BULL. AUSTRAL. MATH. SOC. 22A05, 54D55
VOL. 33 (1986), 103-112.

METRIZABILITY OF SUBGROUPS OF FREE TOPOLOGICAL GROUPS

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It is shown that any sequential subgroup of a free topological group is either sequential of order ω_1 or discrete. Hence any metrizable subgroup of a free topological group is discrete.

1. Introduction

It is known that a free topological group is metrizable if and only if it is discrete. Ordman and Smith-Thomas [9] generalized this to show that any non-discrete free topological group which is sequential, is sequential of order ω_1. We extend this much further by showing that any sequential subgroup of a free (free abelian) topological group is either discrete or sequential of order ω_1. Thus any metrizable (or even Frechet) subgroup of a free (free abelian) topological group is discrete. We do this by showing that if a subgroup G of a free (free abelian) topological group has a non-trivial sequence $y_1, y_2, \ldots$ converging to e and G contains the free (free abelian) topological group on $(\{\cup_{i=1}^{\infty}\{y_i\}\}\cup\{e\})$ and hence also contains the Arhangel'skiĭ-Franklin space S_ω [1,9] which is sequential of order ω_1. This observation also answers

Received 2 May 1985.

Question 3.10 of [9] in the affirmative.

2. Preliminaries

DEFINITIONS. Let X be a topological space with distinguished point e, and $F(X)$ a topological group which contains X as a subspace and has e as its identity element. Then $F(X)$ is said to be the *Graev free (free abelian) topological group on* X if for any continuous map ϕ of X into any topological (abelian topological) group H such that $\phi(e)$ is the identity element of H, there exists a unique continuous homomorphism $\Phi : F(X) \to H$ with $\Phi|X = \phi$.

For a recent survey of free topological groups see [8].

DEFINITION. We say $x_1,\ldots,x_n$ are the *essential elements* of the word $w \in F(X)$ if each $x_i \in X$ and $w \in gp\{x_1,\ldots,x_n\}$ but $w \notin gp\{x_{i_1},\ldots,x_{i_k}\}$ for any proper subset $\{x_{i_1},\ldots,x_{i_k}\}$ of $\{x_1,\ldots,x_n\}$.

The following definitions and examples are based on Franklin [3,4]. See also Engelking [2].

DEFINITIONS. A subset U of a topological space X is said to be *sequentially open* if each sequence converging to a point in U is eventually in U. The space X is said to be *sequential* if each sequentially open subset of X is open.

Remarks. A closed subspace of a sequential space is sequential. A subspace of a sequential space need not be sequential (See Example 1.2 of [3].)

DEFINITIONS. For each subset A of a sequential space X, let $s(A)$ denote the set of all limits of sequences of points of A. The space X is said to be sequential of order 1 if $s(A)$ is the closure of A for every A.

The higher sequential orders are defined by induction. Let $s_0(A) = A$, and for each ordinal $\alpha = \beta + 1$, let $s_\alpha(A) = s(s_\beta(A))$. If α is a limit ordinal, let $s_\alpha(A) = \bigcup\{s_\beta(A) : \beta < \alpha\}$. The *sequential order* of X is defined to be the least ordinal α such that $s_\alpha(A)$ is the closure of A for every subset A of X.

Remarks. The sequential order always exists and does not exceed the first uncountable ordinal ω_1. Sequential spaces of order 1 are also known as *Frechet spaces*. Clearly any metrizable space is a Frechet space however there exist sequential spaces which are not Frechet and Frechet spaces which are not metrizable. Indeed, for each ordinal $\alpha \leq \omega_1$ there exists a sequential space of that order. The key example is due to Arhangel'skiĭ and Franklin [1].

By S_1 we mean a space consisting of a single convergent sequence $s_1, s_2, \ldots,$ together with its limit point s_0 taken as the basepoint.

The space S_2 is obtained from S_1 by attaching to each isolated point s_n of S_1 a sequence $s_{n,1}, s_{n,2}, \ldots,$ converging to s_n. Thus S_2 can be viewed as a quotient of a disjoint union of convergent sequences; we give it the quotient topology. Inductively, we obtain the space S_{n+1} from S_n by attaching a convergent sequence to each isolated point of S_n and giving the resultant set the quotient topology.

