Free compact groups III: Free semisimple compact groups

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Abstract

For each compact connected simple Lie group G and each compact connected pointed space X we explicitly construct a compact connected group $F_G X$ containing X such that every continuous basepoint preserving map $X \to G$ whose image topologically generates G extends uniquely to a morphism $F_G X \to G$. We show that $F_G X$ is isomorphic to $G^{w(X)^{\aleph_0}}$, where $w(X)$ denotes the weight of X, and describe the embedding of X. For each cardinal $\aleph$, this allows the construction of compact connected semisimple groups S whose generating rank $s(G)$ does not exceed $\aleph$ while the weight $w(G)$ is $\aleph^{\aleph_0}$.

AMS 1980 Subject Classification: Primary: 22C05; Secondary: 22D10, 54A25.

Key Words and Phrases: Free compact group, semisimple compact Lie group, weight, generating rank, cardinal number.

0. Introduction

Free compact groups are defined by the left adjoint of the forgetful functor $\mathbf{KG} \to \mathbf{TOP_0}$ from the category of pointed topological spaces into the category of Hausdorff compact topological spaces. Thus for each topological space X we get a natural continuous map $e_X: X \to FX$ from X into a compact group sending the basepoint to the identity which has the well known universal property. This universal property allows us to conclude that e_X factors through the ČECH-STONE compactification $X \to \beta X$ in such a fashion that βX is topologically embedded into FX (see [7], Proposition 1.4). Thus, in fact, it is no loss of generality to assume that X is compact Hausdorff and we do this throughout.

Compact groups have a rich structure theory which is, in the final evaluation, based on the theory of LIE groups and PONTRYAGIN duality and which we have summarized in [8] and [9]; accordingly, the free compact groups FX have an intricate algebraic and topological structure depending on the cohomology and the weight of X. The project addressed in this sequence of papers is to clarify this structure.

For the purpose of introducing the reader to this present paper we shall summarize what we know for the case of a *connected* compact pointed space X; for in this case FX is itself connected. Then let Z_0FX denote the identity component of the center of FX and $F'X$ the commutator subgroup (which we know to be closed!). Then we have an exact sequence

$$0 \to (H^1(X,\mathbb{Z}))^\frown \to Z_0FX \times F'X \to FX \to 0, \tag{1}$$

which, intuitively, expresses the fact that FX is nearly the direct product of the center component and the commutator group except for a zero dimensional compact abelian group in their intersection which is the dual of the (discrete!) first ČECH cohomology of X. Thus, if, for instance, this cohomology vanishes, the direct product decomposition of FX is true on the nose. (For these and more general pieces of information see [9].) The factor group $FX/F'X$ is the free compact *abelian* group F_aX on which [8] gives complete information. From (1) we obtain therefore an exact sequence

$$0 \to (H^1(X,\mathbb{Z}))^\frown \to Z_0FX \to F_aX \to 0 \tag{2}$$

which we have identified as the projective resolution of F_aX in [9], Theorem 2.2. Since, on the other hand, we have discussed the projective cover of F_aX extensively in [8], we may consider the "abelian aspects of FX", namely, the center component Z_0FX and the factor group $F_aX = FX/F'F$ and their relationship as satisfactorily clarified.

So far we know little on $F'X$. This paper contributes to the structure theory of this portion of the free compact group. For compact connected spaces X, this group is a semisimple compact connected group, and one has general information on the structure of such groups. Hence we were able at an early stage to offer questions and speculations on $F'X$ (see [7]). Some of these issues will be clarified in this paper. Given the structure of $F'X$ we know the structure of FX satisfactorily for all compact connected pointed spaces X.

The approach we take is to investigate the basic "molecules" from which $F'X$ is made up. Simple compact Lie groups serve as an index set for these molecules. (A simple Lie group is by definition nonsingleton.) We shall therefore fix a compact connected simple Lie group G and define a new type of free compact group as follows

0.1. Definition. (i) A basepoint preserving function $f: X \to G$ from a pointed space into a topological group is said to be *essential* if it is basepoint preserving and G is topologically generated by $f(X)$, that is, G is the smallest closed subgroup containing $f(X)$.

(ii) For any compact group G, and any compact pointed space X, the *essential G-free compact group* F_GX is a compact group together with a natural map $e_X: X \to F_GX$ such that for every essential function $f: X \to G$ mapping the basepoint of X to the identity of G there is a unique continuous homomorphism $f': F_GX \to G$ such that $f = f' \circ e_X$.

Notice that $F_G X = \{1\}$ if $\operatorname{card} X \le 2$ and G is nonabelian, since one needs at least 2 nonidentity elements to topologically generate a nonabelian compact group. However, we shall see that for connected compact simple Lie groups G and spaces X with at least 3 points, e_X is an embedding so that f' is simply a homomorphic extension of a continuous function on the subspace X of $F_G X$ (after natural identification).

