gp(Y) *and Y is regularly situated with respect to X, then* gp(Y) *is F(Y).*

3. Results

We record here a result which is probably known, but does not appear to be in the literature. The k_ω non-commutative case appears in [3].

Note that a space X with distinguished point e will be called *contractible relative to e* if there is a continuous function ϕ : Xx $I \rightarrow$ X such that $\phi(x, 0) = x$ and $\phi(x,1)=e$ for all $x \in X$, and $\phi(e,t)=e$ for all $t \in I$.

Proposition. *Let X be a completely regular Hausdorff space with distinguished point e. Zf X is contractible relative to e, then so is F(X).*

Proof. Firstly observe that, for any topological group G, and any locally compact Hausdorff space X, the group G^X of continuous functions from X to G, with the compact-open topology, is a topological group. The proof of this fact is routine, and hence omitted.

Let $\phi: X \times I \to X$ be a contraction relative to e. We shall construct a contraction $\Phi: F(X) \times I \to F(X)$.

By [1, Theorem 3.1(1) of Chapter XII], the mapping $\hat{\phi}: X \to X^I$ defined by $\hat{\phi}(x)(t) = \phi(x, t)$, $x \in X$, $t \in I$, is continuous. As X is a subspace of F(X), we can, by [1, 1.2(b) of Chapter XII], regard X^I as a subspace of the topological group $F(X)^I$, and so, by the freeness of F(X), $\hat{\phi}$ extends to a continuous homeomorphism $\hat{\Phi}: F(X) \to F(X)^I$. It is easy to see that the function $\Phi: F(X) \times I \to F(X)$ defined by $\Phi(w, t) = \hat{\Phi}(w)(t)$, $w \in F(X), t \in I$, which by [1, Theorem 3.1(2) of Chapter XII] is continuous, is the contraction we require. $\square$

We now prove our main result.

Theorem. *If, for some $n \in \mathbb{N}$, $F(B^n)$ has $F(X)$ as a closed topological subgroup, then $F(B^n)$ also has $F(X \sqcup_f B^n)$ as a closed topological subgroup, where $f: S^{n-1} \to X$ is any continuous map, and S^{n-1} is regarded as the boundary of B^n.*

Proof. Let C_1 and C_2 be subsets of B^n both homeomorphic to B^n, such that $C_1 \cap C_2 = \{e\}$, and without loss of generality, let $F(X)$ be embedded in $\mathrm{gp}(C_1) = F(C_1)$. Since $F(C_1)$ and $F(B^n)$ are topologically isomorphic, and the latter is contractible, the map $f: S^{n-1} \to X \subseteq F(C_1)$ extends continuously to a function $p: B^n \to F(C_1)$.

Also, by [5], there is an embedding $s: S^n \to \mathrm{gp}(C_2)$ extending to a topological isomorphism of $F(S^n)$ into $F(C_2)$. Let x_0 be the point of S^n satisfying $s(x_0) = e \in F(C_2)$, and let $r: B^n \to S^n$ be any continuous function which maps $S^{n-1} \subset B^n$ to $x_0 \in S^n$, maps no other points to x_0, and is one-to-one on $B^n \setminus S^{n-1}$.

We now claim that the embedding g of $X \sqcup_f B^n$ in $F(B^n)$ which we require is the map induced by the function

$$h(x) = \begin{cases} x, & x \in X, \\ p(x) + s(r(x)), & x \in B^n \end{cases}$$

of $X \sqcup B^n$ into $F(B^n)$.

Note firstly that g is well defined, since if $x \in S^{n-1} \subset B^n$, then $p(x) + s(r(x)) = f(x) + 0 = f(x) = h(f(x))$, so that the two possible expressions for $h(x)$ coincide. Hence, by the definition of the adjunction space, and since h is continuous, g is continuous. It is easy to check, further, that g is one-to-one, using the definitions of p, s, r and the fact that $\mathrm{gp}(C_1) \cap \mathrm{gp}(C_2) = \{e\}$. Using the facts that g is a homeomorphism of X onto its image, that $g(X) = X$ is closed in $F(B^n)$, and that B^n is compact, we see that g *is* a closed mapping. Hence g is a homeomorphism of $X \sqcup_f B^n$ onto its image in $F(B^n)$.

It remains to show only that $g(X \sqcup_f B^n) = Y$ is regularly situated with respect to B^n, and that $Y \setminus \{e\}$ is algebraically a free basis for the subgroup it generates.

Since $p(B^n)$ is a compact subset of the k_ω-space $F(B^n)$, $p(B^n) \subseteq \mathrm{gp}_l(B^n)$ for some $l \in \mathbf{N}, l \geq 2$. Consider now $w \in \mathrm{gp}(Y), w \neq e$, and write

$$w = \sum_{i=1}^{t} \varepsilon_i x_i + \sum_{i=t+1}^{k} \eta_i(p(y_i) + s(r(y_i))) \tag{1}$$

where each ε_i and η_i is -1 or 1, $x_i \in X$ for $i = 1, \ldots, t$, $y_i \in g(B^n) \setminus X$ for $i = t + 1, \ldots, k$, and the representation for w is reduced.

