BULL. AUSTRAL. MATH. SOC. 22D05, 54A25
VOL. 38 (1988) [105–111]

TOPOLOGIES ON LOCALLY COMPACT GROUPS

JOAN CLEARY AND SIDNEY A. MORRIS

Using the Iwasawa structure theorem for connected locally compact Hausdorff groups we show that every locally compact Hausdorff group G is homeomorphic to $\mathbf{R}^n \times K \times D$, where n is a non-negative integer, K is a compact group and D is a discrete group. This makes recent results on cardinal numbers associated with the topology of locally compact groups more transparent. For abelian G, we note that the dual group, $\widehat{G}$, is homeomorphic to $\mathbf{R}^n \times \widehat{K} \times \widehat{D}$. This leads us to the relationship card $G = \omega_0\left(\widehat{G}\right) + 2^{\omega_0(G)}$, where ω (respectively, ω_0) denotes the weight (respectively local weight) of the topological group. From this classical results such as card $G = 2^{\text{card}\, \widehat{G}}$ for compact Hausdorff abelian groups, and $\omega(G) = \omega\left(\widehat{G}\right)$ for general locally compact Hausdorff abelian groups are easily derived.

1. INTRODUCTION

In his major article on topological groups [**3**], Comfort records some results on the topology of locally compact Hausdorff groups, or more specifically on various cardinal invariants associated with the topology of locally compact groups. By and large the proofs are in two parts — firstly, the results are proved for compact Hausdorff groups, and then they are generalised to locally compact Hausdorff groups. This method is not very attractive, because even though there are nice structure theorems for some classes of groups, for example, connected locally compact groups or almost connected groups, there are no such structure theorems for general locally compact Hausdorff groups. So instead one uses results such as every locally compact Hausdorff group contains an open σ-compact subgroup which in turn contains a compact normal subgroup such that the quotient group is metrisable.

We suggest a more elegant approach to obtain these results. As we are seeking to prove topological results rather than topological group results, it suffices to know the structure of a general locally compact Hausdorff group up to homeomorphism. We show that every locally compact Hausdorff group is homeomorphic to $\mathbf{R}^n \times K \times D$ where K is compact and D is discrete. As the results we look for are trivial for $\mathbf{R}^n$ and D, the problem is completely reduced to the compact case.

Received 12 October 1987

2. Topologies

Our first result is the key to the topological structure theorem for locally compact Hausdorff groups.

Lemma 1. *Let G be a topological group and let H be an open subgroup of G. Then G is homeomorphic to $H \times D$, where D is a discrete space.*

Proof: As H is open in G, the set $G/H = \{gH : g \in G\}$ with the quotient topology is discrete and the map $p: G \longrightarrow G/H$ given by $p(g) = gH$ is a continuous open mapping. Let S be a complete set of coset representatives of H in G. Then the map p restricted to S is a continuous open bijection, and so S is discrete. Therefore S and G/H are homeomorphic.

Now as H and S are both subsets of G and group multiplication is continuous, the map $f: H \times S \longrightarrow G$ given by $f((h,s)) = hs$, is a one-to-one continuous mapping of $H \times S$ onto G. To complete the proof that f is a homeomorphism, we need to show that f is an open mapping. Take an open set U in $H \times S$. Then $U = \bigcup_{i \in I} O_i$ where each $O_i = O_{i_1} \times O_{i_2}$ with O_{i_1} open in H, and therefore open in G, and O_{i_2} is open in S. Now $f(U) = \bigcup_{i \in I} f(O_i) = \bigcup_{i \in I} O_{i_1} O_{i_2}$. As each O_{i_1} is open in G, $O_{i_1} O_{i_2}$ is open, and therefore $f(U)$ is open. Hence $H \times G/H$ is homeomorphic to G. ∎

We now prove the topological structure theorem. Its proof uses the notion of an almost connected group. (Recall that a locally compact Hausdorff group, G, is said to be *almost connected* if the group G/G_0 is compact, where G_0 is the connected component of the identity [1].)

