Journal of Pure and Applied Algebra 68 (1990) 181-194
North-Holland


# WEIGHT AND $c$

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Received 28 May 1986

Dedicated to Bernhard Banaschewski on his 60th birthday

We show that every locally compact group $G$ contains a discrete subspace $X$ which is closed in $G\setminus\{1\}$ and has the property that $G$ is the smallest closed subgroup containing $X$. The minimum of the set of all cardinals card $X$ for these $X$ is called the generating rank $s(G)$ of $G$. We show that $s(G)\le w(G)\le s(G)^{\aleph_0}$ for all $G$ which are not monothetic. We calculate $s(G)$ for $G$ with $w(G)\le c$. For compact groups $G$ with $w(G)\le c$ and finitely many components, $s(G)$ is finite.

## Introduction

We say that $G$ is topologically generated by $X$ if $G$ is the smallest closed subgroup containing $X$. One would like to know something about 'thin' closed generating sets. Thin sets should be discrete. But a compact group cannot contain infinite closed discrete subsets. Hence we consider discrete subsets $X$ of a topological group $G$ which generate $G$ topologically and are closed in $G\setminus\{1\}$. Equivalently, these are subsets whose only possible accumulation point is the identity and which generate the group topologically. Such generating sets were considered by Tate in the context of Galois cohomology (see [5]) and by Mel'nikov in the context of free profinite groups (see [13]). We shall call them *suitable subsets of G* for the purposes of this paper. It is not at all clear that suitable subsets exist in general. Our first main result secures the fact that every locally compact group contains a suitable subset. This permits us to define, for all locally compact groups, a new cardinality invariant, *the generating rank*

$$s(G)=\min\{\aleph\colon \text{there is a suitable subset } X \text{ of } G \text{ such that card } X=\aleph\}.$$

The *weight*

$$w(G) = \min\{\aleph: \text{there is a base } \mathscr{X} \text{ for the topology } \mathscr{O}(G) \text{ such that card } \mathscr{X} = \aleph\}$$

of a topological space $G$ is a purely topological invariant, whereas the generating rank $s(G)$ is an invariant for locally compact topological groups. We calculate $s(G)$ for compact groups $G$ in terms of their weight $w(G)$ and discover a curious dichotomy involving the cardinal $c$ of the continuum. Indeed, if $w(G) > c$, then $s(G) = w(G)$. On the other hand, if $w(G) \leq c$, then $s(G)$ is much smaller in general. In fact, $s(G)$ is finite whenever $G$ has only finitely many components. If not, then $s(G) = w(G/G_0)$, where $G_0$ denotes the identity component of $G$.

It is worth noting that our proofs rely heavily on various structure theorems for compact groups.

## 1. Suitable generating sets

**1.1. Definition.** A subset $X$ of a topological group is called *suitable* if

(i) $G$ is the smallest closed subgroup of $G$ containing $X$; that is, topologically generates $G$.

(ii) The identity element $1 \notin X$ and $X$ is discrete and closed in $G \setminus \{1\}$; that is, 1 is the only possible accumulation point.

All topological groups are assumed to be Hausdorff. Observe that in a compact group, if $X$ is suitable, then $X \cup \{1\}$ is compact. Indeed, $X \cup \{1\}$ is the one-point-compactification of the discrete space $X$.

The main result of this section is that every locally compact group contains a suitable subset. This will be accomplished in a sequence of lemmas.

The following simple lemma, whose proof we shall leave to the reader, will be helpful:

**1.2. Lemma.** *Let $G$ be a compact Hausdorff space and $A$ a closed subset. For a subset $X$ of $G \setminus A$ the following conditions are equivalent*:

(1) *$X$ is discrete and closed in $G \setminus A$.*

(2) *For each open subset $U$ of $G$ containing $A$ the set $X \setminus U$ is finite.* □

**1.3. Lemma.** *If $G$ is a topological group which is the product $NH$ of two subgroups $N$ and $H$ each of which has a suitable subset, then $G$ contains a suitable subset.*

**Proof.** If $X$ and $Y$ are suitable subsets of $N$ and $H$, respectively, then $X \cup Y$ is discrete and closed in $G \setminus \{1\}$ and generates $G$ topologically, hence is a suitable subset of $G$. □

Of course, this lemma generalizes to the case of any finite number of subgroups.

**1.4. Lemma.** *If $f: G \to H$ is a surjective morphism of compact groups, and if $G$ has a suitable subset $X$, then $H$ has a suitable subset $f(X) \setminus \{1\}$.*

**Proof.** Let $X$ be a suitable subset of $G$ and set $Y = f(X) \setminus \{1\}$. Clearly, $Y$ topologically generates $H$. In order to show that $Y$ is suitable, we verify condition (2) of Lemma 1.2. Let $V$ denote an open neighborhood of 1 in $H$. Then $U \overset{\text{def}}{=} f^{-1}(V)$ is an open neighborhood of 1 in $G$. Since $f$ is surjective, $f(U) = V$ and $f(G \setminus U) = H \setminus V$. But $X \setminus U$ is finite as $X$ is suitable, and so $Y \setminus V = f(X \setminus U)$ is finite. Hence $Y$ is suitable. (Thanks to the referee for this short proof!) □

