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FREE COMPACT GROUPS I: FREE COMPACT ABELIAN GROUPS

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As the first step in understanding the structure of free compact groups, the structure of free compact abelian groups is here exposed. For example the following result is proved: *Theorem.* Let G be a compact abelian group. Then G is a free compact abelian group if and only if $G \cong K \times (\mathbb{Q}^\wedge)^a \times \prod_{p \text{ prime}} \mathbb{Z}_p^b$, where $\mathbb{Q}$ is the discrete group of rational numbers, K is a compact connected abelian group with dense torsion subgroup, and a and b are cardinal numbers such that $a \geqslant \max\{2^{\aleph_0}, b, \dim K\}$. Furthermore, if G is of this form, then it is the free compact abelian group on the underlying space of the compact abelian group $X = K \times (\mathbb{Q}^\wedge)^a \times F$ where F is a discrete abelian group of order $b-1$, if b is finite, and $\mathbb{Z}(2)^b$, otherwise. En route it is proved that compact Hausdorff spaces X and Y have isomorphic free compact abelian groups if and only if $\operatorname{red} H^1(X, \mathbb{Z}) \cong \operatorname{red} H^1(Y, \mathbb{Z})$, $w(X/\text{conn}) = w(Y/\text{conn})$, and $\max(2^{\aleph_0}, w(X)) = \max(2^{\aleph_0}, w(Y))$, where $\operatorname{red} H^1(X, \mathbb{Z})$ denotes the first Alexander-Spanier cohomology group modulo its largest divisible subgroup, $w(X)$ denotes the weight of the space X, and X/conn denotes the quotient space of X modulo the connectivity relation.

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free compact group	Prüfer group
compact abelian group	projective resolution
free abelian group	

1. The structure of free compact abelian groups

1.1. Introduction and preliminaries

Although free groups, free topological groups and even free profinite groups have been studied extensively, there has been virtually no study of free compact groups, though they are mentioned, for example, in [9]. Free compact abelian groups played a key role in Enochs' proof [6] that for any compact connected abelian group, G, $\pi_1(G)$ is isomorphic to $\operatorname{Hom}(G^\wedge, \mathbb{Z})$, the group of homomorphisms of the character group, $G^\wedge$, into $\mathbb{Z}$, but except for that which appears in [15] and [9], their structure too is unknown. In this and subsequent papers we expose the structure of free compact groups and free compact abelian groups. Indeed we shall see that free compact groups and free compact abelian groups are related in a way for which there is no parallel in the theory of abstract groups: for example, if X is any contractible space then the free compact group FX on X is isomorphic to the direct product of the free compact abelian group F_aX on X and the commutator group $(FX)'$. When X is a more general space, the relationship is a little more complicated and the projective cover of F_aX enters the picture.

In this paper we describe the structure of both F_aX and its projective cover. For example we show that if X is an infinite compact connected metric space then $F_aX \cong (H^1(X, \mathbb{Z}))^\wedge \times G^\wedge$, where G is a (discrete) rational vector space of dimension $2^{\aleph_0}$, and $H^1(X, \mathbb{Z})$ denotes the first Alexander–Spanier cohomology group (with the discrete topology). We also characterize the compact abelian groups which are free compact abelian groups, and show how to construct generating spaces.

For over 30 years the question of what topological spaces have isomorphic free abelian topological groups has been investigated. We completely solve the analogous problem for free compact abelian groups. Indeed since free compact abelian groups are Bohr compactifications of free abelian topological groups, we are able to apply our results to obtain some new information about the question for free abelian topological groups too.

The main results in this paper and its sequels were announced in [8].

Now let us proceed to the investigation of free compact abelian groups. We begin with some notation.

1.1.1. Definitions. (i) We denote by TOP_0 the category of topological spaces with base point, and base point preserving continuous functions. The category of compact Hausdorff groups and continuous group morphisms is denoted KG, the full subcategory of compact abelian groups KAB.

(ii) The forgetful functors

$$\mathrm{KAB} \longrightarrow \mathrm{KG} \xrightarrow{|\,|} \mathrm{TOP}_0$$

have left adjoints

$$\mathrm{TOP}_0 \xrightarrow{F} \mathrm{KG} \xrightarrow{A} \mathrm{KAB}_0.$$

For a pointed space X we say that FX is the *free compact group* on X and that AFX is the *free compact abelian group on X*. We denote the latter by F_aX.

We now recall the characteristic properties of the left adjoints in our concrete situation. These facts are routine, but it is useful to record the details for later reference.

1.1.2. Remarks. (i) If G is a compact group and G' the closure of its commutator subgroup then $AG = G/G'$, and every morphism $G \to M$ from G into a compact abelian group factors uniquely through the quotient $G \to G/G'$. In particular $F_aX = FX/FX'$.

(ii) For each X in TOP_0 there is a natural map $e_X : X \to FX$ of pointed spaces such that for any map $f: X \to G$ of pointed spaces for a compact group G there is a unique KG-morphism $f': FX \to G$ with $f'e_X = f$. The assignment $f \mapsto f': \mathrm{TOP}_0(X, |G|) \to \mathrm{KG}(FX, G)$ is an isomorphism of sets which is natural in X and G.

(iii) If $e_X : X \to F_aX$ is the canonical map, then for any pointed map $f: X \to M$, for a compact abelian group M, there is a unique KAB-morphism $f^*: F_aX \to M$ with $f^*e_X = f$. The assignment $f \mapsto f^*: \mathrm{TOP}_0(X, |M|) \mapsto \mathrm{KAB}(F_aX, M)$ is an isomorphism of sets which is natural in X and M and which we denote $\alpha_{X,M}$.

(iv) When the sets $\mathrm{TOP}_0(X, |M|)$ and $\mathrm{KAB}(F_aX, M)$ are equipped with the addition inherited from M^X, respectively, M^{F_aX}, then $\alpha_{X,M}$ is an isomorphism of abelian groups.

The pivot around which the character theory of abelian groups turns is the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. From Remark 1.1.2(iii), upon substituting $M = \mathbb{T}$ we obtain the following proposition which, in principle, determines F_a. For the group $\mathrm{TOP}_0(X, |M|)$ we shall write the more familiar expression $C(X, M)$, but we must keep in mind that the functions in this collection always preserve base points.

1.1.3. Notations. (i) We denote the category of all abelian groups and homomorphisms by AB. The product in AB is denoted by $\oplus$ and the product in KAB is denoted by $\times$.

(ii) If $G \in \mathrm{KAB}$ then we regard the dual group $G\hat{}$ as an object in AB. On the other hand if $H \in \mathrm{AB}$ then we denote by $H\hat{}$ the compact abelian group which is the character group of the group H endowed with the discrete topology.

(iii) We denote by $[X, \mathbb{T}]$ the group of all homotopy classes $[f]$ of pointed maps $f: X \to \mathbb{T}$.

1.1.4. Proposition. *For any pointed space X, there exists a natural isomorphism of abelian groups.*

$$\alpha_{X,\mathbb{T}}:\ C(X, \mathbb{T}) \to (F_aX)\hat{}. \tag{1}$$

Dualizing and identifying $F_aX\hat{}\hat{}$ and F_aX via the natural isomorphism yields

$$\alpha_{X,\mathbb{T}}\hat{}:\ G_aX \to C(X, \mathbb{T})\hat{}. \tag{2}$$

Denoting the evaluation map $\mathrm{ev}_X : X \to C(X, \mathbb{T})\hat{}$, given by $\mathrm{ev}_X(x)(f) = f(x)$, gives the commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "e_X"'] \arrow[dr, "\mathrm{ev}_X"] & \\ F_aX \arrow[rr, "\alpha_{X,\mathbb{T}}"'] & & C(X, \mathbb{T})\hat{}; \end{tikzcd} \tag{3}$$

In other words, using the notation of Remark 1.1.2(iii)

$$\alpha_{X,\mathbb{T}}\hat{} = \mathrm{ev}_X^*. \tag{3'}$$

Proof. (1) is immediate from Remark 1.1.2(iii) and (2) follows from (1) upon applying the functor $\hat{}$. In order to prove (3), let $x \in X$; then $e_X(x)$ is a character $(F_aX)\hat{} \to \mathbb{T}$ by the identification $F_aX = F_aX\hat{}\hat{}$. From the definition of the functor $\hat{}$ we note $(\hat{\alpha} \circ e)(x) = \hat{\alpha}(e(x)) = e(x) \circ \alpha : C(X, \mathbb{T}) \to \mathbb{T}$; take $f \in C(X, \mathbb{T})$ and calculate $(\hat{\alpha} \circ e)(x)(f) = e(x)\alpha(f) = e(x)f^* = f(x)$ by Remark 1.1.2(iii). But $f(x) = \mathrm{ev}_X(f)$. This shows $\hat{\alpha} \circ e = \mathrm{ev}$, as asserted. □

Proposition 1.1.4 tells us, then, that F_aX is the dual group of $C(X, \mathbb{T})$, and thus we are led to study the fine structure of $C(X, \mathbb{T})$.

1.2. On $C(X, \mathbb{T})$

In Section 1.1 we observed that the free compact abelian group F_aX has as its dual group the discrete group $C(X, \mathbb{T})$. In this section we analyse the structure of $C(X, \mathbb{T})$ and show that to understand it, one must examine the Alexander–Spanier cohomology group $H^1(X, \mathbb{Z})$.