Now r is one-to-one on $B^n \setminus S^{n-1}$, so $\sum_{i=t+1}^{k} \eta_i s(r(y_i))$ is a reduced word in elements of the free basis $s(S^n) \setminus \{e\}$. Also, each $s(r(y_i))$ lies in $\mathrm{gp}(C_2)$, while each x_i and each $p(y_i)$ lies in $\mathrm{gp}(C_1)$. It follows that the reduced length of w relative to B^n is at least k-t, so $w \neq e$ unless k= t. If $k = t$, then $w = \sum_{i=1}^{k} \varepsilon_i x_i$, which is a reduced word in elements of the free basis X \ {e} , and w is therefore not e. Hence Y \ {e} is a free basis.

To prove regular situation of Y, in the expression (1) for w, without loss of generality assume $k \geq 21$. Then by the argument of the above paragraph the reduced length of w with respect to B^n is at least k-t.

Consider the case when $k - t < k/(3l)$. *Now* the only terms in the representation (1) of w which can cancel terms $\varepsilon_i x_i$ are the terms $\eta_j p(y_j)$, and there are k-t of the latter, each of length at most I with respect to B''. *So* the reduced length of w with respect to B^n is at least $t - l(k-t) > t\text{-}k/3 > k/(31)$.

Thus, irrespective of the value of $k - t$, the reduced length of w with respect to B^n is at least $k/(3l)$.

Now let X have k_ω-decomposition X= $\bigcup X_m$ so that F(X) has k_ω-decomposition F(X) = $\bigcup \mathrm{gp}_m(X_m)$. For b any positive integer, $\mathrm{gp}_b(B^n) \cap$ F(X) is compact and so there exists a positive integer a such that

$$\mathrm{gp}_b(B^n) \cap \mathrm{F(X)} \subseteq \mathrm{gp}_a(X_a).$$

As g is a closed mapping, Y is a k_ω-space with k_ω-decomposition Y= $\bigcup Y_m$ where $Y_m = X_m \cup g(B^n) \cup [p(B^n) \cap X]$. (Note that X is closed in F(X) and hence also in $F(B^n)$.)

To prove regular situation of X with respect to B^n it suffices to verify that

$$\mathrm{gp}_b(B^n) \cap \mathrm{gp(Y)} \subseteq \mathrm{gp}_c(Y_c) \quad \text{where } c = a + 6lb.$$

So let w $\in \mathrm{gp}_b(B^n) \cap$ gp(Y). Then, using the representation (1), we have shown above that $k \leq 3lb$.

Now put $w' = \sum_{i=1}^{t} \varepsilon_i x_i$. As observed above, the only terms in the representation (1) of w which can cancel terms $\varepsilon_i x_i$ are the terms $\eta_j p(y_j)$, and there are $k - t \leq k \leq 3lb$ of these. So

$$\begin{aligned} w' &\in \mathrm{gp}_{3lb}(Y_1) + [\mathrm{gp}_b(B^n) \cap F(X)] \\ &\subseteq \mathrm{gp}_{3lb}(Y_{3lb}) + \mathrm{gp}_a(X_a) \\ &\subseteq \mathrm{gp}_{3lb}(Y_{3lb}) + \mathrm{gp}_a(Y_a) \end{aligned}$$

$$\subseteq \mathrm{gp}_{a+3lb}(Y_{a+3lb}).$$

This implies that

$$\begin{aligned} w &\in \mathrm{gp}_{a+3lb}(Y_{a+3lb}) + \mathrm{gp}_{k-t}(Y_1) \\ &\subseteq \mathrm{gp}_{a+3lb}(Y_{a+3lb}) + \mathrm{gp}_{3lb}(Y_{3lb}) \\ &\subseteq \mathrm{gp}_c(Y_c). \end{aligned}$$

Hence, Y is regularly situated with respect to B^n, as required. The theorem follows. ❑

Remark. En route we have used the fact, proved in [5], that $F(B^n)$ contains $F(S^n)$. It should be noted that this is a special case of the theorem just proved.

An interesting consequence is derived by observing that any finite graph can be obtained from the closed unit interval by successive adjunctions of other closed unit intervals.

Corollary 1. *Let X be any finite graph. Then F(X) is topologically isomorphic to a closed subgroup of F(I).* □

Extending the above argument to higher dimensions, we see that, for any $n \in \mathbb{N}$, $F(B^n)$ contains *F(X),* where X is obtained from B^n by a finite number of successive adjunctions of balls of dimension at most n. In particular:

Corollary 2. $F(B^n)$ *contains F(X) for any finite cell complex X of dimension at most* n. □

Indeed, we easily see the following:

Corollary 3. *Let X be any cell complex of dimension* n. *Then F(X) contains F(Y), for any cell complex Y of dimension at most* n. □

Definition. Let Y be obtained from the topological space X by adjoining a finite number of cells. If the greatest dimension of the cells is n, then (Y,X) is said to be a *finite relative C W complex of dimension* n [9, p. 401].

The following generalizations of our theorem follow without extra work:

Corollary 4. *Let* (Y, X) *be a finite relative* CW *complex of dimension* n, *and F(X) a closed subgroup of* $F(B^n)$. *Then F(Y) is also a closed subgroup of* $F(B^n)$.

Corollary 5. *Let K be a closed subset of* B^n *and* $f: K \cap S^{n-1} \to X$ *a continuous function. If F(X) is a closed subgroup of* $F(B^n)$, *then* $F(X \sqcup_f K)$ *is also a closed subgroup of* $F(B^n)$. □

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