Theorem 1. *Let G be an infinite topological group. Then G is locally compact Hausdorff if and only if it is homeomorphic to $\mathbb{R}^n \times K \times D$, where K is a compact Hausdorff group, D is a discrete space, $\mathbb{R}$ is the topological group of real numbers and n is a non-negative integer.*

Proof: Let G be a locally compact Hausdorff group. Then it contains an open almost connected subgroup, H, [2, page 7]. So, by Lemma 1, G is homeomorphic to $H \times D$, where D is a discrete space. By [8], H is homeomorphic to $H_0 \times H/H_0$, where H_0 is the component of the identity of H. As H is almost connected, $H/H_0 = K_1$ is a compact group. But, by the Iwasawa Structure Theorem [7, page 118], H_0 is homeomorphic to $\mathbb{R}^n \times K_2$, where K_2 is a compact Hausdorff subgroup of H_0, $\mathbb{R}$ is the topological group of real numbers with the usual topology, and n is a non-negative integer. So H is homeomorphic to $\mathbb{R}^n \times K$, where $K = K_1 \times K_2$, and therefore G is homeomorphic to $\mathbb{R}^n \times K \times D$.

Conversely, as each of $\mathbb{R}^n$, K and D is locally compact and Hausdorff, it follows that the product, $\mathbb{R}^n \times K \times D$ is an infinite locally compact Hausdorff group. ∎

Remarks.

(i) In the above theorem we can, without loss of generality, assume that D is a discrete group as every discrete space admits a group structure.

(ii) Karl Hofmann has informed us that a stronger result than Theorem 1 is true and known to some people. Our theorem does not say that K can be chosen to be a subgroup of G and our proof does not allow this deduction. However Hofmann points out that this is indeed true. Using Iwasawa's original proof and Theorem 3.1 of Hochschild [**6**], one can extend Iwasawa's structure theorem [**7**, page 118] from connected to almost connected groups. Theorem 1, in the stronger form, then follows immediately from Lemma 1, using the fact that every locally compact group has an open almost connected subgroup.

Notation. If G is a topological group we denote

(a) the cardinality of G by card G;

(b) $\min\{\text{card}\mathcal{B}\colon \mathcal{B}$ is an open basis for $G\}$ by $\omega(G)$;

(c) $\min\{\text{card}\mathcal{B}\colon \mathcal{B}$ is an open basis at the identity for $G\}$ by $\omega_0(G)$;

(d) $\min\{\text{card}\mathcal{F}\colon$ each $F \in \mathcal{F}$ is an open subset of G and $\bigcap_{F\in\mathcal{F}} F = \{1\}\}$ by $\theta(G)$;

(e) $\min\{\text{card}\mathcal{K}\colon$ each $K \in \mathcal{K}$ is a compact subset of G and $\bigcup_{K\in\mathcal{K}} K = G\}$ by $\gamma(G)$;

(f) the cardinality of the set of all open subsets of G by $o(G)$.

The following theorem contains a number of known results on cardinal invariants associated with a compact Hausdorff group.

THEOREM 2. [**3**] *Let K be an infinite compact Hausdorff group. Then*

(i) $\omega(K) = \omega_0(K)$;

(ii) *card* $K = 2^{\omega(K)} = 2^{\omega_0(K)} = o(K)$.

COROLLARY 1. [**3**, Theorem 3.12; **2**, Theorem 4] *Let G be an infinite locally compact Hausdorff group. Then* card $G = 2^{\omega_0(G)} + \gamma(G)$.

PROOF: By Theorem 1, G is homeomorphic to $\mathbf{R}^n \times K \times D$ and from Theorem 2, card $K = 2^{\omega_0(K)}$.

If K is infinite, $\omega_0(K)$ is infinite, and so $2^{\omega_0(K)} \geqslant 2^{\aleph_0}$. Also card $D = \gamma(D) = \gamma(G)$. So

$$\begin{aligned}\text{card } G &= \text{card } K + \text{card } D\\ &= 2^{\omega_0(K)} + \gamma(D)\\ &= 2^{\omega_0(G)} + \gamma(G).\end{aligned}$$

If K is finite, then as G is infinite, either $n \geqslant 1$ or D is infinite, and $\omega_0(G) = \aleph_0$. If $n \geqslant 1$ then

$$\begin{aligned}\text{card } G &= 2^{\aleph_0} + \gamma(D)\\ &= 2^{\omega_0(G)} + \gamma(D)\\ &= 2^{\omega_0(G)} + \gamma(G).\end{aligned}$$

If $n = 0$ then G is homeomorphic to $K \times D$ which is discrete. So $\omega_0(G) = 1$ and D is infinite. Therefore card $G =$ card $D = \gamma(D) = \gamma(G)$ and so card $G = 2^{\omega_0(G)} + \gamma(G)$. ∎

The abelian version of Theorem 1 can be proved directly and is somewhat stronger.