1.2.1. Lemma. *Let X be any pointed space. Then the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{p} \mathbb{T} \to 0$$

induces an exact sequence

$$0 \to C(X, \mathbb{Z}) \xrightarrow{C(X,j)} C(X, \mathbb{R}) \xrightarrow{C(X,p)} C(X, \mathbb{T}) \xrightarrow{[\cdot]} [X, \mathbb{T}] \to 0. \tag{4}$$

Proof. Only exactness at $C(X, \mathbb{T})$ requires a closer look. A function $f : X \to \mathbb{T}$ belongs to im $C(X, p)$ if and only if there is a lifting $\bar{f} : X \to \mathbb{R}$ with $p\bar{f} = f$. But any function into $\mathbb{R}$ is null-homotopic, since $\mathbb{R}$ is contractible. Thus $\bar{f}$ and then $f = p\bar{f}$ are null-homotopic, whence f is in $\ker[\cdot]$. Thus im $C(X, p) \subseteq \ker[\cdot]$. Conversely, a function $f : X \to \mathbb{T}$ is in $\ker[\cdot]$ if and only if it is null-homotopic, and this is the case if and only if f extends to a function $F : \text{cone } X \to \mathbb{T}$, where cone X is the reduced cone over X and $i : X \to \text{cone } X$ is the embedding onto the base; then we have $f = Fi$. Assume momentarily that X is Hausdorff; then cone X is Hausdorff and contractible; thus F lists to a map $\phi : \text{cone } X \to \mathbb{R}$ with $p\phi = F$. Thus $f = Fi = p\phi i = C(X, p)(\phi i)$. Hence $f \in \text{im } C(X, p)$. If X is not Hausdorff, we let $h : X \to Y$ be the complete regularization of X; then $f = f_0 h$ with a unique $f_0 : Y \to \mathbb{T}$ which is contractible if f is. By the preceding arguments we have $f_0 \in \text{im } C(Y, p)$; that is $f_0 = pg_0$ for some $g_0 : X \to \mathbb{R}$, and so

$$f = f_0 h = pg_0 h = C(X, p)(g_0 h) \in \text{im } C(X, p). \quad \square$$

1.2.2. Remarks. (i) We have $C(X, \mathbb{Z}) = \mathrm{TOP}_0(X, \mathbb{Z}) = H^0(X, x_0; \mathbb{Z}) = \tilde{H}^0(X, \mathbb{Z})$, where H^* is Alexander–Spanier cohomology and $\tilde{H}^0$ the reduced cohomology group in dimension zero.

(ii) If X is paracompact Hausdorff, then $[X, \mathbb{T}] \cong H^1(X, \mathbb{Z})$ since $\mathbb{T}$ is a $K(\mathbb{Z}, 1)$ (see [12] and [5, p. 114]).

Motivated by this Remark we introduce the following definition:

1.2.3. Definition. We write $\tilde{H}^0(X, \mathbb{Z}) = C(X, \mathbb{Z})$ and $H^1(X, \mathbb{Z}) = [X, \mathbb{T}]$.

There is no problem with this notation as long as we deal with paracompact Hausdorff spaces. However, there are locally compact locally contractible 2-dimensional non-paracompact spaces X with $H^1(X, \mathbb{Z}) = \mathbb{Z}$ and $[X, \mathbb{T}] = 0$ (see [4]).

The notation of Definition 1.2.3 allows the following formulation of Lemma 1.2.1:

1.2.4. Theorem. *Let X be any pointed space. Then*

(i) *there are natural exact sequences*

$$0 \to \tilde{H}^0(X, \mathbb{Z}) \to C(X, \mathbb{R}) \xrightarrow{C(X,p)} C(X, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to 0, \tag{4'}$$

$$0 \to H^1(X, \mathbb{Z})\hat{} \to F_a X \to C(X, \mathbb{R})\hat{} \to \tilde{H}^0(X, \mathbb{Z})\hat{} \to 0; \tag{4''}$$

(ii) *there are two short exact sequences*

$$0 \to C(X, \mathbb{Z}) \to C(X, \mathbb{R}) \to \operatorname{im} C(X, p) \to 0, \tag{5}$$

$$0 \to \operatorname{im} C(X, p) \to C(X, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to 0, \tag{6}$$

of which the second one (6) *splits* (*although not naturally*).

Proof. (4') is a reformulation of (4), and (4'') follows from (4') by duality. The exact sequences (5) and (6) follow directly from (4). The splitting of (6) is a consequence of the divisibility of $\operatorname{im} C(X, p)$ which in turn derives from the divisibility of $C(X, \mathbb{R})$. □

1.2.5. Remarks. (i) Theorem 1.2.4 relates the desired group $C(X, \mathbb{T})$ which is $(F_a X)\hat{}$, to $H^1(X, \mathbb{Z})$ and $\tilde{H}^0(X, \mathbb{Z})$.

(ii) Since we may (and will) readily identify $C(X, \mathbb{Z})$ with a subgroup of $C(X, \mathbb{R})$ we may write (5) in equivalent form as

$$\operatorname{im} C(X, p) = C(X, \mathbb{R}) / C(X, \mathbb{Z}). \tag{5'}$$

Thus (6) and (5') together may be expressed by saying that $C(X, \mathbb{T})$ *is the direct sum of* $C(X, \mathbb{R})/C(X, \mathbb{Z})$ *and* $H^1(X, \mathbb{Z})$. It is therefore important to have additional information about these summands. In Section 1.3 we examine the structure of $H^1(X, \mathbb{Z})$.

(iii) Observe that if X is connected, then $C(X, \mathbb{Z})$ is trivial, and so $C(X, \mathbb{T}) \cong H^1(X, \mathbb{Z}) \oplus C(X, \mathbb{R})$.

(iv) Note that for any pointed space X, the group $C(X, \mathbb{R})$ is a rational vector space and so is completely determined by its rank.

1.3. On $H^1(X, \mathbb{Z})$

1.3.1. Proposition. *For any pointed space X, $H^1(X, \mathbb{Z})$ is torsion free.*

Proof. Let $s: H^1(X, \mathbb{Z}) \to C(X, \mathbb{T})$ be a morphism which splits (6); that is, which satisfies $hs = 1$ with the morphism $h: C(X, \mathbb{T}) \to H^1(X, \mathbb{Z})$, $h(f) = [f]$. Let $t \in H^1(X, \mathbb{Z})$ be a torsion element, say, $nt = 0$ for some $n \in \mathbb{Z}$, $n \neq 0$. Set $f = s(t)$. Then $nf = 0$ in $C(X, \mathbb{T})$; that is, $f(X) \subseteq (1/n)\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{T}$. This yields a decomposition of X into open closed subsets $X_k = f^{-1}(k/n + \mathbb{Z})$. We define a continuous function $\bar{f}: X \to \mathbb{R}$ by $\bar{f}(X_k) = \{k/n\}$. Then $f = p\bar{f}$; that is, $f \in \operatorname{im} C(X, p) = \ker h$. Thus $t = h(s(t)) = h(f) = 0$. □

We now show that every torsion free abelian group is isomorphic to $H^1(X, \mathbb{Z})$, for some compact connected Hausdorff space X.

1.3.2. Proposition. *If A is a torsion free abelian group then A is isomorphic to $H^1(X, \mathbb{Z})$, where X is the underlying compact connected Hausdorff space of the dual group of the discrete abelian group A.*

Proof. If X is a compact connected abelian group then the natural map

$$X\hat{} = \operatorname{Hom}(X, \mathbb{T}) \longrightarrow [X, \mathbb{T}] = H^1(X, \mathbb{Z})$$

is an isomorphism [11, p. 206]. So if A is a discrete torsion free abelian group, then $X = A\hat{}$ is compact and connected and $H^1(X, \mathbb{Z}) \cong X\hat{} \cong A$. □

1.4. The cardinalities of $C(X, \mathbb{T})$ and $H^1(X, \mathbb{Z})$

1.4.1. Proposition. *For any compact Hausdorff space X with at least two points and any infinite abelian group M*

(i) *card* $C(X, \mathbb{R}) = \operatorname{card} C(X, \mathbb{T}) = \max(2^{\aleph_0}, w(X)) = w(F_aX)$, *and*

(ii) card $C_{\text{fin}}(X, M) = \max(w(X/\text{conn}), \operatorname{card} M)$

where $w(X)$ = weight of the space $X = \min\{\text{card } \beta : \beta$ a basis for the topology of $X\}$, $C_{\text{fin}}(X, M)$ is the group of all finitely valued locally constant and basepoint preserving functions from X into M, and $X/$conn denotes the quotient space of X modulo the connectivity relation.

Proof. We establish the result through a sequence of technical steps.

Step 1. card $C(X, \mathbb{R}) = \operatorname{card} C(X, \mathbb{T}) = \operatorname{card} C(X, I) = w(F_aX)$, where I is the closed unit interval $[0, 1]$.

Proof. The second equality follows from the fact that there are embeddings $I \to \mathbb{T} \to I^2$ which induce embeddings $C(X, I) \to C(X, \mathbb{T}) \to C(X, I^2) \cong C(X, I)^2$ and from the fact that card $C(X, I)^2 = \operatorname{card} C(X, I)$. The first equality follows from the fact that suitable embeddings $I \to R \to I$ induce injections $C(X, I) \to C(X, \mathbb{R}) \to C(X, I)$. Finally observe that for any compact abelian group G, $w(G) = \operatorname{card}(G^\wedge)$ and so $w(F_a X) = \operatorname{card}(C(X, \mathbb{T}))$.

Step 2. $2^{\aleph_0} \leq \operatorname{card} C(X, \mathbb{R})$.

Proof. Since X has at least two points this is immediate.

Step 3. If $f: X_1 \to X_2$ is a continuous injection and $g: X_2 \to X_3$ is a continuous surjection of compact spaces, then $\operatorname{card} C(X_1, \mathbb{R}) \leq \operatorname{card} C(X_2, \mathbb{R})$ and $\operatorname{card} C(X_3, \mathbb{R}) \leq \operatorname{card} C(X_2, \mathbb{R})$.

Proof. This is clear from the fact that we have a surjection $C(X_2, \mathbb{R}) \to C(X_1, \mathbb{R})$ and an injection $C(X_3, \mathbb{R}) \to C(X_2, \mathbb{R})$.