The reason that we shall restrict our attention here to essential functions is this: Assuming again that X is connected, every homomorphic image H of FX is a product of the identity component $Z_0(H)$ of its center and its (semisimple!) commutator group H'. The image of $F'X$ under this homomorphism is H'. For every continuous function $f: X \to K$ into any compact group, the subgroup H topologically generated by $f(X)$ is a homomorphic image of FX. The homomorphisms of a compact group into Lie groups separate the points. We see therefore that $F'X$ is completely determined by the homomorphisms into the commutator groups H' of groups H for which we have an essential function $X \to H$. At the very least we *have to* deal with essential functions $X \to G$ into simple connected Lie groups.

We shall give a complete structure theory of $F_G X$ and the way X is embedded into $F_G X$. Remarkably, *if X contains at least 3 points outside the basepoint, the structure of $F_G X$ is that of the power $G^{w(X)^{\aleph_0}}$ where $w(X)$ denotes the weight of X.* In [10] we have defined a cardinal invariant for all compact groups G called the *generating rank* $s(G)$ of G. It is the smallest cardinal of a discrete subset of $G \setminus \{1\}$ which topologically generates G. (The existence of such sets *is* a problem which was settled in [10].) We established the relation $s(G) \le w(G) \le s(G)^{\aleph_0}$. The essential G-free compact group $F_G(I \cup \infty)$ of the one point compactification of an arbitrary infinite (discrete) set I based at ∞ affords an example of a compact group G satisfying $s(G) \le \operatorname{card} I$ and $w(G) = (\operatorname{card} I)^{\aleph_0} = s(G)^{\aleph_0}$.

1. Homomorphically simple groups.

1.1. Definition. A compact group G will be called *homomorphically simple* if each endomorphism of G is either constant or an automorphism.

1.2. Lemma. *If $f: G \to G$ is an endomorphism of a connected Lie group G with finite fundamental group $\pi_1(G)$, and if the morphism $L(f)$ induced on the Lie algebra is an isomorphism then f is an isomorphism.*

Proof. Since $\mathbf{L}(f)$ is an isomorphism, the morphism $\widetilde{f}: \widetilde{G} \to \widetilde{G}$ induced by f on the simply connected covering group $\widetilde{G}$ of G is an isomorphism. If $p: \widetilde{G} \to G$ denotes the covering morphism then $p\widetilde{f} = fp$ by the definition of the lifting $\widetilde{f}$. If $K = \ker f$ then $\widetilde{f}(K) \subseteq K$ follows. Now $K \cong \pi_1(G)$, whence K is finite by hypothesis. Since $\widetilde{f}$ is an isomorphism this implies that $\widetilde{f}|K: K \to K$ is an isomorphism, and, since p is in particular a quotient morphism, this entails that $f: G \to G$ is an isomorphism, too. ∎

It may be of interest to note in passing that a surjective endomorphism $f: G \to G$ of a connected Lie group is an open mapping as a consequence of the

Open Mapping Theorem as G is σ-compact. (See [6], Theorem 5.29.) Hence the endomorphism $\mathbf{L}(f): \mathbf{L}(G) \to \mathbf{L}(G)$ of Lie algebras is surjective and thus is an isomorphism as a morphism between finite dimensional vector spaces.

1.3. Lemma. (i) *Any compact connected simple group and any finite simple group is homomorphically simple.*

(ii) *Conversely, a compact connected homomorphically simple Lie group is semisimple or singleton.*

Proof. (i) Suppose that G is a compact connected simple group and $f: G \to G$ a nonconstant endomorphism. Then $\mathbf{L}(f): \mathbf{L}(G) \to \mathbf{L}(G)$ is a nonzero morphism of simple Lie algebras and is, therefore an isomorphism. Since G is a simple compact connected Lie group, $\pi_1(L)$ is finite (see for instance [1], Chap VII, §3, n°2, Prop. 5). Hence Lemma 1.2 applies and shows that f must be an isomorphism. This proves the claim in the first case, and the second case is trivial.

(ii) Let G denote a compact connected nonsingleton homomorphically simple Lie group. If G is not semisimple, then G/G' is a nondegenerate torus, hence there is nontrivial character $\chi: G \to \mathbb{T}$ onto the circle group. Now let $X: \mathbb{T} \to G$ be any morphism with a finite nontrivial kernel; such morphisms exist since every compact nonsingleton group has circle subgroups. Then $f = X \circ \chi$ is a nonconstant endomorphism which is not injective.

■

The proof of Lemma 1.3 is rather direct, but not elementary. It is instructive to note that a covering morphism $\mathbb{T} \to \mathbb{T}$ in general is by no means an isomorphism even though it induces an isomorphism $\mathbb{R} \cong \mathbf{L}(\mathbb{T}) \to \mathbf{L}(\mathbb{T}) \cong \mathbb{R}$.

A homomorphically simple compact Lie group G may not be simple.

1.4. Example. Let $Z \cong \mathbb{Z}(7)$ denote the center of $\mathrm{SU}(7)$ with generator z. Define the subgroup D of $Z \times Z$ by $D = \{(z, z^3): z \in Z\}$. Then $G = (\mathrm{SU}(7) \times \mathrm{SU}(7))/D$ is homomorphically simple but not simple.