THEOREM 3. *Let G be an infinite locally compact Hausdorff abelian group. Then there is a compact Hausdorff abelian group K, a discrete abelian group D and a non-negative integer n such that G is homeomorphic to $\mathbf{R}^n \times K \times D$ and $\widehat{G}$ is homeomorphic to $\mathbf{R}^n \times \widehat{K} \times \widehat{D}$.*

PROOF: We have that G is homeomorphic to $\mathbf{R}^n \times S$ where S contains a compact open subgroup K [4, Theorem 24.30]. By Lemma 1, S is homeomorphic to $K \times S/K$, and so G is homeomorphic to $\mathbf{R}^n \times K \times D$, where D is the discrete group S/K. There is an exact sequence

$$0 \longrightarrow K \xrightarrow{f_1} S \xrightarrow{f_2} D \longrightarrow 0$$

where f_1 and f_2 are open continuous homomorphisms. Hence, by [7, page 79], the sequence

$$0 \longleftarrow \widehat{K} \xleftarrow{\widehat{f_1}} \widehat{S} \xleftarrow{\widehat{f_2}} \widehat{D} \longleftarrow 0$$

is also exact, and $\widehat{f_1}$ and $\widehat{f_2}$ are open continuous homomorphisms. So $\widehat{D}$ is a compact open subgroup of $\widehat{S}$ and $\widehat{K}$ is the discrete quotient group $\widehat{S}/\widehat{D}$. Therefore $\widehat{S}$ is homeomorphic to $\widehat{D} \times \widehat{K}$. Hence $\widehat{G}$ is homeomorphic to $\mathbf{R}^n \times \widehat{S}$ which is homeomorphic to $\mathbf{R}^n \times \widehat{K} \times \widehat{D}$. ∎

In the abelian case, Corollary 1 of Theorem 1 yields the following striking result.

THEOREM 4. *Let G be an infinite locally compact Hausdorff abelian group. Then card $G = 2^{\omega_0(G)} + \omega_0\left(\widehat{G}\right)$.*

PROOF: By [4, Section 24.48] $\omega_0\left(\widehat{G}\right) = \gamma(G)$. ∎

This theorem allows us to deduce the following classical results.

COROLLARY 1. [4, Section 24.47] *Let G be an infinite compact Hausdorff abelian group. Then* $\text{card}\, G = 2^{\text{card}\widehat{G}}$.

PROOF: By Theorem 4,

$$\begin{aligned}\text{card } G &= 2^{\omega_0(G)} + \omega_0\left(\widehat{G}\right)\\ &= 2^{\omega_0(G)} \quad \text{as } \widehat{G} \text{ is discrete,}\\ &= 2^{\omega_0(G)}.\end{aligned}$$

But

$$\begin{aligned}\text{card } \widehat{G} &= 2^{\omega_0\left(\widehat{G}\right)} + \omega_0\left(\widehat{\widehat{G}}\right)\\ &= \omega_0(G) \quad \text{as } \widehat{G} \text{ is discrete}\\ &= \omega_0(G).\end{aligned}$$

So $\text{card } G = 2^{\omega_0(G)} = 2^{\text{card } \widehat{G}}$. ∎

LEMMA 2. *Let G be an infinite compact Hausdorff abelian group. Then* $\omega\left(\widehat{G}\right) = \omega(G)$.

PROOF: As $\widehat{G}$ is discrete, $\omega\left(\widehat{G}\right) = \text{card } \widehat{G}$. Using Theorem 3 above, we have

$$\begin{aligned}\omega\left(\widehat{G}\right) &= \text{card } \widehat{G}\\ &= 2^{\omega_0\left(\widehat{G}\right)} + \omega_0\left(\widehat{\widehat{G}}\right)\\ &= \omega_0(G) \quad \text{as } \widehat{G} \text{ is discrete and } \widehat{\widehat{G}} = G\\ &= \omega(G)\end{aligned}$$

∎

The following results are already known [4, Theorem 24.14]. We include them here as both can easily be derived from our topological structure theorem for abelian groups, Theorem 3.

THEOREM 5. *Let G be an infinite locally compact Hausdorff abelian group. Then* $\omega(G) = \omega\left(\widehat{G}\right)$.

PROOF: As G is homeomorphic to $\mathbf{R}^n \times K \times D$, we have $\omega(G) = \aleph_0 + \omega(K) + \text{card } D$ and as $\widehat{G}$ is homeomorphic to $\mathbf{R}^n \times \widehat{K} \times \widehat{D}$, $\omega\left(\widehat{G}\right) = \aleph_0 + \text{card } \widehat{K} + \omega\left(\widehat{D}\right)$. But, as $\widehat{K}$ is discrete, $\text{card } \widehat{K} = \omega\left(\widehat{K}\right)$ and, from Corollary 3, $\omega\left(\widehat{K}\right) = \omega(K)$ and $\omega\left(\widehat{D}\right) = \omega(D)$ The result now follows. ∎

PROPOSITION 3. *Let* G *be an infinite locally compact Hausdorff group. Then* $o(G) = 2^{\omega(G)}$.