**1.5. Lemma.** *Every direct product of any family of topological groups with suitable subsets has a suitable subset.*

**Proof.** Let $G_j$, $j \in J$ be a family of topological groups, each with a suitable subset $X_j$. Now let $e_j: G_j \to P$ with the direct product $P$ of the $G_j$ be defined by $e_j(g) = (g_k)_{k \in J}$ with $g_j = g$ and $g_k = 1$, otherwise. Then $X = \bigcup \{e_j(X_j): j \in J\}$ is the required suitable subset of $P$ as is readily verified with the aid of Lemma 1.2. □

**1.6. Lemma.** *If $H$ is a compact abelian group, then $H$ has a suitable subset.*

**Proof.** Let $A$ be the dual of $H$. Then $A$ can be embedded into a divisible group $D$. Consequently, there is a surjective morphism $f: G \to H$ with $G = \hat{D}$. Now $D$ is a direct sum of a $\mathbb{Q}$-vector space and some family of groups $\mathbb{Z}(p^\infty)$. Hence $G$ is a direct product of groups $\hat{\mathbb{Q}}$ and $\mathbb{Z}_p$ (the $p$-adic groups). Since $\hat{\mathbb{Q}}$ and $\mathbb{Z}_p$ are monothetic, they have (one-point) suitable sets. Hence $G$ has a suitable subset by Lemma 1.5. But then $H$ has a suitable subset by Lemma 1.4. □

Recall that a compact connected group is said to be *semisimple* if its center is totally disconnected.

**1.7. Lemma.** *A compact connected semisimple group $N$ has a suitable subset.*

**Proof.** The group $N$ is a quotient of a direct product $\prod \{L_j : j \in J\}$ of simple Lie groups [1]. By a result of Kuranishi [11] (see also [14]), each $L_j$ has a (two-point) suitable subset. Once again, the result follows from Lemma 1.5 and Lemma 1.4. □

**1.8. Lemma.** *Every compact connected group $G$ has a suitable subset.*

**Proof.** The group $G$ is the product of its semisimple compact commutator subgroup $N$ and the identity component $H$ of its center [1]. The result now follows from Lemmas 1.6 and 1.7 via Lemma 1.3. □

We now have to deal with the case of totally disconnected compact groups. Douady [5] reports on a proof of Tate that every compact totally disconnected group has a suitable set. This proof is extremely condensed. We give here a different proof which depends on an interesting structure theorem of Varopolous [15].

**1.9. Lemma.** *Suppose that $G$ is compact and that there is a descending series $G = G_1 \rhd G_2 \rhd \cdots \rhd G_n \rhd \cdots$ such that:*

(i) $\bigcap G_n = \{1\}$.

(ii) *For each $n = 1, 2, \ldots$, the quotient group $G_n/G_{n+1}$ has a suitable subset.*

(iii) *For each $n = 1, 2, \ldots$ there is a compact subspace $Y_n \subseteq G_n$ containing 1 such that $(y, g) \mapsto yg : Y_n \times G_{n+1} \to G_n$ is a homeomorphism.*

*Then $G$ has a suitable subset.*

**Proof.** For $n = 1, 2, \ldots$ let $X_n \subseteq Y_n$ be such that $(X_n G_{n+1})/G_{n+1}$ is suitable in $G_n/G_{n+1}$. Then for every $x \in X_n$, the set $xG_{n+1}$ is isolated in $X_n G_{n+1}/G_{n+1}$, hence in $G/G_{n+1}$, and so $\{x\}$ is isolated in $X_n$. Moreover, if $g \in \bar{X}_n \setminus X_n$, then $gG_{n+1} \in (X_n G_{n+1} \cup G_{n+1})/G_{n+1}$, whence $g \in X_n G_{n+1} \cup G_{n+1}$. But since each point of $X_n$ is isolated, we may conclude that $g \in X_n \cup G_{n+1}$. Because of $g \notin X_n$, we finally have $g \in G_{n+1}$. On the other hand, we know $g \in \bar{X}_n \subseteq Y_n$ and thus $g \in Y_n \cap G_{n+1} = \{1\}$. Therefore $g = 1$ and $X_n$ is discrete in $G \setminus \{1\}$. Now the set $Z_n = X_1 \cup \cdots \cup X_n$ is discrete in $G$. We set $X = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} Z_n$. Let $U \subseteq G$ be open and contain 1. Since $\bigcap_{n=1}^{\infty} G_n = \{1\}$, by the compactness of $G$, we find an $n$ such that $G_{n+1} \subseteq U$. Now $X \setminus U \subseteq (Z_n \cup G_{n+1}) \setminus U = Z_n \setminus U$ is finite. Hence $X$ is discrete $G \setminus \{1\}$ by Lemma 1.2. Finally, we claim that $X$ topologically generates $G$. Indeed, let $H$ be the closed subgroup topologically generated by $X$. Further, let $N$ be an arbitrary compact normal subgroup such that $G/N$ is a Lie group. Since Lie groups satisfy the descending chain condition, we find an $n$ such that $G_{n+1} \subseteq N$. Since $X_k G_{k+1}/G_{k+1}$ topologically generates $G_k/G_{k+1}$, we conclude that $G_{n-j}/N \subseteq HN/N$, $j = 1, \ldots, n-1$ and thus that $G \subseteq HN$. Since $N$ was arbitrary and $G = \lim G/N$, we have $G = H$ as we had claimed. □

