LOCALLY COMPACT GROUPS WITH EVERY CLOSED SUBGROUP OF FINITE INDEX

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Introduction

Armacost [1] characterises Δ_p, the topological group of p-adic integers, where p is any prime number, in the class of locally compact Hausdorff abelian topological groups as a non-discrete group having all of its non-trivial closed subgroups of finite index. We show that the condition 'abelian' can be dropped. As a consequence we have that a compact Hausdorff group in which all closed subgroups are open is topologically isomorphic to Δ_p, for some prime number p.

Results

LEMMA 1. *Let G be a torsion-free group with centre $Z(G)$ algebraically isomorphic to Δ_p, for some prime number p, and $G/Z(G)$ finite. Then G is abelian.*

Proof. First we show that $G/Z(G)$ is a p-group. Suppose, on the contrary, that there is a prime $q \neq p$ which divides the order of $G/Z(G)$. For each non-negative integer r, let $K_r = (Z(G))^{p^r}$. Then

(i) K_r is a normal subgroup of G,
(ii) $Z(G)/K_r$ is algebraically isomorphic to the cyclic group C_{p^r} of order p^r,
(iii) $\bigcap_{r=0}^{\infty} K_r = \{1\}$.

Thus G is algebraically isomorphic to a subgroup of $\prod_{r=0}^{\infty} G/K_r$.

In fact, since the K_r form a chain, $G = \varprojlim G/K_r$, under the natural homomorphisms.

Now each G/K_r has order divisible by q and so contains elements of order q. Let Q_r be the set of all elements of order q in K_r. Then, under the natural map from G/K_r to G/K_{r-1}, Q_r maps onto Q_{r-1}. Hence $\varprojlim Q_r$ is a set of elements of order q in G, which is thus not torsion-free. Hence $G/Z(G)$ is a finite p-group and so has a non-trivial centre, which must contain a group isomorphic to C_p. By the proof of the Lemma in Morris and Oates-Williams [4], $G/Z(G)$ cannot contain a subgroup isomorphic to $C_p \times C_p$ and so can contain no other subgroup isomorphic to C_p. But by Theorem 12.5.2 of Hall [3], a finite p-group containing a unique subgroup isomorphic to C_p is either cyclic or generalised quaternion, with generators and relations

$$\{g, h : g^{2^n} = 1, g^{2^{n-1}} = h^2, hgh^{-1} = g^{-1}\} \quad n \geqslant 2.$$

In the former case G is certainly abelian, so suppose that

$$G = \mathrm{gp}\{a, b, Z(G)\},$$

Received 31 December 1987; revised 14 July 1989.
1980 *Mathematics Subject Classification* 22D05.

where $a^{2^{n-1}} = b^2c$, $c \in Z(G)$. Then both a and b commute with $a^{2^{n-1}}$ which thus belongs to $Z(G)$. Hence $G/Z(G)$ cannot be generalised quaternion. Hence G is abelian.

LEMMA 2. *Let G be a non-discrete locally compact Hausdorff group with the property that each of its non-trivial closed subgroups is of finite index. Then G is torsion-free with centre $Z(G) \neq \{1\}$. Indeed $Z(G)$ is topologically isomorphic to Δ_p, for some prime number p.*

Proof. Suppose that G has a non-trivial element g of finite order. Then the subgroup g generates is finite and hence closed. So this subgroup is of finite index, which implies that G is finite and hence discrete, which is a contradiction. Hence G is torsion-free.

Without loss of generality, we may assume G is not abelian. Let $g_1 \in G$. Then $\overline{\text{gp}\{g_1\}}$, the closure of the subgroup generated by g_1, has finite index in G. Thus there exist $g_2, g_3, \ldots, g_n$ such that $\overline{\text{gp}\{g_1, g_2, \ldots, g_n\}} = G$. As $\overline{\text{gp}\{g_1\}}$ has finite index in G, there exists a positive integer m such that $1 \neq g_2^m \in \overline{\text{gp}\{g_1\}}$. Therefore $g_2^m \in \overline{\text{gp}\{g_1\}} \cap \overline{\text{gp}\{g_2\}}$. By assumption, then, the closed subgroup $\overline{\text{gp}\{g_1\}} \cap \overline{\text{gp}\{g_2\}}$ has finite index in G. Thus there exists a positive integer k such that $g_3^k \in \overline{\text{gp}\{g_1\}} \cap \overline{\text{gp}\{g_2\}}$. So $\overline{\text{gp}(g_1\}} \cap \overline{\text{gp}\{g_2\}} \cap \overline{\text{gp}\{g_3\}} \neq \{1\}$. By induction, $\bigcap_{i=1}^n \overline{\text{gp}\{g_i\}} \neq \{1\}$.

Let $1 \neq x \in \bigcap_{i=1}^n \overline{\text{gp}\{g_i\}}$. Clearly $x \in Z(G)$. So $Z(G)$ is a non-trivial non-discrete locally compact Hausdorff abelian group with each closed subgroup having finite index. By [**2**, Corollary 1.3], $Z(G)$ is topologically isomorphic to Δ_p, for some p.

THEOREM 1. *Let G be a non-discrete locally compact Hausdorff group. Then the following are equivalent:*

(i) *G is topologically isomorphic to Δ_p;*
(ii) *every non-trivial closed subgroup has finite index.*

Proof. Corollary 1.3 of Armacost [**2**] says that (i) implies (ii). If G has property (ii), then by Lemma 2, it satisfies the conditions of Lemma 1 and so is abelian. Thus, by Armacost [**2**, Corollary 1.3], G is topologically isomorphic to Δ_p, for some prime number p.

COROLLARY 1. *Let G be a compact Hausdorff group. Then the following are equivalent:*

(i) *G is topologically isomorphic to Δ_p, for some prime number p;*
(ii) *every closed subgroup of G is open.*

Proof. By [**2**, Theorem 1.6] (i) implies (ii). Conversely, every open subgroup in a compact group has finite index, and so by Theorem 1 we see that (ii) implies (i).

The above results complement that of [**4**] where it was shown that a compact Hausdorff group is topologically isomorphic to Δ_p, for some prime number p, if and only if all of its non-trivial proper closed subgroups are topologically isomorphic.

OPEN QUESTION. If G is a non-discrete locally compact Hausdorff group such that every non-trivial closed subgroup is open, is G necessarily topologically isomorphic either to the topological group of p-adic integers, Δ_p, or to the topological group of p-adic numbers, Ω_p?

References

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