Step 4. For any compact metric space Y (with more than one point) and any infinite set A, $w(Y^A) = \operatorname{card} A$. In particular, $w(I^A) = w(\mathbb{T}^A) = w(2^A) = \operatorname{card} A$, where $2 = \{0, 1\}$ with the discrete topology.

Proof. As observed above, if G is any compact abelian group then $w(G) = \operatorname{card}(G^\wedge)$. Therefore $w(\mathbb{T}^A) = \operatorname{card}(\mathbb{T}^{A\wedge})$. But $\mathbb{T}^{A\wedge}$ is the restricted direct sum of card A copies of $\mathbb{Z}$, and so $w(T^A) = \operatorname{card} A$. Similarly, $w(2^A) = w((\mathbb{Z}/2\mathbb{Z})^A) = \operatorname{card}((\mathbb{Z}/2\mathbb{Z})^{A\wedge})$, and $(\mathbb{Z}/2\mathbb{Z})^{A\wedge}$ is the restricted direct sum of card A copies of $\mathbb{Z}/2\mathbb{Z}$. Consequently, $w(2^A) = \operatorname{card} A$. Finally observe that if Y is any compact metric space with more than one point, then there are embeddings $2 \to Y \to \mathbb{T}^{\aleph_0}$ and hence also embeddings $2^A \to Y^A \to \mathbb{T}^A$ which implies that $w(2^A) \leq w(Y^A) \leq w(\mathbb{T}^A) = \operatorname{card} A$. So $w(Y) = \operatorname{card} A$.

Step 5. $w(X) \leq \operatorname{card} C(X, I)$.

Proof. There is an embedding $X \to I^{C(X,I)}$, given by $x \to (f(x))_{f \in C(X,I)}$. Then $w(X) \leq w(I^{C(X,I)}) = \operatorname{card} C(X, I)$, by Step 4.

Step 6. $\max(2^{\aleph_0}, w(X)) \leq \operatorname{card} C(X, \mathbb{R})$.

Proof. This follows immediately from Steps 1, 2 and 5.

Step 7. If X is infinite and compact zero-dimensional, then $\operatorname{card} C(X, 2) = w(X)$.

Proof. The cardinal of $C(X, 2)$ is the same as that of the set of *all* (not necessarily base point preserving) maps from X to 2 and this is equivalent to the set L of all compact open subsets of X. Let B be a basis for the open sets of X with card $B = w(X)$. Now form the set B' of finite unions of sets in B. Then card $B' = \operatorname{card} B = w(X)$. But then we have $L \subseteq B'$, since every compact open set is a finite union of basic subsets contained in it. Thus card $L \leq w(X)$. On the other hand, L is a basis for the topology since X is zero dimensional. Hence $w(X) \leq \operatorname{card} L$. Thus $\operatorname{card} C(X, 2) = w(X)$.

Step 8. If X is compact and zero dimensional, then $\operatorname{card} C(X, \mathbb{R}) = \max(2^{\aleph_0}, w(X))$.

Proof. We let $\mathbb{K}$ be the standard Cantor set and $j: \mathbb{K} \to I$ the embedding and $k: \mathbb{K} \to I$ the surjective Cantor Caratheodory function. Then $C(X, j): C(X, \mathbb{K}) \to C(X, I)$ is clearly an injection. We claim that $C(X, k): C(X, \mathbb{K}) \to C(X, I)$ is surjective. To see this, observe that since X is zero-dimensional, the set of locally constant and finitely-valued functions in $C(X, I)$ is uniformly dense, and each such function lifts to a function in $C(X, \mathbb{K})$, as k is surjective. Thus im $C(X, k)$ is uniformly dense and also uniformly closed. Thus $C(X, k)$ is surjective. So now we know that card $C(X, I) = \text{card } C(X, \mathbb{K})$. But $\mathbb{K}$ is homeomorphic to $2^{\mathbb{N}}$; whence $C(X, \mathbb{K}) \cong C(X, 2^{\mathbb{N}}) \cong C(X, 2)^{\mathbb{N}}$. Thus card $C(X, \mathbb{K}) = (\text{card } C(X, 2))^{\aleph_0}$. If X is finite, then $w(X)$ is finite and so card $C(X, \mathbb{K}) = 2^{\aleph_0}$. If X is infinite then, by Step 7, card $C(X, \mathbb{K}) = (w(X))^{\aleph_0}$. So, by Step 1, card $C(X, \mathbb{R}) = \max(2^{\aleph_0}, w(X))$, whether X is finite or infinite.

Step 9. For any infinite set A, card $C(I^A, \mathbb{R}) = \max(2^{\aleph_0}, \text{card } A)$.

Proof. As $\mathbb{K}$ is a subspace of I and I is a quotient space of $\mathbb{K}$, Step 3 implies card $C(I^A, \mathbb{R}) = \text{card } C(\mathbb{K}^A, \mathbb{R})$. Steps 8 and 4 then imply card $C(\mathbb{K}^A, \mathbb{R}) = \max(2^{\aleph_0}, w(\mathbb{K}^A)) = \max(2^{\aleph_0}, \text{card } A)$. Thus card $C(I^A, \mathbb{R}) = \max(2^{\aleph_0}, \text{card } A)$.

Step 10. card $C(X, \mathbb{R}) \leq \max(2^{\aleph_0}, w(X))$.

Proof. Let B be a basis for the open sets of X, with card $B = w(X)$. Let $J = \{(U, V) \in B \times B \colon \bar{U} \cap \bar{V} = \emptyset\}$. Then $w(X) \leq \text{card } J \leq w(X^2) = w(X)$. For each $j = (U, V) \in J$ we fix a continuous function $f_j : X \to I$ with $U \subseteq f^{-1}(0)$ and $V \subseteq f^{-1}(1)$. Then we have an embedding $x \to (f_j(x))_{j \in J} : X \to I^J$, whence we have

$$\begin{aligned} \text{card } C(X, \mathbb{R}) &\leq C(I^J, \mathbb{R}), \quad \text{by Step 3} \\ &= \max(2^{\aleph_0}, \text{card } J), \quad \text{by Step 9} \\ &= \max(2^{\aleph_0}, w(X)). \end{aligned}$$

Step 11. Putting together the results of Steps 1, 6 and 10 we have completed the proof of part (i) of the Proposition.

Step 12. The quotient map $X \to X/\text{conn}$ induces an isomorphism $C_{\text{fin}}(X/\text{conn}, M) \to C_{\text{fin}}(X, M)$.

Proof. Every locally constant function must be constant on components. Hence every locally constant function on X factors uniquely through $X \to X/\text{conn}$.

Step 13. If X is zero-dimensional, then card $C_{\text{fin}}(X, M) = \max(w(X), \text{card } M)$.

Proof. If X is finite, then the result is clearly true. So consider the case that X is infinite. We know from the proof of Step 7, that the cardinality of the set of open closed subsets of X is $w(X)$. Hence the cardinality of the set E of equivalence relations on X with open closed equivalence classes is likewise $w(X)$ (since each such relation has finitely many equivalence classes). Each $f \in C_{\text{fin}}(X, M)$ uniquely determines a finite set $f(X)$ of values in M and an element $R_f \in E$ whose equivalence classes are the finitely many sets $f^{-1}(m)$ where m ranges through $f(X)$. Thus card $C_{\text{fin}}(X, M) \leq (\text{card } M)(\text{card } E) = \max(w(X), \text{card } M)$. Conversely, since every open closed set not containing the base point defines a function in $C_{\text{fin}}(X, M)$ by assigning on this set a value $m \in M$ and on the complement the value 0, the cardinality of $C_{\text{fin}}(X, M)$ is also at least $\max(w(X), \text{card } M)$.

Step 14. Steps 12 and 13 together imply that part (ii) of the Proposition is true. □

The following Remark shows that, for our purposes, we may as well restrict our attention to compact Hausdorff spaces.

1.4.2. Remark. Let COMP_0 be the category of compact Hausdorff pointed spaces and $\beta:\mathrm{TOP}_0\to\mathrm{COMP}_0$ the left reflection, with front adjunction $b_X: X\to\beta X$. Then

$$F_a b_X:\ F_a X \longrightarrow F_a\beta X \quad \text{is an isomorphism}$$

and

$$e_{\beta X}:\ \beta X \longrightarrow |F_a\beta X| \quad \text{is a topological embedding.}$$

Proof. By the universal property of β, the function $e_X: X\to|F_aX|$ factors uniquely through b_X; that is, there is a unique map $f_X:\beta X\to|F_aX|$ with $f_X^* e_X = f_X b_X$. By Remark 1.1.2(ii) there is a unique map $f_X^*: F_a\beta X\to F_aX$ with $|f_X^*|e_{\beta X}=f_X$. Then $1_{F_aX}e_X = f_X b_X = |f_X^*|e_{\beta X}b_X = |f_X^*||F_a b_X|e_X = |f_X^* F_a b_X|e_X$, whence $f_X^*\circ(F_a b_X) = 1_{F_aX}$ by uniqueness in Remark 1.1.2(ii). Since b_X is epic and $F_a b_X$ is epic too, we conclude that $F_a b_X$ is an isomorphism.

Since the continuous functions $f:\beta X\to\mathbb{T}$ separate the points, there is a continuous injection $\beta X\to\mathbb{T}^{C(\beta X,\mathbb{T})}$ which, by compactness of βX, is an embedding and which factors through $e_{\beta X}$. Hence $e_{\beta X}$ must be an embedding. □

We can now complement our information on the group $H^1(X,\mathbb{Z})$.