**1.10. Lemma.** *Every compact totally disconnected group $G$ has a suitable subset.*

**Proof.** By a theorem of Varopoulos [15, Theorem 2, p. 458], there is a sequence $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n \rhd \cdots$ with $\bigcap G_n = \{1\}$ and such that $G_n/G_{n+1}$ is a direct product of finite simple groups. Hence $G_n/G_{n+1}$ has a suitable subset by Lemma 1.5. Our assertion then follows from the preceding Lemma 1.9, where we note that condition (iii) is satisfied since every surjective morphism of compact zero dimensional groups allows a continuous cross section (see for instance [10]). □

**1.11. Lemma.** *Every compact group $G$ has a suitable subset.*

**Proof.** By a theorem of Lee [12] there is a compact totally disconnected subgroup

$E$ in $G$ such that $G = G_0E$. The assertion now follows from Lemmas 1.10, 1.8, and 1.3. □

Now we are ready for the first main theorem.

**1.12. Theorem.** *Every locally compact group has a suitable subset.*

**Proof.** Let $H$ be an open subgroup of $G$ such that $H/H_0$ is compact. Let $C$ be a maximal compact subgroup of $H$. Then we find one-parameter subgroups $E_1, \ldots, E_n$, each isomorphic to $\mathbb{R}$ such that $H = CE_1 \cdots E_n$ (see, for example, [7] for the Lie group version, which is extended by lifting one-parameter subgroups). Then all $E_k$ have suitable (two element) subsets, since $\mathbb{R}$ has the dense subgroup $\mathbb{Z} + \mathbb{Z}\sqrt{2}$. By Lemma 1.11, the group $C$ has a suitable subset. By the $(n+1)$-subgroups version of Lemma 1.3, it follows that $H$ has a suitable subset $Y$. Now let $Z \subseteq G \setminus H$ be such that $gH \cap Z$ is singleton for each coset $gH$. Then $X = Y \cup Z$ is the desired suitable subset of $G$. □

## 2. Generating compact groups

We begin with some information on compact connected groups which we record completely for the sake of easy reference. Our first definition fixes a terminology which is suggested by the important concept of a maximal torus in a compact Lie group.

**2.1. Definition.** If $G$ is a compact group, then we shall call any maximal compact connected abelian subgroup of $G$ a *maximal protorus*.

**2.2. Lemma.** *Let $F: G \to A$ be a surjective morphism of a compact connected group onto a compact abelian group. Then $f(T) = A$ for any maximal protorus $T$ in $G$.*

**Proof.** We have $G = Z_0 \cdot G'$ with the identity component $Z_0$ of its center and the commutator subgroup $G'$. Since $f(G') = \{1\}$, we know $f(Z_0) = A$; but then $f(T) = A$, since $Z_0$ is clearly contained in every maximal protorus. □

**2.3. Lemma.** *Let $f: G \to H$ be a morphism of compact connected groups. Then $f(T)$ is a maximal protorus of $H$ for every maximal protorus $T$ of $G$. Moreover, if $S$ is a maximal protorus of $H$, then there is a maximal protorus $T$ of $G$ with $f(T) = S$.*

**Proof.** Let $A$ be a maximal protorus of $H$ containing $f(T)$. Let $G_1$ be the identity component of $f^{-1}(A)$. Since components map onto components under surjective morphisms of compact groups, the restriction $f|_{G_1}: G_1 \to A$ is surjective. By Lemma 2.2, we then conclude $f(T) = A$ which proves the first claim.

If now $S$ is a maximal protorus of $H$, let $G_1$ denote the identity component of $f^{-1}(S)$. If $T_1$ is a maximal protorus of $G_1$, then $f(T_1) = S$ by Lemma 2.2. Now let $T$ be a maximal protorus of $G$ containing $T_1$, then $f(T)$ is a compact connected abelian subgroup of $H$ containing $S$, hence by maximality of $S$, agrees with $S$. This proves the second claim. □

**2.4. Proposition.** *Let $G$ be a compact connected group. Then*:

(i) *The maximal protori of $G$ are conjugate.*

(ii) *The union of all maximal protori is $G$.*

(iii) *If $T$ is a maximal protorus, then there is an element $g \in G$ such that $G$ is topologically generated by $T \cup \{g\}$.*

**Proof.** (i) Since $G = Z_0 \cdot G'$ it clearly suffices to prove the claim for semisimple compact connected groups such as $G'$. So we assume $G$ to be semisimple. Now every compact connected semisimple group is the quotient of a direct product $\prod_{j \in J} L_j$ of simple connected compact Lie groups. In view of Lemma 2.3, it therefore suffices to assume that $G$ itself is such a product. Then any maximal protorus $T$ of $G$ is of the form $\prod_{j \in J} T_j$ with a maximal torus $T_j$ of $L_j$. Suppose that $T^{(n)}$, $n = 1, 2$ are two maximal protori in $G$. Since in any compact connected Lie group the maximal tori are conjugate, for each $j \in J$, there is a $g_j$ such that $T^{(2)} = g_j T^{(1)} g_j^{-1}$. If we now set $g = (g_j)_{j \in J}$, then $T^{(2)} = g T^{(1)} g^{-1}$, as asserted.

