NUMERICAL GEOMETRY . . .

NOT NUMERICAL TOPOLOGY

Joan Cleary and Sidney A. Morris

(received 18 August, 1983)

1. Introduction

In 1964 O. Gross [2] proved the following little known, but very interesting result.

Theorem A *Let* (X,d) *be a compact connected metric space. Then there is a unique constant* $a(X,d)$ *with the property that, given any finite collection of points* $x_1, x_2, \ldots, x_n \in X$, *there is a point* $y \in X$ *such that* $\frac{1}{n} = \sum_{i=1}^{n} d(x_i,y) = a(X,d)$.

In 1981 Wolfgang Stadje [4] proved a stronger version of the above theorem in which he replaced (X,d) by any compact connected Hausdorff space and the metric by any symmetric continuous function $f : X \times X \to \mathbb{R}$.

The work by Gross and Stadje generated much interest, and has resulted in papers by Morris and Nickolas [3], Yost [8], Strantzen [5], Szekeres and Szekeres [6], and Wilson [7]. A survey of the known results is given in Cleary [1] .

If X is a compact connected Hausdorff space and f is any continuous symmetric function $f : X \times X \to \mathbb{R}$ which is not identically zero, then we define $D(X,f)$ to be the real number $\sup\{|f(x,y)| : x\ y \in X\}$. If the space X has at least two points, then the *dispersion constant* $m(X,f)$ (also called the *magic number*) is defined to be $a(X,f)/D(X,f)$, where $a(X,f)$ is the number in Stadje's version of Theorem A which corresponds to $a(X,d)$.

Math. Chronicle 13(1984) 47-57

If d is a metric, then $D(X,d)$ is the diameter of the space X and it is readily seen that $\frac{1}{2} \le m(X,d) \le 1$. Gross [2] shows that $m(X,d) < 1$. For the general case, clearly $-1 \le m(X,f) \le 1$.

Our main result is Theorem 1 :

Theorem 1 *Let X be any compact connected metrizable space. Then for each real number m such that $\frac{1}{2} \le m < 1$, there is a metric d on X such that $m(X,d) = m$.*

This result is particularly interesting as it shows that $m(X,d)$ does not depend on the topology, but rather only on the metric. For this reason this topic has become known as "numerical geometry" rather than, say, "numerical topology".

2. The main results

In this section we will prove Theorem 1. But firstly we prove some lemmas which are needed in the proof.

Notation Let (X,d) be a compact connected metric space of diameter one. For each non-negative real number λ , define

$$\rho_\lambda : (X,d) \times (X,d) \to \mathbb{R} \quad \text{by} \quad \rho_\lambda(x,y) = \frac{(\lambda+1)d(x,y)}{\lambda d(x,y)+1} .$$

Note that (X,ρ_λ) is a metric space and is homeomorphic to (X,d) for each λ .

The proof of the first lemma is straightforward and so is left to the reader.

Lemma 1 *Let (X,d) be a compact connected metric space of diameter 1 . Let ε be any real number such that $0 < \varepsilon < 1$. If $\lambda \ge \frac{1}{\varepsilon^2}$ and $x,y \in X$ are such that $d(x,y) \ge \varepsilon$ then $1 \ge \rho_\lambda(x,y) > 1-\varepsilon$.*

Lemma 2 *Let (X,d) be a compact connected metric space of diameter 1 . Then there is a sequence $x_0,x_1,\ldots,x_n,\ldots$ of points of X satisfying $d(x_i,x_j) \ge 2^{-i}$, whenever $i > j$.*

Proof For $z \in X$ and r any positive real number, let $B(z,r)$ denote the open ball with centre z and radius r. Let a and b be elements of X such that $d(a,b) = 1$. We shall prove by mathematical induction that there exists a sequence of points $x_0, x_1, x_2, \ldots, x_n, \ldots$ in X such that $d(x_i, x_j) \geq 2^{-i}$, whenever $i > j$.

Let $x_0 = a$. Suppose that $x_0, x_1, \ldots, x_n$ are elements of X such that $d(x_i, x_j) \geq 2^{-i}$, whenever $i > j$ and $i,j \in \{0,1,2, \ldots n\}$. Clearly

$$B(x_i, 2^{-n-1}) \cap B(x_j, 2^{-n-1}) = \emptyset, \quad \text{for all } i \neq j .$$

Therefore, as X is connected $X \neq \bigcup_{i=0}^{n} B(x_i, 2^{-n-1})$.