1.4.3. Proposition. *For any pointed space X such that βX has at least two points*

$$\operatorname{card} H^1(X,\mathbb{Z}) = \max(\aleph_0, \operatorname{rank} H^1(X,\mathbb{Z})) \leqslant \max(2^{\aleph_0}, w(\beta X)) = \operatorname{rank} C(\beta X,\mathbb{R}).$$

Proof. The first equality is clear since $H^1(X,\mathbb{Z})$ is a torsion free group by Proposition 1.3.1. By Theorem 1.2.4(6), $H^1(X,\mathbb{Z})$ is a quotient of $C(X,\mathbb{T})$. But $C(X,\mathbb{T})\cong C(\beta X,\mathbb{T})$, and so by Proposition 1.4.1

$$\begin{aligned}\operatorname{rank} H^1(X,\mathbb{Z}) \leqslant \operatorname{card} H^1(X,\mathbb{Z}) \leqslant \operatorname{card} C(X,\mathbb{T}) &= \operatorname{card} C(\beta X,\mathbb{T})\\ &= \max(2^{\aleph_0}, w(\beta X))\\ &= \operatorname{card} C(\beta X,\mathbb{R}).\end{aligned}$$

As noted in Remark 1.2.5 (iv), $C(\beta X,\mathbb{R})$ is a rational vector space, and as $\operatorname{card} C(\beta X,\mathbb{R})\geqslant 2^{\aleph_0}$ we have $\operatorname{card} C(\beta X,\mathbb{R}) = \operatorname{rank} C(\beta X,\mathbb{R})$. □

1.5. The first structure theorem

We are now in a position to determine the structure of $C(X,\mathbb{R})/C(X,\mathbb{Z})$. Recall that we identify $C(X,\mathbb{Z})$ with a subgroup of $C(X,\mathbb{R})$.

1.5.1. Proposition. *Let X be any pointed space such that βX has at least two points.*

(i) *The torsion subgroup of $C(X,\mathbb{R})/C(X,\mathbb{Z})$ is isomorphic to the group $C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})$ of all finitely valued locally constant and base point preserving functions on X with values in $\mathbb{Q}/\mathbb{Z}$. The cardinality of the torsion group is* $\max(w(\beta X/\mathrm{conn}), \aleph_0)$.

(ii) *The group* $C(X,\mathbb{R})/C(X,\mathbb{Z})$ *splits over its torsion group and its torsion free quotient is isomorphic to* $C(X,\mathbb{R})$. *Thus*

$$C(X,\mathbb{R})/C(X,\mathbb{Z})\cong C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})\oplus C(X,\mathbb{R}). \tag{7}$$

The cardinality of the torsion free quotient is $\max(2^{\aleph_0}, w(\beta X))$.

Proof. (i) The image in $C(X,\mathbb{R})/C(X,\mathbb{Z})$ of an element $f\in C(X,\mathbb{R})$ is in the torsion subgroup if and only if there is an integer n such that $nf\in C(X,\mathbb{Z})$: that is $f\in C(X,(1/n)\mathbb{Z})$. Since $C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})=\bigcup\{C(X,(1/n)\mathbb{Z})\colon n\in\mathbb{Z}, n\neq 0\}$ we have the first part of our assertion. The statement about the cardinality is proved as follows: The maps $X\to\beta X\to\beta X/\text{conn}$ induce isomorphisms $C_{\text{fin}}(\beta X/\text{conn},\mathbb{Q}/\mathbb{Z})\to C_{\text{fin}}(\beta X,\mathbb{Q}/\mathbb{Z})\to C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})$. The assertion then follows from Proposition 1.4.1(ii).

(ii) The torsion subgroup of $C(X,\mathbb{R})/C(X,\mathbb{Z})$ is divisible by (i) and hence is a direct summand of $C(X,\mathbb{R})/C(X,\mathbb{Z})$. As the group itself is divisible, the torsion free quotient is a rational vector space D, and we have the relation

$$\text{card } C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})\cdot\text{card } D=\text{card } C(X,\mathbb{R})/C(X,\mathbb{Z}).$$

Now $C(X,\mathbb{R})/C(X,\mathbb{Z})$ contains a subgroup which is isomorphic to $C_{\text{fin}}(X,\mathbb{R}/\mathbb{Z})$. We write $\mathbb{R}/\mathbb{Z}=\mathbb{Q}/\mathbb{Z}\oplus \mathrm{H}$, where H is a subgroup which is isomorphic to $\mathbb{R}$. Then we calculate $C_{\text{fin}}(X,\mathbb{R}/\mathbb{Z})=C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})\oplus C_{\text{fin}}(X,H)$. Thus D has a subgroup which is isomorphic to $C_{\text{fin}}(X,\mathbb{R})=C_{\text{fin}}(\beta X,\mathbb{R})$, whose cardinality is $\max(w(\beta X/\text{conn}),2^{\aleph_0})$, by Proposition 1.4.1(ii). Likewise we have $\text{card } C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})=\max(w(\beta X/\text{conn}),\aleph_0)\leq\text{card } D$. Thus $\text{card } D=\text{card } C(X,\mathbb{R})/C(X,\mathbb{Z})$. Since $\text{card } C(X,[0,\tfrac12[)=\text{card } C(X,\mathbb{R})$ and the quotient map $C(X,\mathbb{R})\to C(X,\mathbb{R})/C(X,\mathbb{Z})$ maps $C(X,[0,\tfrac12[)$ injectively, it follows that

$$\text{card } C(X,\mathbb{R})/C(X,\mathbb{Z})\geq\text{card } C(X,[0,\tfrac12[)=\text{card } C(X,\mathbb{R}).$$

Obviously $\text{card } C(X,\mathbb{R})\geq\text{card } C(X,\mathbb{R})/C(X,\mathbb{Z})$ and so

$$\text{card } C(X,\mathbb{R})=\text{card } C(X,\mathbb{R})/C(X,\mathbb{Z})=\text{card } D.$$

As $C(X,\mathbb{R})$ and D are rational vector spaces, the statement $\text{card } C(X,\mathbb{R})=\text{card } D$ implies that $C(X,\mathbb{R})$ is isomorphic to D, providing $\text{card } C(X,\mathbb{R})\geq 2^{\aleph_0}$, which is of course true. So (7) is proved.

Finally observe that

$$\begin{aligned}\text{card } C(X,\mathbb{R})&=\text{card } C(X,I)\\&=\text{card } C(\beta X,I)\\&=\text{card } C(\beta X,\mathbb{R})\\&=\max(2^{\aleph_0}, w(\beta X)),\quad\text{by Proposition 1.4.1(i).}\end{aligned}\qquad\square$$

The torsion subgroup of $C(X,\mathbb{R})/C(X,\mathbb{Z})$ can be further decomposed in a canonical fashion. Indeed

$$\mathbb{Q}/\mathbb{Z}=\sum_{p\text{ prime}}\mathbb{Z}(p^{\infty})$$

where $\mathbb{Z}(p^\infty)$ is the Prüfer group of all p^n-th roots of unity, $n=2,3,\ldots$. Using this observation we have

1.5.2. Remark. The torsion subgroup of $C(X,\mathbb{T})$ is isomorphic to the torsion subgroup of $C(X,\mathbb{R})/C(X,\mathbb{Z})$ and is

$$C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z}) \cong \sum_{p \text{ prime}} C_{\text{fin}}(X,\mathbb{Z}(p^\infty)) \cong \sum_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(r)} \tag{8}$$

where $r = w_0(\beta X/\text{conn})$; $w_0(Y)=w(Y)$ if $w(Y)$ is infinite and $w_0(Y)=w(Y)-1$ if $w(Y)$ is finite; and for any abelian group M, $M^{(r)}$ denotes the direct sum of r copies of M.

Proof. By Proposition 1.3.1, $H^1(X,\mathbb{Z})$ is torsion free and so by Remark 1.2.5(ii) the torsion subgroup of $C(X,\mathbb{T})$ is isomorphic to the torsion subgroup of $C(X,\mathbb{R})/C(X,\mathbb{Z})$. By Proposition 1.5.1 this torsion subgroup is $C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})$ which, according to the decomposition of $\mathbb{Q}/\mathbb{Z}$ into the direct sum of its Sylow subgroups is isomorphic to

$$\sum_{p \text{ prime}} C_{\text{fin}}(X,\mathbb{Z}(p^\infty)).$$

Now $C_{\text{fin}}(X,\mathbb{Z}(p^\infty)) = C(\beta X/\text{conn},\mathbb{Z}(p^\infty))$. If $\beta X/\text{conn}$ has $n+1$ elements, then this last group is $\mathbb{Z}(p^\infty)^{(n)}$ and $n = w_0(\beta X/\text{conn})$. If $\beta X/\text{conn}$ is infinite, then the rank of $C(\beta X/\text{conn},\mathbb{Z}(p^\infty))$ is equal to its cardinality, which is $r = w(\beta X/\text{conn})$ by Proposition 1.5.1(i). □

1.5.3. Corollary. *Let X be a pointed topological space such that βX has at least two points.*

(i) *The weight of F_aX is $w(F_aX) = \max(2^{\aleph_0}, w(\beta X))$.*

(ii) *The zero dimensional quotient $F_aX/(F_aX)_0$ is isomorphic to*

$$\prod_{p \text{ prime}} \mathbb{Z}_p(w_0(\beta X/\text{conn}))$$

where $\mathbb{Z}_p$ is the additive group of p-adic integers.

(iii) *The map*

$$X \xrightarrow{e_X} F_aX \to F_aX/(F_aX)_0$$

is equivalent to the evaluation map

$$x \to (f \to f(x)) \colon X \to (C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z}))^\wedge.$$

Proof. (i) By Remark 1.4.2, $F_aX \cong F_a\beta X$. Then by Proposition 1.4.1,

$$w(F_aX) = w(F_a\beta X) = \max(w(\beta X), 2^{\aleph_0}).$$

(ii) By Proposition 1.1.4, the character group of $F_aX/(F_aX)_0$ is isomorphic to the torsion subgroup of $C(X,\mathbb{T})$ in view of duality. This torsion subgroup was determined in Remark 1.5.2. Since the character group of $\mathbb{Z}(p^\infty)$ is $\mathbb{Z}_p$, the assertion (ii) follows by duality.