(ii) We shall consider the function $e: G \times T \to G$ given by $e(g, t) = gtg^{-1}$. We must show that $e$ is surjective. But this map is the limit of the maps $e_N: G/N \times T/N \to G/N$ as $N$ ranges through the filterbasis of all compact normal subgroups such that $G/N$ is a Lie group. Since all $e_N$ are surjective, the surjectivity of $e$ follows.

(iii) Again in view of the fact that $G = Z_0 \cdot G'$ and that $Z_0$ is contained in every maximal protorus, it suffices to consider the case that $G$ is semisimple. As $G$ is a quotient of a direct product of a family of simple compact Lie groups, in view of Lemma 2.3, it suffices to assume that $G$ itself is such a product. Thus we have $G = \prod_{j \in J} L_j$ and a maximal protorus $T = \prod_{j \in J} T_j$ with a maximal torus $T_j$ of $L_j$ for each $j \in J$. By Kuranishi's Theorem, there is an element $g_j$ such that $L_j$ is topologically generated by $T_j \cup \{g_j\}$. If we set $g = (g_j)_{j \in J}$, then $G$ is topologically generated by $T \cup \{g\}$. □

**2.5. Corollary.** *Let $G$ be a compact connected group and $T$ a maximal protorus in $G$. If $X$ is a suitable subset of $T$, then there is a $g \in G$ such that $X \cup \{g\}$ is a suitable subset of $G$.* □

## 3. Some background information on the weight

For our discussion of the generating rank $s(G)$ in Section 4 below we need certain information on the weight of compact groups. Observe that for an infinite compact group $G$ the weight $w(G)$ is at the same time the *local weight* in the sense that it is the smallest among all the cardinals $\aleph$ for which there is a neighborhood basis $\mathscr{U}$ of 1 such that card $\mathscr{U}=\aleph$.

We shall first make a few observations concerning the weight of locally compact groups. The identity component of a topological group $G$ will be denoted by $G_0$. Surely there will be many ways of proving these matters; here is the way we do it:

**3.1. Proposition.** *For an infinite locally compact group $G$ we have the following conclusions*:

(i) $w(G)=\max\{w(G_0), w(G/G_0)\}$.

(ii) *If $G$ is connected and $C$ is a maximal compact subgroup, then* $w(G)=\max\{\aleph_0, w(C)\}$.

(iii) *If $G$ is compact and connected, and if $Z_0$, $G'$ and $T$ denote the identity component of the center, the commutator subgroup, and a maximal protorus, respectively, then* $w(G)=\max\{w(Z_0), w(G')\}=w(T)$.

(iv) *If $H$ is a compact open subgroup of $G$, then $w(H)$ is independent of the choice of $H$ if $G$ is nondiscrete and* card $G/H$ *is independent of the choice of $H$, if $G$ is noncompact.*

**Proof.** (i) The space underlying $G$ is homeomorphic to the Cartesian product $G_0\times G/G_0$ (see for instance [10]). The claim follows.

(ii) Here $G$ is homeomorphic to the Cartesian product $C\times\mathbb{R}^n$ (see [7]). This implies the claim.

(iii) The group $G$ is a quotient of $Z_0\times G'$, whence

$$w(G)\le\max\{w(Z_0), w(G')\},$$

and since $w(Z_0)\le w(G)$ and $w(G')\le w(G)$, equality follows.

Since $w(T)\le w(G)$, it remains to show that $w(G)\le w(T)$. For this purpose we first consider an identity neighborhood $V$ of $T$. Since $G$, as a compact group, has a basis of open identity neighborhoods invariant under inner automorphisms, there is an open invariant identity neighborhood $U$ of $G$ such that $U\cap T\subseteq V$. Hence for all $g\in G$ we have $U\cap gTg^{-1}=g(U\cap T)g^{-1}\subseteq gVg^{-1}$. As $G=\bigcup_{g\in G} gTg^{-1}$ by Proposition 2.4, we have $\bigcup_{g\in G} g(U\cap T)g^{-1}=U\subseteq\bigcup_{g\in G} gVg^{-1}$. If we set $\tilde{V}=\bigcup_{g\in G} gVg^{-1}$, then the function $V\mapsto\tilde{V}$ maps any basis of identity neighborhoods of $T$ onto a basis of identity neighborhoods of $G$. Since the weight and the local weight of a compact group agree, $w(G)\le w(T)$ follows. (This line of proof again follows an idea of the referee and replaces the authors' more complicated original argument.) $\square$

**3.2. Proposition.** *Let $G$ be a compact group and $D$ a totally disconnected normal subgroup contained in $G_0$. Then $w(G)=w(G/D)$.*

**Proof.** We first prove the assertion when $G$ is compact connected abelian. In this case, $w(G/D)=\operatorname{card}\widehat{G/D}=\operatorname{card}D^{\perp}$, where $D^{\perp}$ denotes the annihilator of $D$ in $\hat{G}$. Since $D$ is totally disconnected, $\hat{D}\cong\hat{G}/D^{\perp}$ is a torsion group. Hence the pure subgroup generated by $D^{\perp}$ in $\hat{G}$ is $\hat{G}$. Thus $w(G/D)=\operatorname{card}D^{\perp}=\operatorname{card}\hat{G}=w(G)$, which proves the claim in this case.

