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FREE ABELIAN TOPOLOGICAL GROUPS

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The notions of 'free topological group' and 'free abelian topological group' were introduced in 1941 by A.A. Markov [95], with full details appearing in his paper "On free topological groups" [96], which appeared in 1945. He began that paper as follows (but in Russian):

"The primary goal of the following investigation is to study the normality of topological groups, which arises in a natural way in the general theory of topological groups. Indeed as L. Pontryagin proved in an unpublished letter to A. Weil, every (Hausdorff) topological group is completely regular. Since next to complete regularity, normality is the most interesting property of separation of spaces, it is natural to ask whether every (Hausdorff) topological group is a normal space."

Markov answered the question in the negative by showing that: every completely regular Hausdorff space can be embedded as a closed subspace of a Hausdorff topological group - namely the Markov free topological group, $FM(X)$, on X. By letting X be any completely regular Hausdorff space which is not a normal space, we then have that $FM(X)$ is a non-normal topological group (since a closed subspace of a normal space is normal).

Markov's proof was about 50 pages long and relied on 48 lemmas. In 1948, M.I. Graev [47] proved Markov's result, that X completely regular Hausdorff implies $FM(X)$ is Hausdorff and X is closed in $FM(X)$, in 11 pages (with no lemmas!). While today there are other proofs of this fact which are much shorter than Graev's, Graev's approach remains of interest as it gives information about free topological groups, free abelian topological groups and free products of topological groups not obtained otherwise.

Definition Let X be a topological space with distinguished point e, and $FG(X)$ a topological group which contains X as a subspace and has e as its identity element. Then $FG(X)$ is said to be the Graev free topological group on X if for any continuous map ϕ of X into any topological group H such that $\phi(e)$ is the identity element of H, there exists a unique continuous homomorphism $\Phi : FG(X) \to H$ with $\Phi|X = \phi$.

Warning: Our topological groups need not be Hausdorff.

Let Top Gps be the category of all topological groups with the arrows being continuous homomorphisms and CR_0 be the category of all completely regular (not necessarily Hausdorff) pointed topological spaces with the arrows being basepoint preserving continuous maps.

Then the forgetful functor S : Top Gps $\to CR_0$, by the adjoint functor theorem, has a left adjoint $T : CR_0 \to$ Top Gps. Then (as every completely regular space can be embedded as a subspace of a topological group) $FG(X) = T(X)$.

Note that $FG(X)$ is independent of the choice of basepoint in X; that is, if $e,f \in X$ then

$$FG(X_e) \cong FG(X_f).$$

This is proved in Graev [47].

Replacing CR_0 by CR = the category of completely regular spaces and continuous maps we obtain the Markov free topological group, $FM(X)$, on X. Replacing Top Gps by the category of abelian topological groups, AB Top Gps, we obtain the Graev free abelian topological group, $AG(X)$, on X, and the Markov free abelian topological group on X, $AM(X)$.

Of course there is a simple relationship between Markov and Graev free topological groups:

Proposition [111] $FM(X) \cong FG(X) \amalg \mathbb{Z}$ and

$$AM(X) \cong AG(X) \times \mathbb{Z}$$

where $\amalg$ is the coproduct in Top Gps and $\mathbb{Z}$ is the discrete group of integers.

Proposition [47] $FG(X)$ is algebraically the free group on the set $X\setminus\{e\}$; $AG(X)$ is algebraically the free abelian group on the set $X\setminus\{e\}$; $FM(X)$ is the free group on the set X; $AM(X)$ is the free abelian group on the set X.

Proof It suffices to prove the first of these. To do this we show that any map ϕ of $X\setminus\{e\}$ into any group H can be extended uniquely to a homomorphism

$\Phi : FG(X) \rightarrow H$. Put the indiscrete topology on H (that is, the only open sets are $\emptyset$ and H) - so H is now a topological group. Define $\phi(e)$ to be the identity of H. So $\phi : X \rightarrow H$ is continuous (any map into an indiscrete space is continuous) and therefore there exists a unique (continuous) homomorphism $\Phi : FG(X) \rightarrow H$ which extends ϕ. Therefore $FG(X)$ is algebraically free on $X \setminus \{e\}$. //

<u>Proposition [47]</u> $FG(X)$ has the finest group topology on the underlying group which induces the given topology on X.