Let $x_{n+1} \in X \setminus \bigcup_{i=0}^{n} B(x_i, 2^{-n-1})$. Clearly $d(x_{n+1}, x_j) \geq 2^{-(n+1)}$ for all $j \in \{1,2, \ldots, n\}$ and so $d(x_i, x_j) \geq 2^{-i}$, for all $i,j \in \{1,2, \ldots, n+1\}$. This completes the induction and the proof of the Lemma.

Lemma 3 *Let* (X,d) *be a compact connected metric space of diameter* 1. *If* n *is any positive integer and* $\lambda \geq 4^{n+1}$ *then* $a(X, \rho_\lambda) \geq 1 - \frac{2}{n}$.

Proof By Lemma 2, there are points $x_1, x_2, \ldots, x_n$ in X such that

$$d(x_i, x_j) \geq 2^{-n}, \quad \text{for } i \neq j .$$

By Theorem A, there exists a point y in X with the property that

$$\frac{1}{n} \sum_{i=1}^{n} \rho_\lambda(x_i, y) = a(X, \rho_\lambda) .$$

Then $d(x_i, y) \geq 2^{-(n+1)}$, for all but at most one $i \in \{1,2, \ldots, n\}$. Since $\lambda \geq 4^{n+1}$, Lemma 1 implies that $\rho_\lambda(x_i, y) > 1 - 2^{-n-1}$ for all but at most one $i \in \{1,2, \ldots, n\}$.

$$a(X,\rho_\lambda) = \frac{1}{n}\sum_{i=1}^{n} \rho_\lambda(x_i,y) > \frac{n-1}{n}\,(1-2^{-n-1})$$

$$= (1-\frac{1}{n})\,(1-\frac{1}{2^{n+1}}) \geq (1-\frac{1}{n})^2 \geq 1-\frac{2}{n}\,.$$

We also need the following proposition of Yost [8] .

Proposition 1 *Let* X *be a compact connected Hausdorff space and let* $f : X \times X \to \mathbb{R}$ *be a continuous symmetric function. Fix* $a,b \in \mathbb{R}$ *with* $a \leq b$. *If, given any* $x_1,x_2,\dots x_n \in X$, *there is a point* $y \in X$ *such that*

$$a \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i,y) \leq b$$

then $a \leq a(X,f) \leq b$.

Lemma 4 *Let* (X,d) *be a compact connected metric space of diameter* 1 . *Then* $a(X,\rho_\lambda)$ *is a continuous function of the parameter* λ .

Proof Clearly, $\rho_\lambda(x,y) = \dfrac{(\lambda+1)d(x,y)}{\lambda d(x,y)+1}$ is a continuous function of λ ; that is, given any $\varepsilon > 0$, for each $x,y \in X$ there is a $\delta > 0$ such that if $|\lambda_1 - \lambda_2| \leq \delta$ then $|\rho_{\lambda_1}(x,y) - \rho_{\lambda_2}(x,y)| \leq \varepsilon$. Indeed, since X is compact, there is a $\delta > 0$ such that for all $x,y \in X$, if $|\lambda_1 - \lambda_2| \leq \delta$ then $|\rho_{\lambda_1}(x,y) - \rho_{\lambda_2}(x,y)| \leq \varepsilon$. So for all $x,y \in X$, $\rho_{\lambda_1}(x,y) - \varepsilon \leq \rho_{\lambda_2}(x,y) \leq \rho_{\lambda_1}(x,y) + \varepsilon$. Now choose any $x_1,x_2,\dots,x_n \in X$. Then, by Theorem A, there is a point $y \in X$ such that

$$\frac{1}{n}\sum_{i=1}^{n} \rho_\lambda\,(x_i,y) = a(X,\rho_{\lambda_1})\ ,$$

and so, $a(X,\rho_{\lambda_1}) - \varepsilon \leq \frac{1}{n}\sum_{i=1}^{n} \rho_{\lambda_2}(x_i,y) \leq a(X,\rho_{\lambda_1}) + \varepsilon$. It follows from Proposition 1 that $a(X,\rho_{\lambda_1}) - \varepsilon \leq a(X,\rho_{\lambda_2}) \leq a(X,\rho_{\lambda_1}) + \varepsilon$.