(iii) This follows from the fact that $F_aX \to F_aX/(F_aX)_0$ is dual to $C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}) \to C(X, \mathbb{T})$ and from Proposition 1.1.4(3). $\square$

Putting together our findings we obtain:

$$\begin{aligned}
(F_aX)\hat{} &\cong C(X, \mathbb{T}) \\
&\cong C(X, \mathbb{R})/C(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}), && \text{by Remark 1.2.5} \\
&\cong C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}) \oplus C(X, \mathbb{R}) \oplus H^1(X, \mathbb{Z}), && \text{by Proposition 1.5.1(7)} \\
&\cong C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))} \oplus H^1(X, \mathbb{Z}), \\
& && \text{by Proposition 1.5.1(ii)} \\
&\cong \Big(\prod_{p \text{ prime}} \mathbb{Z}_p^{(w_0(\beta X/\text{conn}))}\Big)\hat{} \oplus \mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))} \oplus H^1(X, \mathbb{Z}) \\
& && \text{by Remark 1.5.2 and Corollary 1.5.3.}
\end{aligned}$$

So this is our first structure theorem:

1.5.4. Theorem. *For a pointed topological space X such that βX has at least two points, the free compact abelian group F_aX is isomorphic to the direct product of the three factors:*

(i) *the character group of $H^1(X, \mathbb{Z})$,*

(ii) *the character group of $C(X, \mathbb{R}) \cong \mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))}$,*

(iii) $\prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(\beta X/\text{conn})}$, *where $w_0(Y)+1 = w(Y)$.*

1.5.5. Remark. If A is an abelian group then it has a unique maximal divisible subgroup, div A and the quotient group $A/\text{div}\, A$ is a reduced group, denoted by red A. The group red $H^1(X, \mathbb{Z})$ is then the quotient group $H^1(X, \mathbb{Z})/V$ where V is the largest divisible subgroup of $H^1(X, \mathbb{Z})$. As $H^1(X, \mathbb{Z})$ is torsion free, V is a rational vector space, and

$$H^1(X, \mathbb{Z}) \cong \text{red}\, H^1(X, \mathbb{Z}) \oplus V.$$

Since rank $V \leq$ card $H^1(X, \mathbb{Z}) \leq \max(2^{\aleph_0}, w(\beta X)) = \text{rank}\, C(X, \mathbb{R})$,

$$\begin{aligned}
H^1(X, \mathbb{Z}) \oplus C(X, \mathbb{R}) &\cong \text{red}\, H^1(X, \mathbb{Z}) \oplus V \oplus \mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))} \\
&\cong \text{red}\, H^1(X, \mathbb{Z}) \oplus \mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))}
\end{aligned}$$

So F_aX is also isomorphic to the product of the three factors (ii), (iii) of Theorem 1.5.4, and

(i)′ the character group of red $H^1(X, \mathbb{Z})$.

1.5.6. Remark. It is clear from Remark 1.5.5 that for any compact Hausdorff space X the topological invariants red $H^1(X, \mathbb{Z})$, $w(X)$, and $w(X/\text{conn})$ completely determine the compact abelian group F_aX.

Using Remark 1.2.5(iii) we obtain a particularly nice result when X is connected:

1.5.7. Corollary. *If X is a connected pointed space such that βX has at least two points, then*

$$F_a X \cong H^1(X,\mathbb{Z})^\wedge \times C(X,\mathbb{R})^\wedge \cong H^1(X,\mathbb{Z})^\wedge \times (\mathbb{Q}^{(\max(2^{\aleph_0}, w(\beta X)))})^\wedge. \tag{9}$$

1.5.8. Corollary. *If X is a totally disconnected compact Hausdorff pointed space with at least two points then there is a functorial exact sequence.*

$$0 \to F_a X \to C(X,\mathbb{R})^\wedge \to \tilde{H}^0(X,\mathbb{Z})^\wedge \to 0 \tag{10}$$

and

$$F_a X \cong (\mathbb{Q}^\wedge)^{\max(2^{\aleph_0}, w(X))} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(X)}.$$

We can now calculate the homotopy groups of a free compact abelian group (cf. [1] and [6]).

1.5.9. Proposition. *Let X be any pointed space. Then*

(i) $\pi_0(F_a X) \cong \operatorname{Ext}(C(X,\mathbb{T}),\mathbb{Z})$

$$\cong \operatorname{Ext}(C(X,\mathbb{R}),\mathbb{Z}) \oplus \operatorname{Ext}(H^1(X,\mathbb{Z}),\mathbb{Z}) \oplus \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(\beta X/\mathrm{conn})}.$$

(ii) $\pi_1(F_a X) \cong \operatorname{Hom}(H^1(X,\mathbb{Z}),\mathbb{Z})$.

(iii) $\pi_n(F_a X) = 0$, *for* $n \geq 2$.

Proof. For any compact abelian group G we have

$$\pi_0(|G|) \cong \operatorname{Ext}(G^\wedge,\mathbb{Z}), \quad \text{so } \pi_0(F_a X) \cong \operatorname{Ext}(C(X,\mathbb{T}),\mathbb{Z}).$$

As $\operatorname{Ext}(-,\mathbb{Z})$ is additive, by Remark 1.2.5(ii) and Proposition 1.5.1(ii) which say that

$$C(X,\mathbb{T}) \cong H^1(X,\mathbb{Z}) \oplus C(X,\mathbb{R}) \oplus C_{\mathrm{fin}}(X,\mathbb{Q}/\mathbb{Z})$$

we obtain

$$\pi_0(F_a X) \cong \operatorname{Ext}(H^1(X,\mathbb{Z}),\mathbb{Z}) \oplus \operatorname{Ext}(C(X,\mathbb{R}),\mathbb{Z}) \oplus \operatorname{Ext}(C_{\mathrm{fin}}(X,\mathbb{Q}/\mathbb{Z}),\mathbb{Z}).$$

Since for a torsion group A,

$$\operatorname{Ext}(A,\mathbb{Z}) \cong \operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z}) \cong A^\wedge$$

the formula for $\pi_0(F_a X)$ becomes

$$\pi_0(F_a X) \cong \operatorname{Ext}(H^1(X,\mathbb{Z}),\mathbb{Z}) \oplus \operatorname{Ext}(C(X,\mathbb{R}),\mathbb{Z}) \oplus C_{\mathrm{fin}}(X,\mathbb{Q}/\mathbb{Z})^\wedge$$

from which (i) follows by Remark 1.5.2. $\square$

If X is a connected CW-complex then we have the Hurewicz morphism $\pi_1(X) \to H_1(X, \mathbb{Z})$, which identifies the first singular homology group with the commutator factor group of $\pi_1(X)$. The universal coefficient theorem yields an isomorphism $H^1(X, \mathbb{Z}) \to \operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})$, since $\operatorname{Ext}(H_0(X, \mathbb{Z}), \mathbb{Z}) = 0$. Thus we have the following corollary:

1.5.10. Corollary. *If X is a pointed connected CW-complex, then*

$$\pi_1(F_a X) \cong \operatorname{Hom}(\operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}), \mathbb{Z}).$$

All other homotopy groups are zero.

1.6. Isomorphic free compact abelian groups

In [15] it was pointed out that nonhomeomorphic compact Hausdorff spaces X and Y can have isomorphic free compact abelian groups. We are now able to settle the question: which spaces have isomorphic free compact abelian groups? The result follows immediately from Remark 1.5.5.

1.6.1. Proposition. *The following statements are equivalent for compact Hausdorff spaces X and Y:*

(i) $F_a X \cong F_a Y$,

(ii) $C(X, \mathbb{T}) \cong C(Y, \mathbb{T})$,

(iii) $\operatorname{red} H^1(X, \mathbb{Z}) \cong \operatorname{red} H^1(Y, \mathbb{Z})$, $w(X/\text{conn}) = w(Y/\text{conn})$, *and* $\max(2^{\aleph_0}, w(X)) = \max(2^{\aleph_0}, w(Y))$.

Since Markov's 1948 paper [14] on free topological groups there has been much investigation of when two topological spaces have isomorphic free abelian topological groups (see [2, 3, 7, 13, 16, 18, 19]). Observing that the free compact abelian group on a space X is the Bohr compactification of the free abelian topological group on X, we see that if X and Y have isomorphic free abelian topological groups, then they also have isomorphic free compact abelian groups. So we obtain:

1.6.2. Corollary. *If compact Hausdorff pointed spaces X and Y have isomorphic free abelian topological groups then* $\operatorname{red} H^1(X, \mathbb{Z}) \cong \operatorname{red} H^1(Y, \mathbb{Z})$, $\max(2^{\aleph_0}, w(X)) = \max(2^{\aleph_0}, w(Y))$, *and* $w(X/\text{conn}) = w(Y/\text{conn})$.

1.6.3. Remarks. (i) The converse of Corollary 1.6.2 is false. If X and Y are contractible compact Hausdorff pointed spaces of equal weights (e.g. n-cells for any $n \leq 2^{\aleph_0}$) then $F_a X \cong F_a Y$. However the free abelian topological group functor distinguishes dimension and detects metrizability [7].

(ii) Let X and Y be compact Hausdorff connected abelian groups with dense torsion subgroups. If $F_a|X| \cong F_a|Y|$ then X is isomorphic to Y.