<u>Proof</u> Let $G(X)$ be a topological group which has the same underlying group as $FG(X)$ and which induces the given topology on X. (e is the identity element of $G(X)$.)

$$FG(X) \xrightarrow[\Phi]{} G(X)$$

$$X \xrightarrow[\phi]{} X$$

The identity map $\phi : X(\subseteq FG(X)) \rightarrow X \subseteq G(X)$ is continuous and $\phi(e)$ = the identity of $G(X)$. So there exists a continuous homomorphism $\Phi : FG(X) \rightarrow G(X)$ which extends ϕ. Clearly Φ is the identity map, and as it is continuous we deduce that the topology of $FG(X)$ is finer than the topology of $G(X)$. //

This characterizes the free topology!

<u>THEOREM [47]</u> If X is any completely regular Hausdorff space then $FG(X)$, $FM(X)$, $AG(X)$ and $AM(X)$ are Hausdorff.

Firstly we show that X Hausdorff (and completely regular) implies $AM(X)$ is Hausdorff.

<u>Proof</u> It suffices to show that for each $g \in AM(X)$, $g \neq 0$, there exists a continuous mapping Φ_g of $AM(X)$ into a Hausdorff group such that $\Phi_g(g) \neq 0$.

Let g have reduced representation

$$g = m_1 x + m_2 x_2 + \ldots + m_k x_k$$

where $x_i \in X$, for each i. As X is completely regular and Hausdorff there exists a continuous map $\phi : X \rightarrow \mathbb{R}$, the additive group of reals with the usual topology, such that

$$\phi(x_i) = \begin{cases} 1, & \text{if } i = 1 \\ 0, & \text{if } i \neq 1 \end{cases}$$

Then ϕ extends to a continuous homomorphism $\Phi_g : AM(X) \to \mathbb{R}$. Clearly $\Phi_g(g) = m_1 \neq 0$. //

The non-abelian case is not so easy. One method is to replace $\mathbb{R}$ by various unitary groups. (See [60]).

However I think the nicest proof uses:

Theorem [23] Every topological space X can be embedded in a contractible topological space X^*. Further X Hausdorff implies X^* Hausdorff; X completely regular implies X^* completely regular; there is a natural isomorphism $(X \times Y)^* \cong X^* \times Y^*$; so "any algebraic structure" of X is inherited by X^*.

Corollary [59] Every Hausdorff topological group is a topological subgroup of a path connected Hausdorff group.

Now let us prove that X completely regular Hausdorff implies $FM(X)$ Hausdorff.

Proof Once again it suffices to show that if $g \in FM(X)$, $g \neq 1$, then there exists a continuous mapping Φ_g of $FM(X)$ into a Hausdorff group such that $\Phi_g(g) \neq 1$.

Let g have reduced representation $g = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$. Let the distinct x_i's be $y_1, y_2, \dots, y_\ell$. Consider the free group $F\langle y_1, \dots, y_\ell \rangle$ freely generated by the set $\{y_1, \dots, y_\ell\}$. Put the discrete topology on this group. According to the previous theorem this can be embedded in a path connected Hausdorff group F^*. As X is completely regular and Hausdorff and F^* is path connected, there exists a continuous map $\phi : X \to F^*$ such that $\phi(y_i) = y_i$, for $i = 1, 2, \dots, \ell$. Then ϕ can be extended to a continuous homomorphism $\Phi_g : FM(X) \to F^*$. Clearly $\Phi_g(g) \neq 1$, as required. //

Having settled the question of Hausdorffness we ask whether the topology on $FM(X)$ is locally compact. Unfortunately free topologies are locally compact only in trivial cases.

Theorem [31] If F is a locally compact Hausdorff topological group which is algebraically a free group or a free abelian group then it has the discrete topology.

This result is extended to free products in [123] and amalgamated free products in [2,3,4].

In the case of free abelian topological groups we are able to give an adequate description of the topology. Indeed Graev's construction of a topology on the free abelian group actually yields the free topology. This is not so in the non-abelian case. (See [126,35,88,131]).

Graev's construction [47] Let X be a completely regular Hausdorff space. Then the topology on X is determined by the family $\{\rho_i : i \in I\}$ of all continuous pseudometrics. Extend each ρ_i to the free abelian group F on X as follows:

If $w_1, w_2 \in X$ and

$$w_1 = a_1 + a_2 + \dots + a_k \quad \text{and}$$

$$w_2 = b_1 + b_2 + \dots + b_k$$

where these are not necessarily reduced representations, with $a_i, b_i \in X \cup (-X) \cup \{0\}$, we define

$$\rho_i'(w_1,w_2) = \inf\left\{ \sum_{j=1}^{k} \rho_i(a_j,b_j) \right\}$$

where the infimum is taken over all representations of w_1 and w_2 of equal length, and we put

$$\rho_i(-x_i,0) = \rho(x_i,0) = 1 \quad \text{and} \quad \rho(x_i,-x_j) = 2,$$

for $x_i, x_j \in X$.