We now address the general case. By Proposition 3.1(i), we have $w(G)=\max\{w(G_0), w(G/G_0)\}$. It therefore suffices to consider the case that $G$ is connected. We assume that $G$ is not abelian. Then there are two surjective morphisms

$$Z_0\times\prod_{j\in J}L_j\to G\to Z_0/\Delta\times\prod_{j\in J}L_j/Z_j,$$

with $L_j$ a nonempty family of simply connected simple compact Lie groups having finite centers $Z_j$, and with $\Delta$ a totally disconnected subgroup of $Z_0$. Thus $w(Z_0/\Delta)=w(Z_0)$ by the first part of the proof. So we have

$$\begin{aligned}\max\{w(Z_0/\Delta),\aleph_0,\operatorname{card}J\} &= \max\{w(Z_0),\aleph_0,\operatorname{card}J\}\le w(G)\\ &\le \max\{w(Z_0/\Delta),\aleph_0,\operatorname{card}J\}.\end{aligned}$$

If we write $H=Z_0/\Delta\times\prod_{j\in J}L_j/Z_j$, we now have $w(G)=w(H)$. If $D_1$ is the image of $D$ in $H$, then $D_1$ is central, hence contained in $Z_0/\Delta$. By the first part of the proof again, $w(Z_0)=w(Z_0/\Delta)=w((Z_0/\Delta)/D_1)$ which implies

$$\begin{aligned}w(G)&\ge w(G/D)\ge w(H/D_1)\\ &=\max\{w((Z_0/\Delta)/D_1),\aleph_0,\operatorname{card}J\}\\ &=\max\{w(Z_0),\aleph_0,\operatorname{card}J\}=w(G).\end{aligned}$$

This completes the proof. □

We denote by $N\rtimes H$ a semidirect product of the normal subgroup $N$ by a subgroup $H$.

**3.3. Proposition.** *Let $G$ be a compact group. Then there is a totally disconnected compact normal subgroup $D$ contained in the center of $G_0$ such that*

$$\begin{aligned}G/D&\cong\left(\mathbf{T}^{J_0}\times\prod_{n\in\mathbb{N}}L_n^{J_n}\right)\rtimes E\\ &=\mathbf{T}^{J_0}\rtimes\left(\prod_{n\in\mathbb{N}}L_n^{J_n}\rtimes E\right)\end{aligned}$$

*where $\mathbf{T}$ is the circle group, $L_n$ a simple compact Lie group with trivial center, the $J_n$, $n=0,1,\ldots$ are appropriate sets, and where $E$ is a compact totally disconnected group.*

**Proof.** By Lee's Theorem (see [12]) there is a compact totally disconnected group $F$ such that $G=G_0\cdot F$ and $G_0\cap F$ is in the center of $G_0$. Let $D=(G_0\cap F)\cdot(Z_0\cap(G_0)')\cdot Z((G_0)')$ where $Z_0$ is the identity component of the center of $G_0$ and $Z((G_0)')$ denotes the center of $(G_0)'$. Then $D$ is a compact totally disconnected normal subgroup of $G$ contained in the center of $G_0$, and $G/D\cong(A\times S)\rtimes E$, where $A=(Z_0\cdot D)/D$, $S=((G_0)'\cdot D)/D$, and $E=(F\cdot D)/D$. Now $S\cong\prod_{n\in\mathbb{N}}L_n^{J_n}$ for some simple compact Lie groups $L_n$ with trivial center and with appropriate sets $J_n$. The assertion of the lemma will now be implied by the subsequent lemma. □

**3.4. Lemma.** *Let $A$ be a compact connected abelian group. Then there is a totally disconnected subgroup $D$ such that $A/D\cong\mathbf{T}^J$, for some set $J$.*

**Proof.** The claim follows by duality: Indeed $\hat{A}$ is a torsion free abelian group which then contains a maximal free subgroup $F$ such that $\hat{A}/F$ is a torsion group. Then the annihilator $D=F^\perp$ of $F$ in $A$ satisfies the requirements. □

## 4. Weight and generating rank

Our Theorem 1.12 enables us to introduce a new cardinal invariant for locally compact groups. (For profinite groups, this cardinal was already formulated by Mel'nikov [13].)

**4.1. Definition.** For a locally compact group $G$ we set

$$s(G)=\min\{\aleph\colon \text{there is a suitable subset } X \text{ with } \operatorname{card}X=\aleph\}.$$

We call $s(G)$ the *generating rank of $G$*.