Proof. If $F_a|X| \cong F_a|Y|$ then $C(X, \mathbb{T}) \cong C(Y, \mathbb{T})$ and so red $C(X, \mathbb{T}) \cong$ red $C(Y, \mathbb{T})$. But by Remark 1.2.5(ii) and Remark 1.5.5, red $H^1(X, \mathbb{Z}) \cong$ red $C(X, \mathbb{T}) \cong$ red $C(Y, \mathbb{T}) \cong$ red $H^1(Y, \mathbb{Z})$. Recall that for a compact abelian group G, G has a dense torsion subgroup if and only if $G\hat{}$ is reduced. But as observed in the proof of Proposition 1.3.2, $X\hat{} \cong H^1(X, \mathbb{Z})$. So

$$X\hat{} \cong \text{red } X\hat{} \cong \text{red } H^1(X, \mathbb{Z}) \cong \text{red } H^1(Y, \mathbb{Z}) \cong \text{red } Y\hat{} \cong Y\hat{}.$$

Hence $X \cong Y$, as required. □

1.6.3. Remark. (iii) The condition 'dense torsion subgroup' in (ii) can not be omitted, since, for example $F_a|\mathbb{T}| \cong F_a|\mathbb{T} \times \mathbb{Q}\hat{}|$. Indeed, if X is any compact connected abelian group then by Corollary 1.5.7

$$F_a|X| \cong X \times (\mathbb{Q}^{(\max(2^{\aleph_0},\, w(X)))})\hat{},$$

Further, by Corollary 1.6.1, if X and Y are compact abelian groups, then the following three conditions are equivalent:

(a) $F_a|X| \cong F_a|Y|$

(b) red $X\hat{} \cong$ red $Y\hat{}$ and $w(X) = w(Y)$

(c) $\overline{\text{tor } X} \cong \overline{\text{tor } Y}$ and $w(X) = w(Y)$, where 'tor' denotes the 'torsion subgroup of'.

(iv) From Proposition 1.6.1 it is clear that for compact Hausdorff totally disconnected spaces X and Y, $F_aX \cong F_aY$ if and only if $w(X) = w(Y)$.

1.7. The characterization theorem and the second structure theorem

We begin by recalling some terminology and notation.

If tor A denotes the torsion subgroup of an abelian group A, then

$$\text{rank } A = \text{rank}(A/\text{tor } A) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes A$$

and, for any prime p, we have

$$p\text{-rank } A = p\text{-rank tor } A = \dim_{GF(p)} p\text{-socle } A$$

where p-socle $A = \{a \in A \mid pa = 0\}$.

Dually, if G is a compact abelian group then $\dim G = \dim G_0 = \text{rank } G\hat{}$. We set p-rank $G = p$-rank $G\hat{}$.

Now we introduce a non-standard concept.

1.7.1. Definition. An abelian group A is said to be *co-free* if it satisfies the following conditions:

(i) Its torsion subgroup, tor A is divisible;

(ii) All primary components (p-Sylow subgroups) of tor A have the same p-rank, a;

(iii) rank div $A \geq 2^{\aleph_0}$;

(iv) rank A = rank div $A \geq a$.

Now we are ready for the second structure theorem. This result tells us which compact abelian groups are free compact abelian groups.

1.7.2. Theorem. *Let G be a compact abelian group and $A = G\hat{}$ its character group. Then the following conditions are equivalent*:

(i) *G is a free compact abelian group*;

(ii) *G is a free compact abelian group on some compact Hausdorff space*;

(iii) *A is co-free*;

(iv) $G \cong K \times (\mathbb{Q}\hat{})^a \times \prod_{p \text{ prime}} \mathbb{Z}_p^b$,

where K is a compact connected abelian group with a dense torsion subgroup, and $a \geqslant \max\{2^{\aleph_0}, b, \dim K\}$.

Proof. The condition (iv) clearly implies (iii). Conversely assuming (iii) says that $\operatorname{tor} A \subseteq \operatorname{div} A$ and so

$$A \cong \operatorname{red} A \oplus \operatorname{div}(A/\operatorname{tor} A) \oplus \operatorname{tor} A$$

from which (iv) then follows.

Conditions (i) and (ii) are equivalent by Remark 1.4.2.

By Remark 1.5.5, (i) implies (iv).

Finally assume (iv) is true. Let X be the compact abelian group given by

$$X = K \times (\mathbb{Q}\hat{})^a \times F$$

where F is any finite abelian group of order $b-1$, if b is finite, and $\mathbb{Z}(2)^b$, otherwise. Then $H^1(X, \mathbb{Z}) \cong K\hat{} \oplus \mathbb{Q}^a$ and $w(X) = a$. So by Theorem 1.5.4,

$$\begin{aligned}(F_{\mathrm{a}}|X|)\hat{} &\cong K \oplus \mathbb{Q}^{(a)} \oplus \mathbb{Q}^{(a)} \oplus \sum_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(b)} \\ &\cong K \oplus \mathbb{Q}^{(a)} \oplus \sum_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(b)}\end{aligned}$$

and thus $F_{\mathrm{a}}|X| \cong G$. So G is a free compact abelian group; that is, (i) is true. □

From the proof of Theorem 1.7.2 we see that every free compact abelian group on any space is isomorphic to the free compact abelian group on the underlying space of a compact abelian group. What is more, in the above proof, the compact group is constructed. So given any free compact abelian group G we know how to construct a space Y such that $G \cong F_{\mathrm{a}} Y$.

1.7.3. Corollary. *Let G be a free compact abelian group. Then there exists a compact abelian group X such that $G \cong F_{\mathrm{a}}|X|$.*

We can now state the characterization theorem:

1.7.4. Theorem. *A compact abelian group, G, is a free compact abelian group if and only if $G = G_1 \times G_2 \times G_3$ where*

(a) *$G_1\hat{}$ is a torsion free group,*
(b) *$G_2\hat{}$ is a rational vector group,*
(c) *$G_3\hat{}$ is a divisible torsion group whose Sylow p-subgroups all have the same rank r,*

and

(d) card $(G_2\hat{}) \geq \max\{\mathrm{card}(G_1\hat{}),\ r,\ 2^{\aleph_0}\}$.

Proof. Clearly if G is a free compact abelian group then, by Theorem 1.7.2, G satisfies (a), (b), (c) and (d).

Conversely, let G satisfy (a), (b), (c) and (d). Then putting $K = \mathrm{red}\, G_1\hat{}$ and observing that the cardinality restrictions yield $K \oplus G_2\hat{} \cong G_1\hat{} \oplus G_2\hat{}$ we have that $G \cong K\hat{} \times G_2 \times G_3$ and so the condition (iv) of Theorem 1.7.2 is satisfied, whence G is a free compact abelian group. □

The following proposition gives a method different to that of Theorem 1.7.2 for finding a generating space for a given free compact abelian group.

1.7.5. Proposition. *Let G be a free compact abelian group, so that $G \cong G_1 \times G_2 \times G_3$ where G_1, G_2, and G_3 satisfy the conditions of Theorem* 1.7.4. *Then $G \cong F_a X$, where*

$$X = \begin{cases} G_1 \amalg I^a \amalg 2^r, & \text{if } r \text{ is infinite,} \\ G_1 \amalg I^a \amalg S_r, & \text{if } r \text{ is finite,} \end{cases}$$

r is the rank of the Sylow p-subgroups of $G_3\hat{}$, S_r is a discrete space with r points, $a = \mathrm{card}\, G_2\hat{}$, and $\amalg$ is the coproduct in Top_0.

Proof. The component of the base point of X is $G_1 \times I^a$ which is homotopy equivalent to G_1. Hence $H^1(X, \mathbb{Z}) \cong H^1(G_1, \mathbb{Z}) \cong G_1\hat{}$.

The group $C(X, \mathbb{R})$ is isomorphic to

$$C(G_1, \mathbb{R}) \oplus C(I^a, \mathbb{R}) \oplus C(E_r, \mathbb{R})$$

where E_r is 2^r or S_r, according as r is infinite or finite, and thus by Proposition 1.4.1

$$\mathrm{card}\, C(X, \mathbb{R}) = \max(2^{\aleph_0}, \mathrm{card}\, G_1\hat{}, a, r).$$

Further, the group $C_{\mathrm{fin}}(X, \mathbb{Z}(p^\infty))$ has cardinality $\max(w(X/\mathrm{conn}), \aleph_0)$ and since X/conn is homeomorphic to E_r, $w(X/\mathrm{conn}) = r$. Hence $\mathrm{card}_{\mathrm{fin}}(X, \mathbb{Z}(p^\infty)) = \max(r, \aleph_0) = m$, say, and so $C_{\mathrm{fin}}(X, \mathbb{Z}(p^\infty)) \cong \mathbb{Z}(p^\infty)^{(m)}$. So by Remark 1.2.5(iii) and Proposition 1.5.1(ii), $F_a X \cong G_1 \times G_2 \times G_3$, as required. □

For connected compact abelian groups it is easier to recognize the free compact abelian groups. The following is an easy corollary of Theorem 1.7.4.

1.7.6. Corollary. *A compact connected abelian group G is a free compact abelian group if and only if the maximal divisible subgroup D of $G^\wedge$ has at least continuum cardinality and if the reduced quotient $G^\wedge/D$ has no larger cardinality than D (that is, if the index of D is less than or equal to its order).*

1.7.7. Corollary. *The identity component of a free compact abelian group is a free compact abelian group.*

1.7.8. Corollary. *The quotient group, H, of a free compact abelian group, G, modulo the closure of its torsion subgroup is a free compact abelian group.*

Proof. As G is a free compact abelian group it is isomorphic to $G_1 \times G_2 \times G_3$, where G_1, G_2 and G_3 satisfy the conditions of Corollary 1.7.4. Therefore $H^\wedge \cong \mathrm{div}(G^\wedge)$ and $\mathrm{div}(G^\wedge)$ contains both $G_2{}^\wedge$ and $G_3{}^\wedge$. Clearly then the conditions of Corollary 1.7.4 are satisfied for H to be a free compact abelian group. $\square$

1.7.9. Remark. Observe that by Theorem 1.7.2 a (nontrivial) totally disconnected group is never a free compact abelian group. In particular $F_a X/(F_a X)_0$ is not a free compact abelian group. (It may however, be a free compact zero-dimensional abelian group.)

We conclude this Section with an example.