Let S_ω be the union of the sets $S_1 \subset S_2 \subset S_3 \subset \ldots,$ with the weak union topology (a subset of S_ω is closed if and only if its intersection with each S_n is closed in the topology of S_n).

It is shown in [1] that each S_n is sequential of order n and S_ω is sequential of order ω_1.

DEFINITION. Let $F(X)$ be the Graev free (free abelian) topological group on a Tychonoff space X and Y a subset of $F(X)$. Then Y is said to be *regularly situated with respect to* X if for each positive integer n there exists an integer m such that $gp(Y) \cap F_n(X) \subseteq gp_m(Y)$, where $gp(Y)$ denotes the subgroup generated by Y, $F_n(X)$ denotes the set of all words in $F(X)$ of length $\leq n$ with respect to X, and $gp_m(Y)$ denotes the set of all words in $gp(Y)$ of length $\leq m$ with respect to Y.

THEOREM A. [Graev, 5] *Let X be a compact Hausdorff space and Y a compact subspace of $F(X)$ containing e. If $Y\setminus\{e\}$ is a free algebraic basis for $gp(Y)$ and Y is regularly situated with respect to X,* *then* $gp(Y) = F(Y)$.

In the study of free topological groups the class of k_ω-spaces plays a central role.

DEFINITIONS. A Hausdorff space X is said to be a k_ω-space [7] if it has a countable family of compact subspaces $X_1 \subseteq X_2 \subseteq \ldots$, such that $X = \bigcup_{n=1}^{\infty} X_n$ and a subset A of X is closed if and only if $A \cap X_n$ is closed for all n. We call $X = \cup X_n$ a *k_ω-decomposition.*

Note that if a subspace A of X is compact, then $A \subseteq X_n$ for some n.

THEOREM B. [5,7] *If X is a compact Hausdorff space then $F(X)$ is a k_ω-space with k_ω-decomposition $F(X) = \cup F_n(X)$.*

We shall use the following result.

LEMMA. [6, p. 127] *For any $w \in F(X)\backslash\{e\}$ there is an $l \in F(X)$ and $c \in F(X) \setminus \{e\}$ such that $w = lcl^{-1}$ where c has reduced form $c = x_1 \ldots x_n$ with $x_i \in X\backslash\{e\}$ for $i = 1,\ldots,n$ for some $n \geq 1$, and $x_1 \neq x_n^{-1}$. Further, for any $t \geq 1$, $w^t = lc^t l^{-1}$ and c^t has reduced form $x_1 \ldots x_n x_1 \ldots x_n \ldots x_1 \ldots x_n$.*

Moreover, either $l = e$ or $lc^t l^{-1}$ is the reduced form of w^t.

3. Results

Our first result generalizes Theorem A above and also Lemma 3.6 of [9].

THEOREM 1. *Let $F(X)$ be the Graev free topological group on a Tychonoff space X. Let $Y \ni \{e\}$ be a compact subspace of $F(X)$ such that $Y \setminus \{e\}$ is an algebraic free basis for the group it generates. If Y is regularly situated with respect to X, then $gp(Y)$ is the Graev free topological group on Y.*

Proof. Let $F(\beta X)$ be the Graev free topological group on the Stone-Čech compactification of X and Φ the continuous injective homomorphism of $F(X)$ into $F(\beta X)$ induced by the canonical embedding of X in βX.

Clearly $\Phi(Y)$ is a compact subspace of $F(\beta X)$ such that $\Phi(Y)\setminus\{e\}$ is a free algebraic basis for $gp(\Phi(Y))$ and $\Phi(Y)$ is regularly situated with respect to βX. Therefore by Theorem A, $gp(\Phi(Y)) = F(\Phi(Y)) = F(Y)$.

As Φ is a continuous injective homomorphism of $gp(Y) \subseteq F(X)$ onto $gp(\Phi(Y)) = F(Y)$ the topology of $gp(Y)$ is finer than the free topology of $F(Y)$. But this implies $gp(Y) = F(Y)$, as required.