**4.2. Remark.** *For any locally compact group $G$ we have $s(G)\leq w(G)$.*

**Proof.** If $X$ is a suitable subset of $G$ with cardinality $s(G)$, and if $\mathscr{B}$ is a basis of the topology of $G$ with cardinality $w(G)$ then for every $x\in X$ there is an element $U(x)\in\mathscr{B}$ with $U(x)\cap X=\{x\}$. Then $x\mapsto U(x):X\to\mathscr{B}$ is an injective function and thus $s(X)\leq w(G)$. □

Lemma 1.4 instantaneously translates into the following remark:

**4.3. Lemma.** *For a surjective morphism of compact groups $f:G\to H$ the inequality $s(H)\leq s(G)$ holds.* □

Our objective is to show that for compact nonmonothetic groups $G$ we have $s(G)^{\aleph_0}=w(G)^{\aleph_0}$.

For convenience, we make the following definition:

**4.4. Definition.** A compact group $G$ will be called *good* if it satisfies $s(G)^{\aleph_0} = w(G)^{\aleph_0}$. We shall write $\bar{w}(X) = w(X)^{\aleph_0}$, further $\bar{s}(X) = s(X)^{\aleph_0}$ and finally $\overline{\text{card}}(X) = (\text{card}(X))^{\aleph_0}$.

**4.5. Lemma.** *If $D$ is a totally disconnected compact normal subgroup of a compact group $G$ such that $D$ is contained in $G_0$ and $G/D$ is good, then $G$ is good.*

**Proof.** By Proposition 3.2 and Lemma 4.3, $\bar{w}(G) = \bar{w}(G/D) = \bar{s}(G/D) \leq \bar{s}(G)$. □

The preceding lemma and Proposition 3.3 will allow us to reduce the proof that a given compact group $G$ is good to proving that a group of the form

$$(*) \qquad \mathbf{T}^{J_0} \rtimes \left( \prod_{n \in \mathbb{N}} L_n^{J_n} \rtimes E \right)$$

is good, provided $w(E)$ or any of the cardinals, $\text{card}\, J_n$, is large enough.

By Remark 4.2, in order to show that a group $G$ of type $(*)$ is large, it suffices to prove

$$(**) \qquad \bar{s}(G) \geq \max \left\{ \bar{w}(E), \max_{n=0,1,\ldots} \overline{\text{card}}\, J_n \right\}.$$

This task forces us to handle the generating rank of semidirect products. Not too much can be said. Let $G$ be a locally compact group with $G = N \rtimes H$. Then $s(H) \leq s(G) \leq s(N) + s(H)$. Indeed the first inequality follows from Lemma 1.4. If $Y$ is a suitable subset of $N$ and $X$ is a suitable subset of $H$, then $(Y \times \{1\}) \cup (\{1\} \times X)$ is a suitable subset of $G$. Hence the second inequality holds. In order to see that the bounds are sharp let us consider discrete finite abelian groups. If $N = H = \mathbb{Z}/2\mathbb{Z}$ and the product is direct, then $s(N) = s(H) = 1$ and $s(G) = 2$. So the upper bound is attained. Next let $H = \mathbb{Z}/2\mathbb{Z}$ and $N = \mathbb{Z}/3\mathbb{Z}$ and define $G$ to be the direct product of these two groups. Then $G$ is cyclic and $s(G) = 1 = s(H) < 2 = s(N) + s(H)$. However, the following information concerning semidirect products will be useful to us:

**4.6. Lemma.** *Suppose that $G = N \rtimes H$ is compact. Then the normal subgroup $N \rtimes \{1\}$ contains a compact topological generating subset $Y$ with*

$$w(Y) \leq w(H)s(G).$$

**Proof.** If $X$ is a suitable subset of $G$ of cardinality $s(G)$, we set $X' = X$ if $X$ is closed and $X' = X \cup \{1\}$ otherwise, and thus $s(G) = \text{card}\, X = w(X')$. Let $X_N$ denote the projection of $X'$ into $N$. Then the function $(h, x) \mapsto (h, 1)(x, 1)(h, 1)^{-1}$ maps the compact space $H \times X_N$ surjectively onto a compact subspace $Y$ of $N \times \{1\}$. We have $w(Y) \leq w(H \times X_N) = w(H)w(X_N) \leq w(H)w(X') = w(H)s(G)$. It remains to show that the space $Y$ topologically generates $N \times \{1\}$. For this purpose let $M$ be the compact subgroup of $N \times \{1\}$ which is topologically generated by $Y$. Since $Y$ is invariant under all inner automorphisms of $G$ induced by the elements of $\{1\} \times H$, the same is true for $M$. Hence $M(\{1\}\times H)$ is a compact subgroup of $G$. But it also contains $X'$, whence it agrees with $G$. This implies $M=N\times\{1\}$. □

The estimate in this lemma may appear curious. However, we shall apply it in a situation where we have lower estimates for topological generating sets of $N$ and where $w(H)$ is small by comparison with $s(G)$. Then these lower estimates give lower estimates for $s(G)$.

Mel'nikov [13] has the following result:

**4.7. Lemma.** *If $G$ is an infinite compact totally disconnected group, then*

$$w(G) = \max\{\aleph_0, s(G)\}. \quad \square$$

**4.8. Lemma.** *Let $L$ be a compact infinite Lie group, $X$ a compact pointed nonsingleton space, and $C(X,L)$ the set of continuous functions from $X$ to $L$ mapping the base point to the identity. Then* card $C(X,L)=\bar{w}(X)$.