Proof. G is the product of two elementwise commuting subgroups $A = (\mathrm{SU}(7) \times \{1\})D/D$ and $B = (\{1\} \times \mathrm{SU}(7))D/D$. A nonconstant endomorphism f of G induces an isomorphism $\mathbf{L}(f)$—in which case it is an isomorphism by Lemma 1.2—or has $\mathbf{L}(A)$ or $\mathbf{L}(B)$ as kernel. In the first case $\ker f$ is locally isomorphic to A; but $G/A \cong B/(A \cap B) \cong \mathrm{SU}(7)/Z = \mathrm{PSU}(7)$. Since $\mathrm{PSU}(7)$ is centerfree, $A = \ker f$ and $\operatorname{im} f \cong \mathrm{PSU}(7)$. However, the Lie subgroups of G which are locally isomorphic to $SU(7)$ but are different from A, B and G are all of the form $S_\alpha = \{(g, \alpha(g))D \mid g \in SU(7)\}$ for an automorphism $\alpha \in \mathrm{Aut}(G)$. If K is the kernel of the morphism $g \mapsto (g, \alpha(g)) : \mathrm{SU}(7) \to S_\alpha$, then $S_\alpha \cong \mathrm{SU}(7)/K$. But $k \in K$ if and only if $(k, \alpha(k)) \in D$, that is, if there is a $z \in Z$ such that $(k, \alpha(k)) = (z, z^3)$. This means $k \in Z$ with $\alpha(k) = k^3$. Now an automorphism α of $\mathrm{SU}(7)$ either fixes every element of Z or else is of order 2, and thus $\alpha(k) = k^{-1}$. Therefore $k \in K$ if and only if $k\mathbb{Z}$ and satisfies $k = k^3$ or $k^{-1} = k^3$, that is $k^2 = 1$ or $k^4 = 1$. In both cases we conclude $k = 1$. Thus all subgroups of G which are locally isomorphic to $\mathrm{SU}(7)$ are isomorphic to $\mathrm{SU}(7)$, hence cannot be the image of f. Analogously, $\ker \mathbf{L}(f) = \mathbf{L}(B)$ is impossible, too. Hence any nonconstant endomorphism of G is an automorphism. ■

It is an instructive exercise to show that, for instance, the group $(\mathrm{SU}(2)\times \mathrm{SU}(2))/D$ with the diagonal D of the center $Z\times Z$ *is not homomorphically simple.* (Note that all nonnormal Lie subgroups of this group which are locally isomorphic to $SU(2)$ are isomorphic to $\mathrm{SO}(3)$.)

1.5. Lemma. *Let G be a compact connected homomorphically simple Lie group and J any set. Then any nonconstant morphism $G^J \to G$ is a projection followed by an automorphism of G.*

Proof. Suppose that $f: G^J \to G$ is any morphism. Since G is a Lie group there is an identity neighborhood V in G containing no subgroup other than $\{1\}$. Then any subgroup of G^J contained in the identity neighborhood $f^{-1}(V)$ is in the kernel of f. By the definition of the product topology on G^J, there is a cofinite subset I of J such that the partial product G^I (identified with the obvious subgroup of G^J) is annihilated by f. Hence f factors through the projection $G^J \to G^{J\setminus I}$. We may therefore assume that J is finite. If the kernel N of f meets any of the factors G inside G^J, it must contain this factor since the restriction of f to this factor is either constant or an isomorphism. Now $\mathrm{L}(N)$ is an ideal in $\mathrm{L}(G^J)=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_n$ with $\mathfrak{g}_j\cong \mathrm{L}(G)$. Since G is semisimple by Lemma 3(ii), $\mathrm{L}(G)=\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_p$ with simple ideals $\mathfrak{s}_k$, whence $\mathfrak{g}_j=\mathfrak{s}_{j1}\oplus\cdots\oplus\mathfrak{s}_{jp}$ with $s_{jt}\cong\mathfrak{s}_k$. Now the ideal $\mathrm{L}(N)$ is a sum of the $\mathfrak{s}_{jk}$ and if it contains $\mathfrak{s}_{jk}$ then it contains $\mathfrak{g}_j$. It follows that $\mathrm{L}(N)$ is a sum of the $\mathfrak{g}_j$. Consequently, $\mathrm{L}(f): \mathrm{L}(G)^J \to L(G)$ is a projection, whence $f: G^J \to G$ is a projection. ■

2. The G-free compact group over X

The concept of subdirect products belongs to universal algebra. We formulate it for the variety of compact groups in which we work.

2.1. Definition. A closed subgroup S of a product $P=\prod_{j\in J} G_j$ of a family of compact groups is called *a subdirect product of this family* if $G_j=\mathrm{pr}_j(S)$ for each projection $\mathrm{pr}_j: P\to G_j$.