THEOREM 2. *Let X be any Tychonoff space and $F(X)$ the Graev free topological group on X. Let $y_1,\ldots,y_n,\ldots,$ be a non-trivial sequence in $F(X)$ converging to e. If $Y = (\bigcup_{n=1}^{\infty}\{y_n\}\bigcup\{e\})$ then $gp(Y)$ has a closed subgroup topologically isomorphic to $F(Y)$.*

Proof. By Theorem 1 it suffices to find a subsequence $z_1,\ldots z_n\ldots,$ such that the compact space $Z = (\bigcup_{i=1}^{\infty}\{z_i\}\bigcup\{e\})$ is regularly situated with respect to X and $Z\setminus\{e\}$ is a free algebraic basis for $gp(Z)$.

We choose the subsequence as follows. Let βX, $F(\beta X)$, and Φ be as in the proof of the previous result. As $\Phi(Y)$ is a compact subspace of $F(\beta X)$ and $F(\beta X)$ is a k_ω-space, $\Phi(Y) \subseteq F_N(\beta X)$ for some N, by Theorem B and the note that precedes it. Hence $Y \subseteq F_N(X)$ for this N. Therefore there is a subsequence of distinct words $z_1,\ldots,z_n,\ldots,$ each of which lies in $F_M(X)\setminus F_{M-1}(X)$ for some fixed $M \le N$. By the Lemma in §2 we can find reduced words l_i and c_i with $c_i \ne e$ such that $z_i^t = l_i c_i^t l_i^{-1}$, for $t = 1,2,\ldots,$ and either this is the reduced form of z_i^t or $l_i = e$ and $z_i^t = c_i^t$ in reduced form. Since the l_i have lengths $\le M$ we can choose a subsequence of $z_1,\ldots,z_n,\ldots,$ for which the l_i have the same length. Relabelling, we again denote the subsequence by $z_1,\ldots,z_n,\ldots$. Either there are infinitely many distinct l_i and relabelling we assume the sequence $z_1,\ldots,z_n,\ldots,$ satisfies $l_i \ne l_j$,

$l_i \neq l_j^{-1}$ and $l_i \neq e$ for all i and $j \neq i$, or we can choose a subsequence of the z_i such that, with relabelling, $l_i = l$, a fixed word, for all i.

If $l_i = l$ for all i, the c_i are all distinct and have fixed length and we choose a further subsequence of $z_1, \ldots, z_n, \ldots,$ as follows. Let $a_1, \ldots, a_q$ be the essential elements of l. We now choose a subsequence of the c_i's.

Let $X_1 = \{x \in X \backslash \{a_1, \ldots, a_q\} : x$ is an essential element of z_i, for some $i \geq 1\}$. Since each $z_i \in F_N(X)$ and the z_i are distinct, X_1 is countably infinite. Define $G(z_i)$, $i = 1, 2, \ldots,$ inductively as follows. Let $G(z_1) = \{x \in X_1 : x$ is an essential element of $z_1.\}$ Having defined $G(z_i)$, for $1 \leq i \leq k$, let

$G(z_{k+1}) = \{x \in X_1 \backslash \bigcup_{i=1}^{k} G(z_i) : x$ is an essential element of $z_{k+1}\}$.