**Proof.** In [8, Proposition 1.4.1] we have shown card $C(X,\mathbf{I})=\bar{w}(X)$, where $\mathbf{I}$ denotes the unit interval with base point 0 and $C$ refers to base point preserving continuous functions. Since $C(X,\mathbf{I}^n)\cong C(X,\mathbf{I})^n$ the set $C(X,\mathbf{I}^n)$ has still the same cardinality for all $n=1,2,\ldots$. Now there are topological embeddings $\mathbf{I}\to L\to\mathbf{I}^n$ for an appropriate natural number $n$. Hence there are injections $C(X,\mathbf{I})\to C(X,L)\to C(X,\mathbf{I}^n)$. Thus card $C(X,L)=$ card $C(X,\mathbf{I})=\bar{w}(X)$ as was asserted. □

The following is a key lemma:

**4.9. Lemma.** *Let $L$ be a compact Lie group and $J$ a nonsingleton set. Then* $\bar{s}(L^J)=\overline{\text{card}}\,J$.

**Proof.** Since $\bar{s}(L^J)\le\bar{w}(L^J)=\overline{\text{card}}\,J$ by Remark 4.2, we must show the reverse inequality. This claim will be implied by the following slightly more general lemma, which we will also need in the sequel. □

**4.10. Lemma.** *Let $L$ be an infinite compact Lie group, $J$ a nonsingleton set, and $Y$ a compact subspace of $L^J$ which contains the identity and topologically generates $L^J$. Then* $\overline{\text{card}}\,J=\bar{w}(Y)$.

**Proof.** We consider $Y$ and $L$ as pointed compact spaces with the identities of $L^J$ and $L$, respectively, as base points. Let $C(Y,L)$ denote the set of all base point preserving continuous functions of $Y$ into $L$. Then

$$\text{card}\, C(Y,L) = \bar{w}(Y) \tag{1}$$

by Lemma 4.8. We now consider the compact group $G = L^{C(Y,L)}$ and note

$$w(G) = \bar{w}(Y) \tag{2}$$

in view of (1). We also have

$$w(L^J) = \text{card } J. \tag{3}$$

Hence by (2) and (3), in order to prove $\overline{\text{card}}\, J \le \bar{w}(Y)$ it suffices to establish the following claim:

$$\text{There is a surjective morphism of compact groups } G \to L^J. \tag{4}$$

Since $w(Y) \le w(L^J) = \text{card } J$ implies $\bar{w}(Y) \le \overline{\text{card}}\, J$, this will prove the lemma. In order to prove claim (4) we prove a certain universal property of $G$. First we observe that there is a topological embedding $\varepsilon : Y \to G$ given by $\varepsilon(y)(f) = f(y)$ for $y \in Y$ and $f \in C(Y, L)$. Now we establish the following universal property (U) of $G$:

(U) For each base point preserving continuous function $f : Y \to L$ there is a morphism of compact groups $f' : G \to L$ such that $f = f' \circ \varepsilon$.

Indeed, for an arbitrary element $\gamma \in G$, $\gamma : C(Y, L) \to L$ we simply define $f'(\gamma) = \gamma(f)$. Then $f'$, as a point evaluation, is a morphism of compact groups, and for $y \in Y$ we note $f(y) = \varepsilon(y)(f) = f'(\varepsilon(y)) = (f' \circ \varepsilon)(y)$.

Now we define, for each $j \in J$ a continuous function $f_j : Y \to L$ by $f_j(y) = y(j)$ (remembering that $y \in L^J$ is a function $y : J \to L$). As a point evaluation, $f_j$ is continuous. By the universal property (U), we find a morphism of compact groups $f_j' : G \to L$ with $y(j) = f_j(y) = f_j'(\varepsilon(y))$. By the universal property of the Cartesian product $L^J$ we now find a morphism of compact groups $F : G \to L^J$ given by $F(\gamma)(j) = f_j'(\gamma)$. In particular, $F(\varepsilon(y))(j) = f_j'(\varepsilon(y)) = y(j)$; that is, $F(\varepsilon(y)) = y$. Thus the image of $F$ contains $Y$. Since $Y$ topologically generates $L^J$, the morphism $F$ is surjective and claim (4) is proved. This finishes the proof of the lemma. □

Now we assume that $G$ is a compact group of type (*), and that $w(G) \ge c$. We begin to prove statement (**). If $\bar{w}(E) = \bar{w}(G) \ge c$, then $w(E) \ge \aleph_0$ and then $w(E) = s(E)$ by Lemma 4.7. Thus $s(G) \ge s(E) = w(E)$ and (**) follows. Now assume $\bar{w}(E) < \bar{w}(G)$ (hence $w(E) < w(G)$), but also that at least one of the cardinals $\overline{\text{card}}\, J_m$, $m \ge 1$, equals $\bar{w}(G)$. Set $G_1 = \prod_{n \in \mathbb{N}} L_n^{J_n} \times E$. Then $s(G_1) \le s(G)$ by Lemma 4.3. In view of Lemma 4.6, we find a compact topological generating subset $Y$ of $\prod_{n \in \mathbb{N}} L_n^{J_n} \times \{1\}$ with $w(Y) \le w(E) s(G_1) \le w(E) s(G)$. The projection $Y_m$ of $Y$ into $L_m^{J_m}$ is a compact topological generating subset of $L_m^{J_m}$. Lemma 4.10 implies $\bar{w}(G) = \overline{\text{card}}\, J_m \le \bar{w}(Y_m) \le \bar{w}(Y)$. Thus $\bar{w}(G) \le \bar{w}(Y) \le \bar{w}(E) \bar{s}(G)$, and since $\bar{w}(E) < \bar{w}(G)$, this implies $\bar{w}(G) \le \bar{s}(G)$.

