AN ELEMENTARY PROOF THAT THE HILBERT CUBE IS COMPACT

SIDNEY A. MORRIS
Department of Pure Mathematics, La Trobe University, Bundoora, Victoria, Australia 3083

The Hilbert cube is defined to be the product of a countably infinite family $\{I_n : n = 1, 2, \ldots\}$ of homeomorphic copies of the closed unit interval $[0,1]$. That it is compact follows, of course, from Tychonoff's Theorem which says that any product of compact spaces is compact. Our proof of the compactness of the Hilbert Cube is, however, of a very different flavor from the usual proofs of Tychonoff's Theorem. We call the proof elementary because it can be easily understood by the average student of topology and also because (when written out with some care) it avoids the Axiom of Choice. The approach is as follows.

Define the Cantor space, G, in the usual way so that it is seen to be a closed subspace of $[0,1]$ and so is compact. Next, observe that each point in G has a unique ternary representation $\Sigma_{n=1}^{\infty} a_n/3^n$ with $a_n \in \{0,2\}$, for each n. For each positive integer n, define A_n to be the discrete space $\{0,2\}$. Then it is easily verified that the mapping ϕ from the product space $\prod_{n=1}^{\infty} A_n$ onto G given by

$$\phi((a_1, a_2, \ldots, a_n, \ldots)) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

is a homeomorphism. (See [**1**, p. 104].)

Now we state two lemmas, the proofs of which are quite straightforward.

LEMMA 1. *For each positive integer n, let G_n be homeomorphic to the Cantor space G. Then the product space $\prod_{n=1}^{\infty} G_n$ is homeomorphic to G.*

Lemma 1 follows from the fact that a countable product of a countable product of copies of $\{0,2\}$ is homeomorphic to a countable product of copies of $\{0,2\}$. (There is nothing special about $\{0,2\}$ in this statement, it is equally true for any topological space: [**1**, p. 102].)

LEMMA 2. *There exists a continuous mapping ψ of the Cantor space onto* $[0,1]$.

Lemma 2 is obtained by observing that the mapping $\theta : \prod_{n=1}^{\infty} A_n \to [0,1]$ given by

$$\theta((a_1, a_2, \ldots, a_n, \ldots)) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}$$

is continuous [1, p. 104] and surjective. The required map $\psi = \theta \circ \phi^{-1}$, where ϕ is as above.

We can now prove the main result.

THEOREM. *The Hilbert cube is compact.*

Proof. With G_n and I_n as above, Lemma 2 implies that there is a continuous mapping ψ_n of G_n onto I_n, for each positive integer n. Thus there is a mapping Ψ of $\prod_{n=1}^{\infty} G_n$ into $\prod_{n=1}^{\infty} I_n$ given by

$$\Psi((g_1, g_2, \ldots, g_n, \ldots)) = (\psi_1(g_1), \psi_2(g_2), \ldots, \psi_n(g_n), \ldots)$$

where $g_n \in G_n$, for each n. It is easily verified that Ψ is continuous and surjective. Caution is required when proving surjectivity, so as to avoid the Axiom of Choice. Let $(x_1, x_2, \ldots, x_n, \ldots) \in \prod_{n=1}^{\infty} I_n$. Observing that (i) each G_n has an ordering inherited from $[0,1]$, (ii) each G_n is a closed subset of $[0,1]$, and (iii) each $\psi_n : G_n \to I_n$ is surjective, we can let g_n be the *smallest* element of G_n such that $\psi_n(g_n) = x_n$. Then

$$\Psi((g_1, g_2, \ldots, g_n, \ldots)) = (x_1, x_2, \ldots, x_n, \ldots).$$

Then Lemma 1 says that $\prod_{n=1}^{\infty} G_n$ is homeomorphic to the Cantor space, and so it is compact. Thus the Hilbert cube $\prod_{n=1}^{\infty} I_n$ is a continuous image of a compact space and hence is compact.

The above approach has several advantages. The Cantor space has been introduced not as an oddity, but rather as a tool. Also, it is now but a small step (see [1, p. 104]) to show that the n-cube $[0,1] \times \cdots \times [0,1]$ is a continuous image of $[0,1]$—so space filling curves appear quite naturally. Another advantage, and this is quite subjective, is that it is a good idea to spend some time on countable products before moving on to uncountable products. Finally, one can proceed to prove that every compact metric space is homeomorphic to a subspace of the Hilbert cube, from which one can then deduce that any countable product of compact metric spaces is compact and also that every compact metric space is a continuous image of (a closed subspace of) the Cantor space. (See [2].)

References

1. J. Dugundji, Topology, Allyn and Bacon, Boston, 1968.
2. S. A. Morris, Topology Without Pain, (to appear).