Typically, the diagonal in any power G^J of a compact group G is a subdirect product and is itself isomorphic to G. For any given family, the subdirect products are of a great diversity. Accordingly, the usefulness of this concept depends significantly on the family whose subdirect products we consider. For instance, *every* compact connected abelian group is a subdirect product of a family of circle groups $\mathbb{T}$; indeed the evaluation injection $G\to \mathbb{T}^{\widehat{G}\setminus\{0\}}$ defines such a subdirect product. This information cannot be of much value. The situation is different with subdirect products of powers of simple compact connected groups. One might conjecture that a subdirect product inside a power of a simple connected compact Lie group is itself isomorphic to a power of this group. Some examples are instructive:

2.2. Example. Let G be a simple compact group with nontrivial center Z. Let D denote the diagonal in G^n for $n>1$. Then $S=DZ^n\subsetneq G^n$ is a subdirect product in G^n which is not isomorphic to G^m. ■

This shows that without connectivity assumptions on S the conjecture is false. The following example is more interesting:

2.3. Example. Let S be a simple group with center $\langle z, z' \rangle$, where $\langle z \rangle \cong \langle z' \rangle \cong \mathbb{Z}(2)$ such that there is an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(S)$ with $z' = \widetilde{\alpha}(z)$. Such groups exist, for instance $S = \mathrm{Spin}(2m)$ with $m > 2$. Now we consider the quotient morphisms $\pi: S \to S/\langle z \rangle$ and $\pi': S \to G \stackrel{\mathrm{def}}{=} S/\langle z' \rangle$. Then $\widetilde{\alpha}$ induces an isomorphism $\alpha: S/\langle z \rangle \to G$. Define $\delta: S \to G^2$ by $\delta(g) = \big(\alpha(\pi(g)), \pi'(g)\big)$. Then S is a subdirect product in G^2. ■

This example shows that a simply connected compact simple Lie group may be a subdirect product in a power of simple Lie groups which are not simply connected. Hence the conjecture is invalid unless further hypotheses on the global geometry of G are imposed. On the infinitesimal level, however, the conjecture is true:

2.4. Remark. In the category of finite dimensional real Lie algebras and Lie algebra morphisms, a subdirect product $\mathfrak{s}$ inside a finite power $\mathfrak{g}^n = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $\mathfrak{g}_j \cong \mathfrak{g}$ of a simple algebra $\mathfrak{g}$ is isomorphic to a power of $\mathfrak{g}$. If $\mathfrak{s}$ is subdirect in $\mathfrak{g}^n$ then there is an ideal $\mathfrak{n}$ of $\mathfrak{g}^n$ so that $\mathfrak{g}$ is the semidirect sum of $\mathfrak{n}$ and $\mathfrak{s}$.

Proof. Since the projections onto the simple factors separate the points, the radical of $\mathfrak{s}$ must be zero. Hence $\mathfrak{s}$ is semisimple, that is, is isomorphic to a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of simple ideals, and every ideal of $\mathfrak{s}$ is a sum of a subset of these summands. Since the homomorphisms onto $\mathfrak{g}$ separate points, $\mathfrak{s}_j \cong \mathfrak{g}$ follows for all $j = 1, \ldots, m \leq n$. Reorder indices so that s_j projects faithfully onto the summand $\mathfrak{g}_j$, $j = 1, \ldots, m$. Then $\mathfrak{n} = \sum_{m+1}^{n} \mathfrak{g}_k$ is the required ideal. ■

2.5. Remark. Let G denote a compact connected simple Lie group and n a natural number. If $S \subseteq G^n$ is a subdirect product, and if S is connected then $S \cong G^m$ for some $m = 1, \ldots, n$, provided that G is center free.

Proof. By the preceding remarks, $\mathbf{L}(S) \cong \mathbf{L}(G)^m$. Hence S and G^m are locally isomorphic. If G is center free, then so are G^m and S since the surjective morphisms $S \to G$ separate points. The isomorphy of $L(S)$ and $L(G^m)$ then implies the isomorphy of S and G^m. ■

In the third section we shall generalize this remark. However, we shall not need it for the pursuit of our main objective. In this line we shall find that sometimes other information may allow us to conclude that a subdirect product of a power of a simple group is itself a power of this group.

2.6. Definition. Let X be a pointed compact space and G any compact group. We denote by $E(X, G)$ the set of all essential functions $f: X \to G$. The function $\mathrm{ev}_X: X \to G^{E(X,G)}$ is given by $\mathrm{ev}_X(x)(f) = f(x)$ for $f \in E(X, G)$.

2.7. Lemma. *For any essential function $f: X \to G$ there is a morphism $f': G^{E(X,G)} \to G$ of compact groups such that $f = f' \circ \mathrm{ev}_X$.*

Proof. In view of the definition of the product topology, that is, the topology of simple convergence on $G^{E(X,G)}$, the function ev_X is continuous, and if $f \in E(X, G)$, then $f(x) = \mathrm{ev}_X(x)(f) = (\mathrm{pr}_f \circ \mathrm{ev}_X)(x)$ and thus $f' = \mathrm{pr}_f: G^{E(X,G)} \to G$ satisfies the requirement. ■

2.8. Lemma. *If G is an arcwise connected compact nonsingleton group whose weight is at most $2^{\aleph_0}$, and if X contains at least 3 points then the evaluation map $ev_X: X \to G^{E(X,G)}$ is a topological embedding of the compact pointed space X.*