Thus $G(z_i) \cap G(z_j) = \emptyset$ for all $i \neq j$, $\bigcup_{i=1}^{\infty} G(z_i) = X_1$, and $G(z_i)$ has at most N elements for each i. So $G(z_i) \neq \emptyset$ for an infinite number of z_i. Deleting the z_i for which $G(z_i) = \emptyset$ and relabelling the sequence thus obtained, we can assume that $G(z_i) \neq \emptyset$ for all i. Now given any subsequence T of $z_1, \ldots, z_n, \ldots,$ and $z_{i_1}, \ldots, z_{i_{N+1}}$ there exists $j \in \{1, \ldots, N+1\}$ and $x \in G(z_{i_j})$ and a subsequence T_1 of T such that x is not an essential element of any term z_l of T_1. This follows since $z_k \in F_N(X)$ for all terms z_k of T and the $G(z_{i_j})$ are non-empty and pairwise disjoint for $j \in \{1, \ldots, N+1\}$. Denote the sequence $z_1, \ldots, z_n, \ldots$ by S_1 and let z_{i_1} be the first term of S_1 for which there exists $b_1 \in G(z_{i_1})$ and a subsequence S_2 of S_1 such that b_1 is not an essential element of any term of S_2. Let z_{i_2} be the first term of S_2 for which there exists $b_2 \in G(z_{i_2})$ and a subsequence S_3 of S_2 such that b_2 is not an essential element of any term of S_3. Continue this process inductively. Relabelling z_{i_j} as z_j and c_{i_j} as c_j, we obtain a sequence $z_1,\ldots,z_n,\ldots,$ converging to e. Further, as $b_i \notin \{a_1,\ldots,a_q\}$, b_i is an essential element of c_i but b_i is not an essential element of c_j for $j \neq i$. So $z_i = lc_il^{-1}$ and $c_i = d_i^{-1}f_ig_i$ where b_i is not an essential element of d_i or g_i, and f_i begins and ends with elements from the set $\{b_i,b_i^{-1}\}$. Moreover this is the reduced form of c_i with respect to X provided d_i^{-1} is deleted if $d_i = e$ and g_i is deleted if $g_i = e$.

We now show that in both cases ($l_i = l$ for all i and $l_j \neq l_i \neq l_j^{-1}$ for all $i \neq j$) the set Z is regularly situated with respect to X and $Z\setminus\{e\}$ is a free algebraic basis for $gp(Z)$. We do this by verifying the following: if $w_n \in gp(Z)$ has reduced form $z_{i_1}^{\epsilon_1}\ldots z_{i_n}^{\epsilon_n}$ with respect to Z, where $\epsilon_j = \pm 1$, $1 \leq j \leq n$, then the length of w_n with respect to X is at least n. We proceed by induction.

If all the l_i are distinct the induction hypothesis is that, with respect to X, w_n has reduced form $l_{i_1}u_nc_{i_n}^{\epsilon_n}l_{i_n}^{-1}$ where u_n, $n \geq 2$, contains the words $c_{i_1}^{\epsilon_1},\ldots,c_{i_{n-1}}^{\epsilon_{n-1}}$ and $u_1 = e$. This is clear for $n = 1$. so assume it is true for $n = k$.

Let $w_{k+1} \in gp(Z)$ have reduced form $z_{i_1}^{\epsilon_1}\ldots,z_{i_k}^{\epsilon_k}z_{i_{k+1}}^{\epsilon_{k+1}}$ with respect to Z. Thus $w_{k+1} = w_kz_{i_{k+1}}^{\epsilon_{k+1}} = l_{i_1}u_kc_{ik}^{\epsilon_k}l_{i_k}^{-1}l_{i_{k+1}}c_{i_{k+1}}^{\epsilon_{k+1}}l_{i_{k+1}}^{-1}$. Let $l_{i_k}^{-1}l_{i_{k+1}} = v$ and $u_{k+1} = u_kc_{i_k}^{\epsilon_k}v$. Since l_{i_k} and $l_{i_{k+1}}$ have the same length, w_{k+1} has reduced form $l_i\, u_{k+1} c_{i_{k+1}}\, l_{i_{k+1}}^{-1}$, with respect to X. (Note that if $z_{i_k}^{\epsilon_k} = z_{i_{k+1}}^{\epsilon_{k+1}}$ then $v = e$ and $c_{i_k}^{\epsilon_k} = c_{i_{k+1}}^{\epsilon_{k+1}}$ so no cancellation can occur between $c_{i_k}^{\epsilon_k}$ and $c_{i_{k+1}}^{\epsilon_{k+1}}$.) This completes the proof for the case of distinct l_i.