1.7.10. Example. By Corollary 1.7.6 a compact connected torsion free abelian group G is a free compact abelian group if and only if $w(G) \geq 2^{\aleph_0}$. (This is so since $w(G) = \mathrm{card}\, G^\wedge$ and G torsion free implies $G^\wedge$ is divisible.) In this case $G^\wedge$ is a divisible torsion free group and so is isomorphic to $\mathbb{Q}^{(r)}$, where $r \geq 2^{\aleph_0}$. Thus $G \cong (\mathbb{Q}^\wedge)^r$, for some $r \geq 2^{\aleph_0}$. So the following statements are equivalent for compact connected torsion free abelian groups:

(i) G is a free compact abelian group, and

(ii) $G \cong (\mathbb{Q}^\wedge)^r$, $r \geq 2^{\aleph_0}$.

One might well ask on what space is $(\mathbb{Q}^\wedge)^r$ a free compact abelian group? By Theorem 1.7.2, $(\mathbb{Q}^\wedge)^r$ is isomorphic to the free compact abelian group on the space $|(\mathbb{Q}^\wedge)^r|$; that is $F_a|(\mathbb{Q}^\wedge)^r| \cong (\mathbb{Q}^\wedge)^r$.

2. Projectivity and freeness of compact abelian groups

2.1. Projection resolutions

In Free Compact Groups II we shall show that the identity component of the centre of a free compact group is projective, but not necessarily a free compact abelian group. In this section we compare the concepts of freeness and projectivity in the category of compact abelian groups.

We recall the definition:

2.1.1. Definitions. An object P in a category is said to be *projective* if for any epic $e: A \to B$ and any morphism $f: P \to B$ there is a morphism $F: P \to A$ such that $eF = f$. An object is said to be *injective* in the category if it is projective in the opposite category.

As a consequence of the fact that in the category AB of abelian groups the injectives are precisely the divisible groups and of duality we immediately have the following observation:

2.1.2. Remark. A compact abelian group G is projective (in KAB) if and only if $G\hat{}$ is divisible; that is, if and only if G is torsion free. Thus projectivity in KAB is the same as torsion freeness.

2.1.3. Proposition. (i) *Let G be a projective compact connected abelian group such that $w(G) \geq 2^{\aleph_0}$. Then $G \cong (\mathbb{Q}\hat{})^{w(G)}$, and is a free compact abelian group.*

(ii) *A free compact abelian group F_aX is projective if and only if $H^1(X, \mathbb{Z})$ is divisible.*

(iii) *For any free compact abelian group F_aX, the compact group $F_aX/\overline{\mathrm{tor}(F_aX)}$ is projective.*

Proof. These follow immediately from Example 1.7.10, Theorem 1.5.4, and Corollary 1.7.8 (and its proof). □

2.1.4. Definition. An injective morphism $e: M \to M_1$ between abelian groups M and M_1 is said to be *essential* if the injectivity of any composition fe with any morphism $f: M_1 \to A$ implies the injectivity of f.

A surjective morphism $p: G_1 \to G$ between abelian groups G_1 and G is said to be *coessential* if the surjectivity of any composition pf with any morphism $f: H \to G_1$ implies the surjectivity of f. Equivalently, if H is a subgroup of G_1 such that $p(H) = G$, then $H = G_1$.

Let M be an arbitrary abelian group. Then there exists an injective resolution

$$0 \to M \to M_1 \to M_2 \to 0 \tag{11}$$

where M_2 is a torsion group and $M \to M_1$ is essential.

The resolution (1) is not unique in general. We shall presently discuss the cases in which it is. For the moment let us draw the following conclusion by duality:

2.1.5. Proposition. *Every compact abelian group has a projective resolution*

$$0 \to G_2 \to G_1 \to G \to 0 \tag{12}$$

in which G_2 is zero-dimensional and the quotient map $p: G_1 \to G$ is coessential.

For any abelian group M the tensor product $\mathbb{Q}\otimes M$ over $\mathbb{Z}$ is well defined. There is a canonical morphism $d: M\to\mathbb{Q}\otimes M$ given by $d(m)=1\otimes m$. The group $\mathbb{Q}\otimes M$ is torsion free and divisible and indeed a rational vector space; the cokernel, coker $d=K$, is a torsion group and the kernel, ker $d=T$, is the torsion subgroup of M, so that we have an exact sequence

$$0\to T\to M\xrightarrow{d}\mathbb{Q}\otimes M\to K\to 0. \tag{13}$$

If M is torsion free (that is, $T=0$) then (13) is an injective resolution of M and, in fact, in this case, it is unique and functorial on the category of torsion free groups. The functor $M\mapsto\mathbb{Q}\otimes M$ is a left reflection from the category of abelian groups into the category of torsion free divisible abelian groups.

We observe that $\mathbb{Q}$ is the injective limit of the direct system

$$\mathbb{Z}\xrightarrow{2}\mathbb{Z}\xrightarrow{3}\mathbb{Z}\xrightarrow{4}\mathbb{Z}\to\cdots.$$

Tensoring with M preserves direct limits, and $\mathbb{Z}\otimes M$ is naturally isomorphic to M under the map $m\mapsto 1\otimes m$. Hence we have

$$\mathbb{Q}\otimes M=\lim(M\xrightarrow{2}M\xrightarrow{3}M\xrightarrow{4}M\cdots), \tag{14}$$

and the map d corresponds under this isomorphism to the colimit map from the first cofactor M of the direct system to the limit. We dualize and obtain the following result:

2.1.6. Proposition. (i) *Let G be a compact abelian group and define*

$$PG=\lim(G\xleftarrow{2}G\xleftarrow{3}G\xleftarrow{4}G\leftarrow\cdots). \tag{15}$$

Then PG is a projective connected compact abelian group and the functor P is a right reflection of the category KAB *of compact abelian groups into the category of connected compact projective abelian groups. The back adjunction $P: PG\to G$ is the limit map from PG to the first factor G of the inverse system.*

(ii) *PG can be calculated as $PG=(\mathbb{Q}\otimes G\hat{})\hat{}$, and p dually corresponds to $d: G\hat{}\to\mathbb{Q}\otimes G\hat{}$. If the index of the torsion subgroup of $G\hat{}$ has at least the cardinality of the continuum, then PG is a free compact abelian group.*

(iii) *There is a functorial exact sequence*

$$0\to\Delta G\to PG\xrightarrow{p_G}G\to G/G_0\to 0 \tag{16}$$

where $\Delta G=\ker p_G$ and G/G_0 are zero-dimensional groups. Furthermore we have

$$(\Delta G)\hat{}=\operatorname{coker} d,\qquad d:\ G\hat{}\to\mathbb{Q}\otimes G\hat{}. \tag{17}$$

2.2. Characteristic sequences

2.2.1. Definition. The exact sequence (16) is called the *characteristic sequence of G* and $p_G: PG\to G$ (and sometimes also the group PG itself) is called the *projective cover of G.*

2.2.2. Remark. It is interesting that p_G parallels in many respects the exponential function. This is more evident if we compare (16) with another standard exact sequence for a compact abelian group G:

$$0 \to \pi_1(G) \to L(G) \xrightarrow{\exp} G \to \pi_0(G) \to 0 \tag{18}$$

where $\pi_1(G) = \mathrm{Hom}(G\hat{}, \mathbb{Z})$, $L(G) = \mathrm{Hom}(\mathbb{R}, G)$, $\exp X = X(1)$, and $\pi_0(G) =$ group of arc components $= \mathrm{Ext}(G\hat{}, \mathbb{Z})$. Here $L(G)$ is a real vector space which in many ways deserves the name of Lie algebra of G. The cokernel of p_G is the group of components, while the cokernel of exp is the group of arc components. On the other hand, the domain of p_G is a compact connected group which is torsion free and algebraically a vector space over $\mathbb{Q}$ while the domain $L(G)$ of exp is a locally convex real vector space in the topology of pointwise convergence. In some, perhaps a bit remote, way we can also say that ΔG is a compact analogue of the fundamental group $\pi_1(G)$ of G: indeed we have

$$\Delta G \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z} \otimes G\hat{}, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathrm{Hom}(G\hat{}, \mathbb{R}/\mathbb{Z})) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, G),$$

while $\pi_1(G) \cong \mathrm{Hom}(G\hat{}, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{R}/\mathbb{Z}, G)$, where the last Hom-group, $\mathrm{Hom}(\mathbb{R}/\mathbb{Z}, G)$, refers to morphisms in KAB, while all others refer to morphisms in AB.

At any rate, both exact sequences (16) and (18) are basic for the theory of compact abelian groups. The group ΔG measures how far G_0 is away from being projective, the group $\pi_1(G)$ measures how far G_0 is from being simply connected. The group G/G_0 measures how far G is away from being connected, and the group $\pi_0(G)$ measures how far G is away from being arcwise connected.

If M is an abelian group and $D = \mathrm{div}\, M$, then $M \cong D \oplus M/D$ and an injective resolution is simply obtained by taking an injective resolution of M/D and directly adding the summand D. This is particularly simple if the torsion subgroup of M is divisible, that is if M/D is torsion free. Thus we have:

If M is an abelian group with a torsion subgroup which is contained in the maximal divisible subgroup D of M then an injective resolution of M is obtained as

$$\begin{aligned} &0 \to M \cong D \oplus M/D \to D \oplus (\mathbb{Q} \otimes M/D) \to \mathrm{coker}\, d_{M/D} \to 0, \\ &d_{M/D}\colon\ M/D \to \mathbb{Q} \otimes M/D. \end{aligned} \tag{19}$$

In view of Theorem 1.5.4 this applies in particular to free compact abelian groups and yields the following result:

2.2.3. Proposition. *Let X be any pointed space. Then there is a projective resolution*

$$0 \to \Delta F_a X \to G_1 \to F_a X \to 0 \tag{20}$$

with $(\Delta F_a X)\hat{} = \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z})$. If X is a paracompact Hausdorff space then there *is an exact sequence*

$$0 \to \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Q}/\mathbb{Z}) \to \operatorname{tor} H^2(X, \mathbb{Z}) \to 0, \tag{21}$$

which splits but not naturally.