Proof. If I denotes the unit interval, then the functions of $C(X,\mathsf{I})$ separate the points of X. Thus the evaluation $\eta: X \to G^{C(X,\mathsf{I})}$ is injective. If the compact group G is a compact connected group whose weight does not exceed the cardinality of the continuum, then it is topologically generated by two points a and b (see [10], §2). There is a homeomorphism h of the unit interval I into G with $h(1/2) = 1$, $h(0) = a$ and $h(1) = b$. Since X contains at least three points $x_0 =$ basepoint, x_1 and x_2, the set $C_*(X,\mathsf{I})$ of functions f with $f(0) = 1/2$, $f(x_1) = 0$, $f(x_2) = 1$ still separates the points of X and h induces an injection $C(X,h)$ of $C_*(X,\mathsf{I})$ into $C(X,G)$ such that all functions f in the image satisfy $a, b \in f(X)$ and thus are essential. Hence we have an embbeding of $C_*(X,\mathsf{I})$ into $E(X,G)$. This shows that the functions of $E(X,G)$ separate the points of X. Hence the evaluation $\mathrm{ev}_X: X \to G^{E(X,G)}$ is injective. Since X is compact and $G^{E(X,G)}$ Hausdorff, it is an embedding.

■

This information now readily allows us a first identification of the essential G-free compact group.

2.9. Proposition. *Let X be any compact pointed space and G any compact group. Let F_GX denote the compact subgroup generated by $\mathrm{ev}_X(X)$ in $G^{E(X,G)}$ and $e_X: X \to F_GX$ the essential map obtained by corestricting ev_X. Then F_GX is the essential G-free compact group over X and e_X the universal mapping of X into it. If G is arcwise connected and of weight at most $2^{\aleph_0}$ and if X has at least 3 points, then e_X is a topological embedding.* ■

The proof of this, of course, is routine from the preceding Lemmas 2.7 and 2.8.

2.10. Corollary. *The group F_GX is a subdirect product in $G^{E(X,G)}$.*

Proof. If $f \in E(X,G)$, then $\mathrm{pr}_f(F_GX)$ is the subgroup topologically generated by $f(X)$ in G, hence is G. ■

We now proceed to describe this subdirect product F_GX accurately if G is a homomorphically simple Lie group. We begin with the observation that the automorphism group $\mathrm{Aut}\,G$ acts on the set $E(X,G)$ on the left by

$$(\alpha, f) \mapsto \alpha \circ f : \mathrm{Aut}\,G \times E(X,G) \to E(X,G).$$

We shall denote the orbit space $E(X,G)/\mathrm{Aut}\,G$ by $A(X,G)$ and write $[f]$ for the orbit $\{\alpha \circ f \mid \alpha \in \mathrm{Aut}\,G\}$.

For two functions $f, f' \in E(X,G)$ we shall write $f \sim f'$ if and only if there is an automorphism $\alpha \in \mathrm{Aut}\,G$ such that $\mathrm{pr}_{f'}|F_XG = \alpha\,\mathrm{pr}_f|F_GX$, and this is tantamount to $\mathrm{pr}_{f'} \circ \mathrm{ev}_X = \alpha \circ \mathrm{pr}_f \circ \mathrm{ev}_X$, that is, to $f' = \alpha \circ f$. Thus $f \sim f'$ is equivalent to $[f] = [f']$. Hence $\sim$ is none other than the orbit equivalence of the action of $\mathrm{Aut}\,G$ on $E(X,G)$.

For each $F \in A(X,G)$ we select a function $s_F \in F \subseteq E(X,G)$. Thus $[s_F] = F$ and $f \sim s_{[f]}$. Hence for each $f \in A(X,G)$ there is at least one $\alpha_f \in \operatorname{Aut} G$ such that

$$(5) \qquad f = \alpha_f \circ s_{[f]}.$$

We define $\varepsilon_X : X \to G^{A(X,G)}$ by

$$(6) \qquad \varepsilon_X(x)(F) = s_F(x).$$

Then (5) implies

$$\big(f(x)\big)_{f \in E} = \operatorname{ev}_X(x) = \alpha_{\operatorname{ev}_X(x)} \circ s_{[\varepsilon_X(x)]} = \big(\alpha_f\{s_{[f]}(x)\}\big)_{f \in E}.$$

If we define

$$\varphi_X : G^{A(X,G)} \to G^{E(X,G)}, \quad \varphi_X\big((g_F)_{F \in A(X,G)}\big) = (\alpha_f(g_{[f]})_{f \in E(X,G)},$$

then

$$(7) \qquad \operatorname{ev}_X = \varphi_X \circ \varepsilon_X.$$

In particular, $\varepsilon_X : X \to G^{A(X,G)}$ is an embedding whenever ev_X is an embedding.