Finally, we assume that $\bar{w}(E) < \bar{w}(G)$ and that $\overline{\text{card}}\, J_n < \bar{w}(G)$ for all $n = 1, 2, \ldots$. Now we set $N = \mathbf{T}^{J_0}$ and $H = \prod_{n \in \mathbb{N}} L_n^{J_n} \rtimes E$ and apply Lemma 4.6 again to the semidirect product $NH$. Exactly as before we find $\bar{w}(G) = \overline{\text{card}}\, J_0 \le \bar{s}(G)$, and this finishes the proof of condition (**).

We have now proved the following main theorem of this section:

**4.11. Theorem.** *Let $G$ be a compact group with $w(G) \geq c$. Then $s(G)^{\aleph_0} = w(G)^{\aleph_0}$.* □

Our next theorem describes the situation for compact groups whose weight does not exceed $c$. But first we prove a lemma:

**4.12. Lemma.** *If $f: G \to H$ is a surjective morphism of compact groups, then $s(G) \leq s(H) + s(\ker f)$.*

**Proof.** Let $X_1$ be a suitable subset of $K = \ker f$ and $Y$ a suitable subset of $H$, both of minimal cardinality. Then $K$ acts on $f^{-1}(Y)$ and thus there is a continuous cross-section $\sigma: Y \to f^{-1}(Y)$ (see for instance [10, p. 317, 1.12]). Set $X_2 = \sigma(Y)$. Then $X_1 \cup X_2$ is a suitable subset of $G$. Hence $s(G) \leq s(H) + s(K)$. □

**4.13. Theorem.** *Let $G$ be a compact group with $w(G) \leq c$. Then the following cases can occur:*

(i) $s(G) = 0$ *if* $G = \{1\}$.
(ii) $s(G) = 1$ *if $G$ is monothetic, in particular, if $G$ is connected and abelian.*
(iii) $s(G) = 2$ *if $G$ is connected and nonabelian.*
(iv) $\max\{\aleph_0, s(G)\} = w(G/G_0)$ *if $G$ has infinitely many components.*
(v) $s(G/G_0) \leq s(G) \leq s(G/G_0) + 2$ *if $G$ has at least* 2, *but finitely many components.*

**Proof.** (i) is clear. Since any connected compact abelian group whose weight does not exceed $c$ is monothetic, (ii) follows. From (ii) and Corollary 2.5 we infer (iii). Next we prove (iv). From Lemma 4.12 we know $s(G) \leq s(G_0) + s(G/G_0)$. If $G/G_0$ is infinite, then $\max\{\aleph_0, s(G/G_0)\} = w(G/G_0)$ by Lemma 4.7. Then by (i)–(iii), assertion (iv) follows. Finally, in order to prove (v), let $2 \leq \operatorname{card} G/G_0 < \aleph_0$. Then (v) follows from (ii), (iii) above and Lemma 4.12. □

**4.14. Theorem.** *The equation $s(G)^{\aleph_0} = w(G)^{\aleph_0}$ holds for all nonmonothetic compact groups.*

**Proof.** This is a consequence of Theorems 4.11 and 4.13. □

The cardinal number $c$ seems to play a rather special role. But we shall show elsewhere that for each cardinal $a > c$ there is a compact group $G$ such that $s(G) = a$ and $w(G) = a^{\aleph_0}$ [16, Corollary 2.16].

## 5. Supplementary remarks

We compare the generating rank $s(G)$ introduced here with another cardinality invariant, the *density*

$$d(G) = \min\{\aleph\colon \text{there is a dense subset } X \text{ of } G \text{ with card } X = \aleph\}.$$

Due to a result of Comfort and Itzkowitz (see [3, 4]) one knows that for any locally compact group one has

$$d(G) = \log w(G), \quad \text{where } \log \aleph = \min\{\aleph'\colon \aleph \le 2^{\aleph'}\}.$$

If $X$ is a suitable subset of a locally compact group $G$ of cardinality $s(G)$, and if $D$ denotes the group which is algebraically generated by $X$ in $G$, then $D$ is dense and thus, since $\mathrm{card}(D) \le \max\{\aleph_0, s(G)\}$, we have

$$\log w(G) = d(G) \le \max\{\aleph_0, s(G)\}.$$

## Acknowledgment

We thank the referee for his careful scrutiny of the original manuscript and his helpful comments. Also we thank the editor for allowing the article to mature gently like vintage wine by letting it rest a lot.

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