Proof. By Remark 1.2.5 we have $(F_a X)\hat{} \cong C(X, \mathbb{R})/C(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})$ and the first factor is divisible. Hence we have an injective resolution.

$$0 \to (F_a X)\hat{} \to (C(X, \mathbb{R})/C(X, \mathbb{Z})) \oplus (\mathbb{Q} \otimes H^1(X, \mathbb{Z}))$$
$$\to \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z}) \to 0.$$

Moreover, the last term is, up to a natural isomorphism, the cokernel of the morphism $d_{C(X,\mathbb{T})}: C(X, \mathbb{T}) \to \mathbb{Q} \otimes C(X, \mathbb{T})$. In the light of duality, this proves (20). If X is paracompact and Hausdorff, then we can apply the universal coefficient theorem to prove $\mathbb{Q} \otimes H^n(X, \mathbb{Z}) \cong H^n(X, \mathbb{Q})$ for all n. And

$$0 \to \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Q}/\mathbb{Z}) \to \operatorname{tor} H^2(X, \mathbb{Z}) \to 0.$$

This proves the last assertion. □

2.2.4 Theorem. *For any pointed space X with βX having at least two points, the characteristic sequences of $(F_a X)_0$ and $F_a X$ are as follows:*

(i) $$0 \to \Delta(F_a X)_0 \to P(F_a X)_0 \xrightarrow{p(F_a X)_0} (F_a X)_0 \to 0,$$

(ii) $$0 \to \Delta F_a X \to P F_a X \xrightarrow{p F_a X} F_a X \to F_a X/(F_a X)_0 \to 0$$

with $\Delta(F_a X)_0 \cong \Delta F_a X \cong \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z})\hat{}$, *and*

$$F_a X/(F_a X)_0 = \prod_{p \text{ prime}} \mathbb{Z}_p^{w(\beta X/\text{conn})}.$$

Further, the groups $P(F_a X)_0$ and $PF_a X$ *are isomorphic as compact abelian groups.*

Proof. If we specialize (16) we obtain (i) and (ii) and from Proposition 2.2.3 we know that $\Delta(F_a X)\hat{} \cong \mathbb{Q}/\mathbb{Z} \otimes H^1(X, \mathbb{Z})$ (cf. Definition 2.2.4). We claim that $\Delta(F_a X)_0\hat{}$ is isomorphic to the same group. By (17), $\Delta(F_a X)_0\hat{}$ is the cokernel of the map $(F_a X)_0\hat{} \to \mathbb{Q} \otimes (F_a X)_0\hat{}$, which by duality is tantamount to $C(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q} \otimes (C(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}))$, since $C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z})$ is the torsion subgroup of $C(X, \mathbb{T})$. However, in the diagram

$$\begin{array}{ccc} C(X, \mathbb{T}) & \longrightarrow & \mathbb{Q} \otimes C(X, \mathbb{T}) \\ \downarrow & & \downarrow \cong \\ C(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q} \otimes (C(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z})) \end{array}$$

the left vertical map is surjective, and the right vertical map is an isomorphism, whence the cokernel of the bottom horizontal map is isomorphic to the cokernel of the top horizontal map. This proves the claim.

The structure of $F_aX/(F_aX)_0$ was given in Corollary 1.5.3.

From Proposition 2.1.5(ii) we have $(PF_aX)\hat{} = \mathbb{Q}\otimes C(X,\mathbb{T})$ which is naturally isomorphic to $\mathbb{Q}\otimes(C,\mathbb{T})/C_{\text{fin}}(X,\mathbb{Q}/\mathbb{Z})) = (P(F_aX)_0)\hat{}$. □

We observe that the group PF_aX is a free compact abelian group and has as character group, the group $\mathbb{Q}\otimes C(X,\mathbb{T}) = C(X,\mathbb{R})\oplus(\mathbb{Q}\otimes H^1(X,\mathbb{Z}))$ by (7). By Theorem 1.5.4 the rank of $\mathbb{Q}\otimes H^1(X,\mathbb{Z})$ is dominated by that of $C(X,\mathbb{R})$. It follows that $\mathbb{Q}\otimes C(X,\mathbb{T})$ is isomorphic to $C(X,\mathbb{R})$, although not naturally. Thus we have:

2.2.5. Remark. In Theorem 2.2.4, we have $PF_aX \cong P(F_aX)_0 \cong C(X,\mathbb{R})\hat{}$. In particular, these groups are free compact abelian groups.

We have the following special cases:

2.2.6. Corollary. *For any connected pointed space X with βX having at least two points, the characteristic sequence is a projective resolution and reads*

$$0 \to \mathbb{Q}/\mathbb{Z}\otimes H^1(X,\mathbb{Z})\hat{} \to PF_aX \to F_aX \to 0 \tag{22}$$

with $PF_aX \cong C(X,\mathbb{R})\hat{}$. The group PF_aX is isomorphic to the factor group of F_aX modulo the closure of its torsion subgroup.

Proof. We have everything except for the last statement. Now $(PF_aX)\hat{} \cong C(X,\mathbb{R})$, and this is isomorphic to the maximal divisible subgroup of $(F_aX)\hat{}$ by Remark 1.5.5. Then as a consequence of duality PF_aX is isomorphic to the largest torsion free quotient of F_aX. □

This corollary shows that for connected paracompact Hausdorff spaces X the cohomology group $\mathbb{Q}/\mathbb{Z}\otimes H^1(X,\mathbb{Z})$ measures how far F_aX is away from being projective.

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CORRECTION

FREE COMPACT GROUPS I: FREE COMPACT ABELIAN GROUPS

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W. Wistar Comfort and Jan van Mill alerted us to the fact that one of the cardinality arguments in the above paper is incorrect. We should like to rectify the situation and are grateful to Prof. Comfort and Prof. van Mill for drawing our attention to this error. (See notably their article [2]).

Statement (i) of Proposition 1.4.1. on p. 46 should read as follows:

(i) card $C(X, \mathbb{R}) = \operatorname{card}(X, \mathbb{T}) = \max(2^{\aleph_0}, w(X)^{\aleph_0}) = w(F_a X)$.

Here the cardinal $w(X)^{\aleph_0}$ replaces the cardinal $w(X)$ in the original statement. As soon as $w(X)$ is infinite, formula (i) simply reads

$$\operatorname{card} C(X, \mathbb{R}) = C(X, \mathbb{T}) = w(X)^{\aleph_0} = w(F_a X).$$

The function $\aleph \mapsto \aleph^{\aleph_0}$ is increasing and idempotent. It retracts the class of cardinals onto the subclass of those cardinals $\aleph$ which satisfy $\aleph = \aleph^{\aleph_0}$. If $w(X)$ takes one of those cardinals as value, then the original version agrees with the correct one, otherwise not.

In the proof of Proposition 1.4.1, *Step 8* should indeed read:

Step 8. If X is compact and zero dimensional, then card $C(X, \mathbb{R}) = \max(2^{\aleph_0}, w(X)^{\aleph_0})$.

The error in the proof of *Step 8* actually occurs in line 5 of p. 48, where it is correctly shown that card $C(X, \mathbb{K}) = w(X)^{\aleph_0}$ for infinite X, but where the exponent is erroneously omitted in the subsequent formula. *Steps 9 and 10* are to be corrected accordingly.

As a consequence, the reader should implement the following corrigenda:

(1) Replace $w(\beta X)$ by $w(\beta X)^{\aleph_0}$ on p. 49 in lines 16 and 11 from below, on p. 50 in lines 4 from above and 4 from below, on p. 51 in lines 14 and 5 from below, on p. 52 in lines 7, 9, 16, 24, 25, 26 from above, and on p. 53 in lines 4 and 9 from above.

(2) Replace $w(X)$ by $w(X)^{\aleph_0}$ and $w(Y)$ by $w(Y)^{\aleph_0}$ in lines 10 and 11 of the Abstract p. 41, on p. 54 in lines 16, 8 and 7 from below and on p. 55 in lines 11, 15 and 16 from above. (Notice that line 18 of p. 55 remains intact!)
(3) Replace card$(G_2\hat{})$ by card$(G_2\hat{})^{\aleph_0}$ on p. 57 in line 9 from above.
(4) In Definition 1.7.1 on p. 55 we add in condition (iii) the statement

$$\ldots \text{and } (\text{rank div } A)^{\aleph_0} = \text{rank div } A.$$

(5) The last line of the statement of Theorem 1.7.2 on p. 56 we replace by "*and* $a \geqslant \max\{2^{\aleph_0}, b, \dim K\}$, *where* $a^{\aleph_0} = a$."
(6) Line 3 of p. 58 should begin "cardinality $a = a^{\aleph_0}$ and . . . ".
(7) Line 6 of the Abstract on p. 41 should begin "that $a^{\aleph_0} = a \geqslant \max\{2^{\aleph_0}, b, \dim K\} \ldots$".
(8) In line 19 on p. 58 and in line 13 of p. 59, we replace $w(G) \geqslant 2^{\aleph_0}$ by $w(G)^{\aleph_0} = w(G) \geqslant 2^{\aleph_0}$. In lines 21, 22 and 25 on p. 58, we replace $r \geqslant 2^{\aleph_0}$ by $r^{\aleph_0} = r \geqslant 2^{\aleph_0}$.

All of these corrigenda simply amount to saying that *the weight of a free compact abelian group is always a cardinal a satisfying* $a = a^{\aleph_0}$.

The relation card $C(X, \mathbb{R}) = w(\beta X)^{\aleph_0}$ for infinite spaces X was established in [1]. Further references will be found in [2].

Further corrigenda: On p. 56, in line 11 from below, replace $\mathbb{Q}^a$ by $\mathbb{Q}^{(a)}$, in lines 10 and 9 from below replace K by $K\hat{}$.

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