Now suppose that $f : X \to G$ is an essential map. Set $f^* : G^{A(X,G)} \to G$, $f^*\big((g_F)_{F \in A(X,G)}\big) = \alpha_f(g_{[f]})$. Then $f^*\big(\varepsilon_X(x)\big) = f^*\big\{\big((s_F(x)\big)_{F \in A(X,G)}\big\} = \alpha_f\big(s_{[f]}(x)\big) = f(x)$ in view of (6) and (5). We notice that the following diagram is commutative.

$$\begin{array}{ccccc} X & \xrightarrow{\varepsilon_X} & G^{A(X,G)} & \xrightarrow{\varphi_X} & G^{E(X,G)} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f'} \\ G & \xrightarrow[\operatorname{id}_G]{} & G & \xrightarrow[\operatorname{id}_G]{} & G \end{array}$$

Now suppose that $\mu_j : G^{A(X,G)} \to G$, $j = 1, 2$ are two morphisms such that $\mu_1 \circ \varepsilon_X = \mu_2 \circ \varepsilon_X$. *At this point we assume that G is homomorphically simple.* Then Lemma 1.5 implies that there are automorphisms $\beta_j \in \operatorname{Aut} G$, $j = 1, 2$ such that $\mu_j = \beta_j \circ \operatorname{pr}_{F_j}$. Now $\beta_j \operatorname{pr}_{F_j}\big(\varepsilon_X(x)\big) = \beta_j\big(s_F(x)\big)$ by (6). Hence $s_{F_2} \circ \operatorname{ev}_X = (\beta_2^{-1}\beta_1) s_{F_1} \circ_X$ and this implies $s_{F_1} \sim s_{F_1}$. Thus $F_1 = [s_{F_1}] = [s_{F_2}] = F_2$. But now, setting $F = F_1$, via (6) we find $s_F = \varepsilon_X(F) = \operatorname{pr}_F \varepsilon_X = (\beta_2^{-1}\beta_1) \operatorname{pr}_F \operatorname{ev}_X = (\beta_2^{-1}\beta_1) s_F$. Since $s_F \in E(X,G)$, the subset $s_F(X)$ topologically generates G. Thus $\beta_1 = \beta_2$ follows, and we have $\mu_1 = \mu_2$. Therefore, the following universal property of $G^{A(X,G)}$ is proved:

2.11. Lemma. *Let G be a compact connected homomorphically simple Lie group, further X a compact pointed space with at least 3 points, and $\varepsilon_X : X \to G^{A(G,X)}$ the embedding defined by (6). Then for each essential function $f : X \to G$ there is a unique morphism $f^* : G^{A(X,G)} \to G$ such that $f = f^* \circ \varepsilon_X$.* ∎

Hence $G^{A(X,G)}$ is in fact the essential G-free compact group over X. This immediately entails the following principal result for whose formulation we use the embedding $\varepsilon_X : X \to G^{A(X,G)}$ of (6) and the injective morphism $\varphi_X : G^{A(X,G)} \to G^{E(X,G)}$ of (3).

2.12. Theorem. *Let G be a compact connected homomorphically simple compact Lie group and X a compact pointed space with at least 3 points. Then φ_X corestricts to an isomorphism $\psi_X: G^{A(G,X)} \to F_G X$ such that $\psi_X \circ \varepsilon = e_X$.* ∎

We now apply the structure information contained in this theorem to compute the weight $w(F_G X)$and the generating rank $s(F_G X)$ of the G-free compact group.

2.13. Lemma. *If G is a compact connected nonsingleton Lie group and X a pointed compact space with at least 5 points, then* $\operatorname{card} E(X,G) = w(X)^{\aleph_0}$.

Proof. If $C(X,G)$ denotes the set of basepoint preserving continuous functions $X \to G$, then $E(X,G) \subseteq C(X,G)$. Now $\operatorname{card} C(X,G) = w(X)^{\aleph_0}$ (see for instance [10], Lemma 4.8, which uses [8]). We have seen in the proof of Lemma 2.8 that G is topologically generated by two points a and b and noted that there is a homeomorphism h of the unit interval I into G which induces an injection of $C_*(X, \mathsf{I})$ into $E(X,G)$ such that all functions f in the image satisfy $a, b \in f(X)$ and thus are essential. Hence we have an embeding of $C_*(X, \mathsf{I})$ into $E(X,G)$ where $C_*(X, \mathsf{I})$ denotes the set of functions $f: X \to \mathsf{I}$ with $f(0) = 1/2$, $f(x_1) = 0$, $f(x_2) = 1$. Hence $\operatorname{card} C_*(X, \mathsf{I}) \leq \operatorname{card} E(X,G)$. Let I_* denote the figure 8 obtained by collapsing the points 0, 1/2 and 1 into one point and X_* the space obtained from X by collapsing x_0, x_1, and x_2 into the basepoint. Then there is a surjection $C_*(X, \mathsf{I})$ onto $C(X_*, \mathsf{I}_*)$ and thus $\operatorname{card} C_*(X, \mathsf{I}) \geq \operatorname{card} w(X_*)^{\aleph_0} = w(X)^{\aleph_0}$ since X has at least 5 points. Hence $w(X)^{\aleph_0} \leq \operatorname{card} E(X,G)$. ∎