So, given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|\lambda_1 - \lambda_2| \le \delta$ then $|a(X,\rho_{\lambda_1}) - a(X,\rho_{\lambda_2})| \le \varepsilon$. Hence $a(X,\rho_\lambda)$ is a continuous function of λ.

We now prove a special case of Theorem 1.

Lemma 5 *Let* (X,d) *be a compact connected metric space with diameter* 1. *Then there is a metric* ρ *on* X *such that* (X,ρ) *is homeomorphic to* (X,d) *and* $a(X,\rho) = \frac{1}{2}$.

Proof Let a and b be points in X such that $d(a,b) = 1$. As $d : \{a\} \times X \to \mathbb{R}$ is continuous and X is connected, there is a point $c \in X$ such that $d(a,c) = \frac{1}{2}$. So $d(b,c) \ge \frac{1}{2}$. Define $\rho : X \times X \to \mathbb{R}$ by $\rho(c,x) = \min\{\frac{1}{2}, \rho(x,y)\}$, for all $x \in X$ and $\rho(x,y) = \min\{d(x,y), \rho(c,x) + \rho(c,y)\}$, for all $x,y \in X$. It can be easily verified that ρ is a metric on X.

We now show that $\frac{1}{2}d(x,y) \le \rho(x,y) \le d(x,y)$, for all $x,y \in X$, from which it follows that (X,ρ) is homeomorphic to (X,d). Suppose that $\rho(x,y) = \rho(x,c) + \rho(y,c)$; that is, $d(x,y) \ge \rho(x,y) = \rho(x,c) + \rho(y,c)$. If $\rho(x,c) = d(x,c)$ and $\rho(y,c) = d(y,c)$ then $\rho(x,c) + \rho(y,c) = d(x,c) + d(y,c) \ge d(x,y)$, and so $\rho(x,y) = d(x,y)$. If $\rho(x,c) \ne d(x,c)$ then $\rho(x,y) \ge \frac{1}{2} \ge \frac{1}{2}d(x,y)$. Similarly, if $\rho(y,c) \ne d(y,c)$, then $\rho(x,y) \ge \frac{1}{2} \ge \frac{1}{2}d(x,y)$. So $\frac{1}{2}d(x,y) \le \rho(x,y) \le d(x,y)$ for each $x,y \in X$. Hence (X,ρ) is homeomorphic to (X,d).

Now $\rho(a,b) = \min\{d(a,b), \rho(a,c) + \rho(b,c)\} = 1$, and $\rho(x,y) \le d(x,y)$ for all $x,y \in X$, so $D(X,\rho) = 1$.

Since $\rho(x,c) \le \frac{1}{2}$ for all $x \in X$, $a(X,\rho) \le \frac{1}{2}$. Hence $a(X,\rho)$ must be equal to $\frac{1}{2}$.

Proof of Theorem 1 Since X is metrizable, there is a metric, d, on X with $D(X,d) = 1$. Let $a(X,d) = b$. So $b \in [\frac{1}{2}, 1)$.

We shall prove the theorem in two parts. In (i) we show that for each real number $a \in [\frac{1}{2}, b)$, there is a metric δ_t on X such that

$a(X,\delta_t) = a$, and (X,δ_t) is homeomorphic to (X,d) . In (ii) we prove the analogous result for each $a \in [b,1)$.

(i) By Lemma 5, there is a metric ρ on X such that $a(X,\rho) = \frac{1}{2}$ and (X,ρ) is homeomorphic to (X,d) . For each real number, t , such that $t \in [0,1]$, define the metric δ_t by $\delta_t(x,y) = td(x,y) + (1-t)\rho(x,y)$, where $x,y \in X$. Let a and b be the points of X defined in the proof of Lemma 5 such that $d(a,b) = 1$. Then $\rho(q,b) = 1$. So $\delta_t(a,b) = t.1 + (1-t).1 = 1$. Also $\delta_t(x,y) = td(x,y)+(1-t)\rho(x,y) \leq d(x,y)$ for all $x,y \in X$, since $\rho(x,y) \leq d(x,y)$ for all $x,y \in X$. Therefore $D(X,\delta_t) = 1$.