Graev proceeds to show that the infimum is actually achieved.

The family $\{\rho_i' : i \in I\}$ of pseudometrics on F defines a Hausdorff group topology on F which induces the given topology on X. What is more, this topology is the free topology [131,121].

Warning: To obtain the free topology one must extend all continuous pseudometrics. For example, if X is a metric space with metric d the free topology will not be obtained by extending d alone since the free topology is not metrizable (unless it is discrete). //

So in the abelian case we can describe the free topology via Graev's construction. As remarked earlier, this is not so in the non-abelian case. A very useful tool is the family of k_ω-spaces. These are useful for both the abelian and non-abelian free groups.

Definition A Hausdorff space X is said to be a k_ω-space [37,38,75,93] if $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is compact, ($X_n \subseteq X_{n+1}$ for all n), and a subset of X is closed iff $S \cap X_n$ is compact for all n. e.g. $\mathbb{R} = \bigcup_{1}^{\infty} [-n,n]$. Every countable CW-complex and every connected locally compact Hausdorff topological group is a k_ω-space.

k_ω-spaces have very nice properties, e.g.

(i) X and Y $k_\omega \Rightarrow X \times Y$ k_ω,

(ii) If A is a compact subset of the k_ω-space $X = \cup X_n$, then $A \subseteq X_n$, for some n.

Theorem [93] If $X = \cup X_n$ is a k_ω-space, then

$$FG(X) = \cup\, gp_n(X_n) \text{ is a } k_\omega\text{-space, where } gp_n(X_n) = (X_n \cup X_n^{-1})^n$$

(i.e. all words of reduced length $\leq n$). Similar results are true for $FM(X)$, $AM(X)$ and $AG(X)$.

Indeed as every Hausdorff group topology on the underlying group of $FG(X)$ yields the same topology on $gp_n(X_n)$ we see that this k_ω-decomposition is a characterization of the free topology.

Theorem X countable CW-complex implies $FG(X)$ is a countable CW-complex.

We now turn to subgroup questions where our knowledge is far from complete or even satisfactory. Unlike the case for free groups and free abelian groups, a subgroup of a free topological group is not necessarily a free topological group, and a subgroup of a free abelian topological group is not necessarily a free abelian topological group. For example it is shown in [69] that in $FM[0,1]$ the subgroup $gp(0,1)$, generated by the open unit interval is not a free topological group. As a first step towards this, we use the following:

Theorem [69] Every k_ω-topological group is a complete topological group.

Therefore $gp(0,1)$ is not free on any k_ω-space, since it is not closed in $FM[0,1]$. (In particular it is not free on $(0,1)$.) But it is also shown in [69] that $gp(0,1)$ is not a free topological group on any topological space.

So then we ask if a closed subgroup of a free topological group is necessarily a free topological group. The answer is still in the negative. One approach to this is as follows:

Theorem [19] If for a topological space X, $FG(X)$ has open components then $\pi_0(FG(X))$ is a free group. If $AG(X)$ has open components then $\pi_0(AG(X))$ is a free abelian group.

[Recall that π_o is the canonical functor from the category of topological spaces to the category of totally disconnected spaces. It is observed in [19] that if the components in FG(X) are open, then $\pi_o(FG(X)) = FG(\pi_o(X))$.]

Using the above theorem we can easily construct an example of a closed subgroup of a free topological group which is not a free topological group.

Example [19] Consider FG[0,1] and let the identity element be 0. Let H be the subgroup $gp\{\{1\},x^2 : x \in [0,1]\}$, generated by the element 1 and the set of squares of elements in [0,1]. Then $\pi_o(H)$ is evidently $\mathbb{Z}_2$ - which is not a free group. Therefore H is not a free topological group. It is easy to check that H is closed in FG[0,1].

Indeed more is shown in [19], namely that a closed subgroup of a free abelian topological group need not be even projective. For further discussion of projective topological groups see [135,93].

In the positive direction there are some nice results:

Theorem [22] If H is an open (and hence closed) subgroup of FM(X) (or FG(X)), where X is any k_ω-space, then H is a free topological group.

The proof uses topological groupoids and also applies to free products of topological groups. A more general result for free products is obtained in [133].

This brings us to the first of our open questions:

OPEN QUESTION 1 If X is a k_ω-space and H is an open subgroup of AG(X), is H necessarily a free abelian topological group?

A related question is suggested by the next theorem.

Theorem [93] If H is any k_ω-group and ϕ is the canonical continuous homomorphism of FG(H) onto H, then the kernel of ϕ is a free topological group.

OPEN QUESTION 2 If H is an abelian k_ω-group (or even an abelian compact Hausdorff group) and ϕ is the canonical continuous homomorphism of AG(H) onto H, is the kernel of ϕ a free abelian topological group?