2.14. Lemma. *If X is a compact space with at least 5 points, then*

$$\operatorname{card} A(X,G) = w(X)^{\aleph_0}.$$

Proof. Since the orbit map is a surjective function $E(X,G) \to A(X,G)$ we have $\operatorname{card} A(X,G) \leq \operatorname{card} E(X,G)$. Now $\operatorname{Aut} G$ is a compact Lie group and contains all inner automorphisms of G, whence $\operatorname{card}(\operatorname{Aut} G) = 2^{\aleph_0}$. The orbits $[f]$ therefore have at most continuum cardinality. Thus $\operatorname{card} C(X,G) \leq 2^{\aleph_0} \operatorname{card} A(X,G)$. Now $\operatorname{Aut} G$ is a finite extension of the group $\operatorname{Int} G$ of inner automorphisms, and G has a continuum of conjugacy classes, hence a continuum of $\operatorname{Aut} G$-orbits F. If $x \in X$ is different from the basepoint, for each F there is an $f_F \in E(X,G)$ such that $f_F(x) \in F$. Then $\{[f_F] \mid F \in G/\operatorname{Aut} G\}$ is a subset of $A(X,G)$ of continuum cardinalty, whence $\operatorname{card} A(X,G) \geq 2^{\aleph_0}$. Therefore $\operatorname{card} E(X,G) \leq \operatorname{card} A(X,G)$ and the lemma is proved. ∎

2.15. Corollary. *Under the hypotheses of* Theorem 2.12, *if X has at least* 5 *points,* $w(F_G X) = w(X)^{\aleph_0}$.

Proof. We have $w(F_G X) = w(G^{A(X,G)}) = \operatorname{card} A(X,G) = w(X)^{\aleph_0}$ by Lemma 2.14. ∎

2.16. Corollary. *For each infinite cardinal $\aleph_\nu$ there are compact connected groups G with $s(G) \leq \aleph_\nu$ and $w(G) = \aleph_\nu^{\aleph_0} = s(G)^{\aleph_0}$.*

Proof. Let X be the one point compactification of a discrete space of cardinality $\aleph_\nu$ based at ∞. Since $e_X(X)$ has cardinality $\aleph_\nu$ and generates $G = F_{\mathrm{SO}(3)}X$ topologically, we have $s(G) \leq \aleph_\nu$. By Corollary 2.15 we know $w(G) = w(X)^{\aleph_0} = \aleph_\nu^{\aleph_0}$. From [10] we know $w(G) \leq s(G)^{\aleph_0}$.

We observe that $s(G) < \aleph_\nu$ may occur in the preceding construction: Let $\nu = 0$. Then $w(F_{\mathrm{SO}(3)}X) = w(X)^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Hence $s(F_{\mathrm{SO}(3)}X) = 2 < \aleph_0$ by Theorem 4.13 of [10].

3. Supplements on subdirect products

The following result generalizes Remark 2.5.

3.1. Proposition. *Let G be a compact connected simple Lie group which is center free and J an arbitrary set. If S is a subdirect product in G^J and if S is connected, then there is a surjective function $\sigma: J \to I$ and a function $j \mapsto \alpha_j: J \to \mathrm{Aut}(G)$ such that the morphism*

(3) $$\varphi: G^I \to G^J, \qquad \varphi\big((x_i)_{i\in I}\big) = \big(\alpha_j(x_{\sigma(j)})\big)_{j\in J}$$

maps G^I isomorphically onto S.

Proof. We write $\delta: S \to G^J$ for the inclusion and define a binary relation $\sim$ on J by writing $j \sim k$ if and only if there is an $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \circ \mathrm{pr}_j \circ \delta = \mathrm{pr}_k \circ \delta$. One observes at once that $\sim$ is an equivalence relation. This allows us to set $I = J/\sim$ and to define $\sigma: J \to I$ as the quotient map. For each $i \in I$ let $G_i = G^{\sigma^{-1}(i)}$ with projections $\mathrm{pr}_{ji}: G_i \to G$ for $j \in i$, and let S_i denote the projecton of S into the partial product G_i with $\delta_i: S_i \to G_i$ the inclusion. Then S_i is a subdirect product in G_i since S is a subdirect product in G^J.

We claim that each morphism $\mathrm{pr}_{ji} \circ \delta_i : S_i \to L$ with $j \in i$ is an isomorphism; that is, S_i behaves in G_i like a diagonal. Indeed by the definition of $\sim$, for each $k \in i$ we know $k \sim j$, and thus there is a $\beta_k \in \mathrm{Aut}(G)$ such that $\mathrm{pr}_k \circ \delta = \beta_k \circ \mathrm{pr}_j \circ \delta$. The morphism $\psi_j: L \to G_i$ defined by $\psi_j(x) = \big(\beta_k^{-1}(x)\big)_{k\in I}$ is injective. Further, if $(x_k)_{k\in i} \in S_i$, then $x_k = \beta_k(x_j)$ for all $k \in i$ and $\psi_j(\mathrm{pr}_{ji}\{(x_k)_{k\in i}\}) = \psi_j(x_j) = \big(\beta_k^{-1}(x_j)\big)_{k\in i} = (x_k)_{k\in i}$. Since $\mathrm{pr}_{ji}(S_i) = G$, the image of ψ is S_i, and its corestriction to this image is an inverse of $\mathrm{pr}_{ji} \circ \delta_i$. This proves the claim.