When $t = 0$ we have $\delta_0(x,y) = \rho(x,y)$. so $a(X,\delta_0) = \frac{1}{2}$. When $t = 1$ we have $\delta_1(x,y) = d(x,y)$ and thus $a(X,\delta_1) = b$. Clearly δ_t is a continuous function of t , and it can easily be shown using Proposition 1 that $a(X,\delta_t)$ is also a continuous function of t . Therefore, as t varies over $[0,1]$, $a(X,\delta_t)$ takes on all values between $\frac{1}{2}$ and b .

(ii) When $\lambda = 0$, $\rho_\lambda(x,y) = \rho_0(x,y) = d(x,y)$ and so $a(X,\rho_0) = b$. By Lemma 2 , as λ tends to infinity , $a(X,\rho_\lambda)$ approaches 1 . Since $a(X,\rho_\lambda)$ is a continuous function of λ , for each real number $a \in [b,1)$, there is a metric ρ_λ on X (with $D(X,\rho_\lambda) = 1$) such that $a(X,\rho_\lambda) = a$ and (X,ρ_λ) is homeomorphic to (X,d) .

Hence, from (i) and (ii) , for any real number a such that $\frac{1}{2} \leq a < 1$, there is a metric d on X such that $a(X,d) = a$. As $D(X,d) = 1$, $a(X,d) = \dot{m}(X,d)$, which completes the proof.

We now tackle the analogous problem when the metric d is replaced by any continuous symmetric function $f : X \times X \to \mathbb{R}$.

Theorem 2 *Let X be a compact connected Hausorff space. Then for each real number $m \in [-1,1]$ there is a continuous symmetric function $f : X \times X \to \mathbb{R}$ such that $m(X,f) = m$.*

Proof Firstly, X is a compact Hausdorff space and therefore is completely regular. So there is a continuous pseudo-metric ρ on X such that $D(X,\rho) = 1$. Since X is compact, there are points $a,b \in X$ such that $\rho(a,b) = 1$. As $\rho : \{a\} \times X \to \mathbb{R}$ is continuous and X is connected there is a point $c \in X$ such that $\rho(a,c) = \frac{1}{2}$. So $\rho(b,c) \geq \frac{1}{2}$. Now define the function $g : X \times X \to \mathbb{R}$ by $g(x,y) = \rho(x,c)\rho(y,c)\rho(x,y)$ for $x,y \in X$. Then g is continuous and symmetric. Now $g(a,b) \geq \frac{1}{4}$ and so $D(X,g) > 0$. So we can define the continuous symmetric function $f_0 : X \times X \to \mathbb{R}$ by $f_0(x,y) = \dfrac{g(x,y)}{D(X,g)}$. Hence $D(X,f_0) = 1$. Now choose any $x_1, x_2, \ldots, x_n \in X$. Then we have $f\,(x_i,c) = 0$ for each $i = 1, 2, \ldots, n$, and so

$$\frac{1}{n}\sum_{i=1}^{n} f_0\,(x_i,c) = 0 .$$

So, for any given collection of points $x_1, x_2, \ldots, x_n \in X$, the point $c \in X$ has the property that

$$\frac{1}{n}\sum_{i=1}^{n} f_0\,(x_i,c) = 0 .$$

So the number zero satisfies the conditions of Theorem A. As $a(X,f_0)$ is unique, $a(X,f_0)$ must equal zero.

Let $f_1 : X \times X \to \mathbb{R}$ be the continuous symmetric function defined by $f_1(x,y) = 1$ for all $x,y \in X$. Then clearly $a(X,f_1) = m(X,f_1) = 1$.

For each $t \in [0,1]$ define the continuous symmetric function $p_t : X \times X \to \mathbb{R}$ by $p_t(x,y) = t\, f_1(x,y) + (1-t)\, f_0(x,y)$ for $x,y \in X$. Since X is compact there are points $x_1, x_2 \in X$ such that $f_0(x_1,x_2) = 1$. Also $p_t(x,y) \leq 1$ for all $x,y \in X$. Therefore $D(X,p_t) = 1$.