[This was claimed to be true in [135,93] but the proofs are faulty.]

We now move to analysis of subgroups of FM[0,1] and AM[0,1]. Our knowledge about the abelian case is much less than about the non-abelian case.

Proposition [130] $FM([0,1]^2)$ is topological isomorphic to a subgroup of $FM[0,1]$. (We write this as $FM([0,1]^2) \leqslant FM[0,1]$.)

The required embedding is obtained by extending the mapping $[0,1]^2 \rightarrow FM[0,1]$ given by $(x,y) \rightarrow xyx$. One has to verify that $gp\{xyx : x,y \in [0,1]\}$ is a free topological group on $\{xyx : x,y \in [0,1]\}$. This is routine using the following key proposition.

Proposition 93 If X is a k_ω-space and Y is a closed free algebraic basis in $FM(X)$ for $gp(Y)$, and Y is "regularly situated" with respect to X (e.g. $Y \subseteq X$) then $gp(Y) = FM(Y)$.

We can then proceed to see how rich a topological group $FM[0,1]$ is.

Theorem [93] Let X be a compact Hausdorff space. Then $FM(X) \leqslant FM[0,1]$ if and only if X is metrizable and finite-dimensional.

OPEN QUESTION 3 If X is a finite-dimensional metrizable k_ω-space, is $FM(X)$ necessarily topologically isomorphic to a subgroup of $FM[0,1]$.

For abelian groups, we are very much in the dark; for example:

OPEN QUESTION 4 Is $AM([0,1]^2) \leqslant AM[0,1]$?

Conjecture Let X be a compact metrizable space. Then $AM(X) \leqslant AM[0,1]$ if and only if X is 1-dimensional or 0-dimensional.

There are, however, some recent positive results about subgroups of $AM[0,1]$.

Theorem [83] $AM(S^1) \leqslant AM[0,1]$.

Extending to higher dimensions is made possible by the following theorem:

Theorem [83] If $AM(X) \leqslant AM(Y)$, where X and Y are k_ω-spaces, and the embedding is "orderly", then $AM(\Sigma X) \leqslant AM(\Sigma Y)$ where Σ is the reduced suspension.

Corollary [83] $AM(S^n) \leqslant AM([0,1]^n)$.

Extending from compact Hausdorff results to k_ω-results, we mention the following:

Theorem [133] $FM(0,1) \leqslant FM[0,1]$.

This can be proved using the following somewhat surprising result:

Theorem [82] If X is a k_ω-space and Y is a closed subspace of $FM(X)$, then $FM(Y) \leqslant FM(X)$.

Warning The above theorem does not say that $gp(Y) = FM(Y)$, but rather that there exists a homeomorphic copy of Y in $FM(X)$ which generates $FM(Y)$.

So to prove $FM(0,1) \leqslant FM[0,1]$ it suffices to find in $FM[0,1]$ any closed homeomorphic copy of $(0,1)$.

OPEN QUESTION 5 If X is a k_ω-space and Y is a closed subspace of $AM(X)$, is $AM(Y) \leqslant AM(X)$?

One positive result we do have is:

Theorem [81] $AM(0,1) \leqslant AM[0,1]$.

Conclusion and Apology I have tried in this talk to say enough to give the flavour of free abelian topological groups. Of course I could not cover all known results. For example, I have not had time to deal with the following very interesting question: What can be said about the relation between X and Y if $AM(X) \cong AM(Y)$?

Graev [47] showed for example that if X is compact metrizable then so is Y and $\dim(X) = \dim(Y)$. Much more has been done on this by the Russian school and in particular A.V. Arhangelskii and his students. Indeed Vladimir Pestov, in a private communication claims to have generalized Graev's result stated above and those of C. Joiner and L.G. Zambahidze by showing that if X and Y are completely regular spaces with $FM(X) \cong FM(Y)$, then $\dim(X) = \dim(Y)$. A considerable amount of information on this question can also be obtained by examining the Bohr compactification of free abelian topological groups; that is, the free compact abelian groups $AFK(X)$. The structure of free compact abelian groups is fully exposed in [66] and, in particular, the problem of characterizing which X and Y have topologically isomorphic free compact abelian groups is solved. This leads to new information about free abelian topological groups since $AM(X) \cong AM(Y)$ implies $AFK(X) \cong AFK(Y)$.

Finally I mention that while I have concentrated here on the structure of free topological groups and free abelian topological groups, similar questions can be investigated in any variety of topological groups. For a survey of this topic see [119].

I have endeavoured to compensate for my not discussing various results by including a very substantial bibliography.

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