In particular, each S_i is isomorphic to G. In other words, there is an isomorphism $\gamma_i: L \to S_i$. For each $j \in J$ we define

(4) $$\alpha_j: G \to G, \qquad \alpha_j = \mathrm{pr}_{j\sigma(j)} \circ \gamma_{\sigma(j)}.$$

The map $\varphi: G^I \to$ given by

$$\begin{aligned}\varphi\big((x_i)_{i\in I}\big) &= \big(\alpha_j(x_{\sigma(j)})\big)_{j\in J}\\ &= \big(pr_{j\sigma(j)}\{\gamma_{\sigma(j)}(x_{\sigma(j)})\}\big)_{j\in J}\end{aligned}$$

is clearly an injective morphism of compact groups. Hence it is an isomorphism onto its image.

We claim that this image is S. A proof of this claim will finish the proof of the proposition. Firstly, we observe that $\operatorname{im}\varphi = \prod_{i\in I} S_i$ where we have identified G^J and $\prod_{i\in I} G_i$ in the obvious fashion. Clearly $S \subseteq \prod_{i\in I} S_i$, and we have to show equality. This product is subdirect by the definition of S_i and each S_i is isomorphic to G. If $i \neq i'$ in I, then there is no isomorphism $\rho: S_i \to S_{i'}$ such that $\mathrm{pr}_{i'}|S = \rho \circ \mathrm{pr}_i|S$, for such a ρ would give us for $j \in i$ and $j' \in i'$ an $\alpha \in \mathrm{Aut}(G)$ given by $\alpha = (\mathrm{pr}_{ji}|S_i)^{-1}\rho(\mathrm{pr}_{j'i'}$ such that $\mathrm{pr}_{j'}|S = \mathrm{pr}_{j'i'}(\mathrm{pr}_{i'}|S) = \mathrm{pr}_{j'i'}\rho(\mathrm{pr}_i|S) = \alpha\,\mathrm{pr}_{ji}(\mathrm{pr}_i|S) = \alpha(\mathrm{pr}_j|S)$, and this would imply $j \sim j'$, that is, $i = i'$ contrary to the assumption.

Hence in order to prove our last claim, in simplified notation, we have to prove that for a subdirect product $S \subseteq G^J$ we have $S = G^J$ if the relation $\sim$ on J is equality. Let E denote any finite subset of J and S_E the projection of S into G^E. Then S_E is a subdirect product of G^E and the relation $\sim$ on E is likewise trivial. If these circumstances imply $S_E = G^E$ for all finite subsets E of J, then $S = G^J$ follows, since G^J is the projective limit of its projections onto all finite partial products.

Therefore it suffices to prove the claim when J is finite. We assume that the claim is false and derive a contradiction. Let us assume that $S \subseteq G^n$ is a counterexample to the claim with a minimal natural number n. Evidently, $n \geq 2$. The projection S^* into G^{n-1} (after identifying G^n with $G^{n-1} \times G$) satisfies all hypotheses and cannot be a counterexample. Thus $S^* = G^{n-1}$. If we denote by $F \cong G$ the subgroup $\{1\} \times G$ in $G^{n-1} \times G$, then we have an exact sequence

$$1 \to S \cap F \to S \xrightarrow{\pi} G^{n-1} \to 1.$$

Since $S \cong G^m$ for some m by Lemma 2.5, $m \geq n-1$. If $m = n$, then $\dim S = \dim G^n$ and $S = G$ contrary to our assumption. Thus $S \cong G^{n-1}$. Accordingly, $\mathbf{L}(\pi)$ is an isomorphism and thus π a covering morphism. Now Lemma 1.2 implies that π is an isomorphism and thus $S \cap F = \{1\}$. Now $S \subseteq G^{n-1} \times G$ is the graph of a morphism $\theta: G^{n-1} \to G$ and since S is subdirect, θ is surjective. We use Lemmas 1.3 and 1.5 to conclude, that θ is a projection of G^{n-1} followed by an automorphism α of G; let us say that the projection maps onto the last factor of G^{n-1}. This means that $S = \{(1, \ldots, 1, s, \alpha(s)) \mid s \in G\} \subseteq G^n$. If $p: G^n \to G$ is the projection of G^n onto the next to last factor and q the projection on the last factor, then $\alpha \circ p|S = q|S$ in violation of our assumption. This contradiction completes the proof. ∎

3.2. Remark. If $S \subseteq P = \prod_{j\in J} G_j$ is a subdirect product of compact connected semisimple groups G_j, then so is the identity component S_0 of S.

Proof. If $\mathrm{pr}_j: P \to G_j$ is the projection, then $N_j = \mathrm{pr}_j(S_0)$ is a normal subgroup of G_j. If $N_j \neq G_j$, then N_j is contained in the finite center Z_j of G_j. Then the surjective projection $\mathrm{pr}_j|S: S \to G_j$ gives us a surjective morphism $S/S_0 \to G_j/Z_j$. Since S/S_0 is compact zero dimensional, the image of this map is a compact zero dimensional subgroup of a Lie group hence is finite. But if G_j/Z_j is infinite since $N_j \neq G_j$. This contradiction shows that $N_j = G_j$. ∎

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