When $t = 0$ we have $p_0(x,y) = f_0(x,y)$, and so $a(X,p_0) = 0$. When $t = 1$ we have $p_1(x,y) = f_1(x,y)$. So $a(X,p_1) = 1$. Using Proposition 1 it can be readily shown that $a(X,p_t)$ is a continuous function of the parameter t. Therefore, as t varies over the interval $[0,1]$, $a(X,p_t)$ takes on all values between zero and one inclusive.

Now, if $a \in [-1,0]$ then $-a \in [0,1]$. So, from the above argument, there is a continuous symmetric function $h : X \times X \to \mathbb{R}$ such that $a(X,h) = -a$. Define the continuous symmetric function $f : X \times X \to \mathbb{R}$ by $f(x,y) = -h(x,y)$ for all $x,y \in X$. Then clearly $a(X,f) = a$.

So we have proved that for each number $a \in [-1,1]$ there is a continuous symmetric function $f : X \times X \to \mathbb{R}$ such that $a(X,f) = a$. Also, each of these functions satisfies $D(X,f) = 1$. So $a(X,f) = m(X,f)$. This proves the theorem.

3. Extension of Yost's results

In this section we extend some results of Yost [8]. We begin with a result of Yost but give a new more elementary proof. The result is then generalized below.

Proposition 2 *Let (X,d_1) be a compact connected metric space with $D(X,d_1) = D > 0$ and $m(X,d_1) > \frac{1}{2}$. Let (I,d_2) be a closed interval of length D with the Euclidean metric. Then there is a wedge $X \vee I$ which has a metric ρ such that $\rho|X = d_1$, $\rho|I = d_2$ and $m(X \vee I,\rho) = \frac{1}{2}$.*

Proof Since X is compact there are points $a,b \in X$ such that $d_1(a,b) = D$. Identify $a \in X$ with an endpoint of I. Then $Y = X \vee I$ is a compact connected space with topology τ.

Define $\rho : Y \times Y \to \mathbb{R}$ by $$\rho(x,y) = \begin{cases} d_1(x,y) & \text{, for } x,y \in X \\ d_2(x,y) & \text{, for } x,y \in I \\ d_1(x,a)+d_2(a,y) & \text{, for } x \in X,\ y \in I. \end{cases}$$

Then ρ on Y which induces the topology τ. Now $\rho(a,y) \leq D$ for all $y \in Y$ and so from Proposition 1, it follows that $a(Y,\rho) \leq D$. Clearly the diameter of Y is $2D$. Therefore $m(Y,\rho) \leq \frac{1}{2}$. Hence $m(Y,\rho) = \frac{1}{2}$.

This can be generalized as follows:

Corollary *Let (X,d_1) be a compact connected metric space with $D(X,d_1) = D > 0$ and $m(X,d_1) = b > \frac{1}{2}$. Then for each $m \in [\frac{1}{2}, b]$ there is a wedge $X \vee I$, where (I,d_2) is a closed interval with the Euclidean metric, with metric δ such that $\delta|X = d_1$, $\delta|Y = d_2$ and $m(X \vee I,\delta) = m$.*

Proof Let (Y,ρ) be the compact connected metric space in the proof of Proposition 2 with $m(Y,\rho) = \frac{1}{2}$. For each $\lambda \in [0,1]$ let (I_λ,d_2) be a closed interval of length λD with the Euclidean metric, and let $Y_\lambda = X \vee I_\lambda$ be obtained by identifying an endpoint of I_λ with $a \in X$ where $d_1(a,b) = D$ for some $b \in X$. For each $\lambda \in [0,1]$ define $\delta : Y_\lambda \times Y_\lambda \to \mathbb{R}$ by

$$\delta(x,y) = \begin{cases} d_1(x,y) & , \text{ for } x,y \in X \\ d_2(x,y) & , \text{ for } x,y \in I_\lambda \\ d_1(x,a) + d_2(a,y) & , \text{ for } x \in X,\ y \in I_\lambda \end{cases}$$

Then for each $\lambda \in [0,1]$, δ is a metric on Y_λ. When $\lambda = 0$, $m(Y_\lambda,\delta) = m(X,d_1) = b$. When $\lambda = 1$, $m(Y_\lambda,\delta) = m(Y,\rho) = \frac{1}{2}$.