PROOF: As G is homeomorphic to $\mathbf{R}^n \times K \times D$, $\omega(G) = \omega(\mathbf{R}^n) + \omega(K) + \omega(D) = \aleph_0 + \omega(K) + \text{card } D$. Also, by Theorem 2 , $o(K) = 2^{\omega(K)}$; so

$$\begin{aligned} o(G) &= o(\mathbf{R}^n) + o(K) + o(D) \\ &= 2^{\aleph_0} + 2^{\omega(K)} + 2^{\text{card } D} \text{ as } \omega(K) \geqslant \aleph_0 \text{ card } D \geqslant \aleph_0 \text{ or } n \geqslant 1 \\ &= 2^{\aleph_0 + \omega(K) + \text{card } D} \\ &= 2^{\omega(G)} \end{aligned}$$

∎

Remarks.

(i) For G, an infinite locally compact Hausdorff group, we see from Theorem 1, that if D is infinite, then $\gamma(G) = \text{card } D = \gamma(D)$.

(ii) Let G be an infinite locally compact non-compact Hausdorff group. Then, in the notation of Theorem 1, $\gamma(G) = \aleph_0 + \gamma(D)$.

(iii) Let G be an infinite locally compact Hausdorff group. Then, in the notation of Theorem 1,

$$\omega_0(G) = \begin{cases} \omega_0(K) & \text{if } K \text{ is infinite;} \\ \aleph_0 & \text{if } K \text{ is finite and } n \neq 0; \\ 1 & \text{if } K \text{ is finite and } n = 0 \text{ (that is } G \text{ is discrete).} \end{cases}$$

PROPOSITION 4. *Let* G *be an infinite locally compact Hausdorff group. Then* $\omega(G) = \omega_0(G) + \gamma(G)$.

PROOF: Firstly, note from Theorem 2 that $\omega(K) = \omega_0(K)$ for compact groups. Then, using the results in Remarks 2 and 3 above, we see that

$$\begin{aligned} \omega(G) &= \omega(\mathbf{R}^n \times K \times D) \\ &= \aleph_0 + \omega(K) + \omega(D) \\ &= \aleph_0 + \omega_0(K) + \gamma(D) \\ &= \omega_0(G) + \gamma(G). \end{aligned}$$

∎

Remark. If G is an infinite almost connected group, $\gamma(G) \leqslant \aleph_0$, and so card $G = 2^{\omega_0(G)}$.

Remark. To summarise, we have the following results.

Let G be an infinite locally compact Hausdorff group. Then

(i) card $G = 2^{\omega_0(G)} + \gamma(G) = o(G)$;

(ii) $\omega(G) = \omega_0(G) + \gamma(G)$;

If, in addition, G is abelian, then

(iii) card $G = 2^{\omega_0(G)} + \omega_0\left(\widehat{G}\right)$;

(iv) $\omega(G) = \omega\left(\widehat{G}\right)$;

and if G is also compact,

(v) card $G = 2^{\text{card } \widehat{G}}$. ∎

References

[1] R.W. Bagley and M.R. Peyrovian,, 'A note on compact subgroups of topological groups', *Bull. Austral. Math. Soc.* **33** (1986), 273–278.

[2] Joan Cleary and Sidney A. Morris, 'Trinity ... A tale of three cardinals', *Proc. Centre Mathematical Analysis* **14** (1986), 117–127.

[3] W.W. Comfort, 'Topological Groups', in *Handbook of set-theoretic topology*, K. Kunen and J.E. Vaughan eds., pp. 1143–1264 (North-Holland, Amsterdam, 1984).

[4] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmoic Analysis* (Springer-Verlag, Berlin, 1963).

[5] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis II* (Springer-Verlag, Berlin, 1970).

[6] G. Hochschild, *The Structure of Lie Groups* (Holden-Day Inc., San Francisco, 1965).

[7] Sidney A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups* (Cambridge University Press, Cambridge, 1977).

[8] Paul S. Mostert, 'Sections in principle fibre spaces', *Duke Math. J.* **23** (1956), 57–71.

Ms. J. Cleary
Department of Mathematics
La Trobe University
Bundoora, Vic. 3803
Australia

Prof. S.A. Morris
Department of Mathematics
Statistics and Computing Science
The University of New England
Armidale, N.S.W. 2351
Australia