Assume now that $l_i = l \neq e$ for all i. Let $h_{i_n} = g_{i_n}$ if $\epsilon_n = 1$ and $h_{i_n} = d_{i_n}$ if $\epsilon_n = -1$. The induction hypothesis is that w_n has representation $lu_n f_{i_n}^{\epsilon_n} h_{i_n} l^{-1}$ where u_n, $n \geq 2$, contains the words $f_{i_1}^{\epsilon_1}, \ldots, f_{i_{n-1}}^{\epsilon_{n-1}}$ and $u_1 = t^{-1}$ where $t = d_{i_1}$ if $\epsilon_1 = 1$ and $t = g_{i_1}$ if $\epsilon_1 = -1$. The induction hypothesis further asserts that this representation is reduced, with respect to X, provided the term h_{i_n} is deleted if $h_{i_n} = e$ and the term u_1 is deleted if $u_1 = e$. Let $w_{k+1} \in gp(Z)$ have reduced representation $z_{i_1}^{\epsilon_1} \ldots z_{i_k}^{\epsilon_k} z_{i_{k+1}}^{\epsilon_{k+1}}$ with respect to Z. Thus $w_{k+1} = w_k z_{i_{k+1}}^{\epsilon_{k+1}}$. We consider the case $\epsilon_{k+1} = 1$; the case $\epsilon_{k+1} = -1$ is similar. Thus $w_{k+1} = lu_k f_{i_k}^{\epsilon_k} h_{i_k} l^{-1} l d_{i_{k+1}}^{-1} f_{i_{k+1}} g_{i_{k+1}} l^{-1}$.

Let $h_{i_k} d_{i_{k+1}}^{-1} = v$ in reduced form with respect to X and $u_{k+1} = u_k f_{i_k}^{\epsilon_k} v$. If $f_{i_k}^{\epsilon_k} = f_{i_{k+1}}$ then $c_{i_k}^{\epsilon_k} = c_{i_{k+1}}$. Then by choice of c_i and f_i, $f_{i_k}^{\epsilon_k} v f_{i_{k+1}}$ is in reduced form with respect to X, except possibly $v = e$, and the result follows. Otherwise the result follows by noting that $f_{i_k}^{\epsilon_k}$ ends in $b_{i_k}^{\delta_k}$ and $f_{i_{k+1}}$ begins with $b_{i_{k+1}}^{\delta_{k+1}}$, where

$\delta_k, \delta_{k+1} \in \{-1,1\}$ and $b_{i_k} \neq b_{i_{k+1}}$.

If $l_i = e$ for all i we repeat the previous argument deleting the l's and l^{-1}'s. This completes the proof. □

The following Theorem generalizes Theorem 3.9 of [9].

THEOREM 3. *Let $F(X)$ be the Graev free topological group on a Tychonoff space and G a subgroup of $F(X)$. If G is a sequential space then it is sequential of order ω_1 or is discrete.*

Proof. As G is sequential its sequential order is $\leq \omega_1$. Either G is discrete or G contains a non-trivial sequence $y_1, \ldots y_n \ldots,$ convergent to a point $y \in G$. Multiplying the y_i's by y^{-1} and relabelling $y^{-1}y_i$ as y_i we can assume the sequence $y_1, \ldots, y_n, \ldots,$ converges to e. By Theorem 2, $G \supseteq F(Z)$ which is a k_ω-group and hence closed. Thus by Theorem 3.7 of [9], G contains S_ω a space of sequential order ω_1. Hence G is sequential of order ω_1.

COROLLARY 1. *Let $F(X)$ be the Graev free topological group on a Tychonoff space X and G a metrizable or Frechet subgroup of $F(X)$. Then G is discrete.*

Remark. The analogue of Theorem 2 for Graev free abelian topological groups is also true.

Proof. Once again there exists an integer N such that $y_i \in F_N(X)$, for all i. As in the proof of Theorem 2, since each y_i has only a finite number of essential elements it is possible to choose a subsequence $z_1, \ldots, z_n \ldots,$ such that b_i is an essential element of z_i but not of any z_j, $j \neq i$. It is obvious in the abelian case that if $Z = \{z_1, \ldots, z_n, \ldots\} \cup \{e\}$, any word w in $gp(Z)$ has reduced length with respect to X greater than or equal to its reduced length with respect to $Z \setminus \{e\}$. Hence $gp(Z)$ is the free abelian topological group on Z, as required. □

As a consequence of this we see that the analogues for Graev free abelian topological groups of Theorem 3 and Corollary 1 are also true. (Note that the proof of the abelian analogue of Theorem 3.7 of [9] is similar to the non-abelian case.)

Finally we note that it is easily verified that the analogues for Markov free topological groups [8] of Theorems 2 and 3 and Corollary 1 are also valid.

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