Using Proposition 1, it can be easily shown that the mapping $\lambda \to m(Y_\lambda,\delta)$ is continuous. Hence for each $m \in [\frac{1}{2}, b]$ there is a compact connected metric space (Y_λ,δ) with $m(Y_\lambda,\delta) = m$.

Theorem 3 *Let E be any normed space with d being the metric determined by the norm. Let X be any compact connected subset of E with $m(X,d) = b$. Then for each $m \in [\frac{1}{2}, b]$ there exists* *a closed interval* $I = [e,f] \subset E$ *such that (i)* $d|I$ *is the Euclidean metric, (ii)* $X \cap I = \{e\}$, *and (iii)* $m(X \cup I,d) = m$.

Proof Without loss of generality, let $D(X,d) = 1$, $0,e \in X$ and $d(0,e) = 1$. For each $\lambda \in [0,1]$, let I_λ be the closed interval $\{x : x = e + \lambda ke : 0 \le k \le 1\}$. Clearly $I_\lambda \cap X = \{e\}$ for each $\lambda \in [0,1]$, since if it were not the case $D(X,d)$ would be greater than one. When $\lambda = 0$, $I_0 \cup X = X$ and so $m(X \cup I_0,d) = b$. When $\lambda = 1$, $d(0,2e) = 2$ and $d(x,y) \le d(x,e) + d(e,y) \le 2$ for each $x,y \in X \cup I_1$. Therefore $D(X \cup I_1,d) = 2$. Now $d(e,x) \le 1$ for all $x \in X \cup I_1$, and so $a(X \cup I_1,d) \le 1$. It then follows that $m(X \cup I_1,d) = \frac{1}{2}$. Again, using Proposition 1 it is easily shown that $m(X \cup I_\lambda,d)$ varies continuously with λ. Hence, for each $m \in [\frac{1}{2}, b]$ there is a closed interval $I = [e,f]$ with $m(X \cup I ,d) = m$; $X \cap I = \{e\}$ and when restricted to I, the metric d is the Euclidean metric.

Remark 1 Since every metric space can be isometrically embedded in a normed vector space, Corollary 1 can be deduced from the above theorem.

Remark 2 Yost [8] shows that if E is any finite dimensional normed vector space then the set $\{m(X,d) : X$ is a compact connected subset of E and d is the metric induced by the norm$\}$ has an upper bound strictly less than one. He calls the supremum of this set $k(E)$. Yost then proceeds to show that the set $\{m(X,d) : X$ is a compact connected subset of $E\}$ is the whole interval $[\frac{1}{2},k(E)]$. This result is a trivial consequence of Theorem 3.

Acknowledgement The authors wish to express their gratitude to David Yost for helpful comments.

REFERENCES

[1] J. Cleary, *Numerical Geometry*, Honours project, La Trobe University, 1982.

[2] O. Gross, *The rendezvous value of a metric space*, Ann. of Math. Studies 52 (Advances in Game Theory), 49-52.

[3] S.A. Morris and P. Nickolas, *On the average distance property of compact connected metric spaces*, Archiv der Math 40(1983), 459-463.

[4] W. Stadje, *A property of compact connected spaces*, Archiv der Math 36 (1981), 275-280.

[5] J. Strantzen, *An average distance result in Euclidean n-space*, Bull. Austral. Math. Soc. 26 (1982), 321-330.

[6] E. Szekeres and G. Szekeres, *The average distance theorem for compact convex regions*, Preprint 1982.

[7] D. Wilson, *A game with squared distance as pay off*, to appear.

[8] D. Yost, *Average distance in compact connected spaces*, Bull. Austral. Math. Soc. 26 (1982), 331-342.

Added in Proof David Yost has pointed out that part (i) of the proof of Theorem 1 can be omitted simply by applying Lemma 5, at the beginning of the proof, to find a metric ρ such that $a(X,\rho) = \frac{1}{2}$ and hence $b = \frac{1}{2}$.

La Trobe University
Bundoora
Victoria, 3083
AUSTRALIA