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# METRIZABILITY OF FREE PRODUCTS OF TOPOLOGICAL GROUPS

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### Abstract

It is shown that the free product, $G * H$, of topological groups $G$ and $H$ is a metrizable topological group if and only if $G$, $H$ and $G * H$ have the discrete topology. It is also shown that if $A$ is a closed central subgroup of topological groups $G$ and $H$, then the free product of $G$ and $H$ with $A$ amalgamated is metrizable precisely when $A$ is open in $G$ and in $H$.


## § 1. Introduction and Preliminaries

**Definition.** Let $A$ be a common subgroup of topological groups $G$ and $H$. The topological group $G*_A H$ is said to be *the free product of the topological groups G and H with amalgamated subgroup A* if

(i) $G$ and $H$ are topological subgroups of $G*_A H$,

(ii) $G \cup H$ generates $G*_A H$ algebraically, and

(iii) every pair $\phi_1, \phi_2$ of continuous homomorphisms of $G$ and $H$ respectively into any topological group $D$, which agree on $A$, extend to a continuous homomorphism of $G*_A H$ into $D$.

It is readily seen that if $G*_A H$ exists, then it is unique. Existence, itself, is not so easy to deal with. When $A = \{e\}$, $G*_A H$ is simply *the free product of the topological groups G and H,* and a standard category theory argument yields that $G * H$ exists for all $G$ and $H$. In this paper we shall concern ourselves only with the case that $A$ is a closed central subgroup of Hausdorff groups $G$ and $H$. For this case, existence of $G*_A H$ was proved in Khan and Morris [10]. Other cases have been handled in Katz and Morris [6, 7, 8].

Having established the existence of $G*_A H$, the main question, then, is to describe the topology of $G*_A H$ in terms of $G$, $H$ and $A$. Even when $A = \{e\}$, Graev [4] found that it was a non-trivial task to establish that $G$ and $H$ Hausdorff implies $G * H$ is Hausdorff. Extending Graev's argument, Khan and Morris [10] showed that if $A$ is a closed central subgroup of Hausdorff groups $G$ and $H$, then $G*_A H$ is Hausdorff. In the proofs of our two theorems, we use the details of the Graev proof [4] and the Khan-Morris proof [10].

The free product $G * H$ has been extensively examined and it is known, for example, that $G * H$ is not a locally compact group or a complete metric group unless it is discrete. On the other hand, if $G$ and $H$ are $k_\omega$-groups than $G * H$ is a $k_\omega$-group [5, 17].

**Definition.** A Hausdorff space $X$ is said to be a *$k_\omega$-space* and have *$k_\omega$-decomposition* $X = \cup X_n$, if $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$, each $X_n$ is compact and a subset $A$ of $X$ is closed in $X$ if and only if $A \cap X_n$ is compact for all $n$.

As examples of $k_\omega$-spaces we have all compact Hausdorff spaces, all countable $CW$-complexes and all locally compact $\sigma$-compact Hausdorff spaces [2].

We record the following result:

**Theorem A [12].** Let $X = \cup X_n$ be a $k_\omega$-space and $G$ a Hausdorff group which is generated algebraically by $X$ and is such that the topology of $G$ is the finest group topology which induces the same topology on $X$. Then $G$ is a $k_\omega$-space with $k_\omega$-decomposition $G = \cup gp_n(X_n)$, where $gp_n(X_n)$ is the product in $G$ of $n$ copies of $X \cup X^{-1} \cup \{e\}$, and $e$ denotes the identity element.

So if $G$ and $H$ are $k_\omega$-groups then, knowing that $G * H$ is Hausdorff and that the topology of $G * H$ is the finest topology on the underlying group which induces the given topology on $G \cup H$, we see that $G * H$ is a $k_\omega$-group.

**Definition.** Let $X$ be a topological space with distinguished point $e$. Then the topological group $F(X)$ is said to be the *(Graev) free topological group on $X$* if

(i) $X$ is a subspace of $F(X)$ with $e$ the identity element of $F(X)$,

(ii) $X$ generates $F(X)$ algebraically, and

(iii) for every continuous map $\phi$ of $X$ into any topological group $G$ such that $\phi(e)$ is the identity of $G$, there exists a continuous homomorphism of $F(X)$ into $G$ which extends $\phi$.

If in the above definition we replace "group" everywhere by "abelian group" then we have the definition of the (Graev) *free abelian topological group on $X$*, denoted by $FA(X)$.

It is known [14] that $F(X)$ and $FA(X)$ exist and are unique (up to isomorphism) if and only if $X$ is completely regular. As a corollary of Theorem $A$ we have the result which we shall use later:

**Theorem B.** If $X$ is a $k_\omega$-space then $F(X)$ and $FA(X)$ are $k_\omega$-spaces.

If $G$ and $H$ are topological groups then the underlying group of $G * H$ is the algebraic free product of the underlying groups [17, 13]. The kernel of the canonical homomorphism of $G * H$ onto the direct product $G \times H$ is called the *cartesian subgroup* of $G * H$. It clearly contains all the elements $g^{-1}h^{-1}gh$, $g \in G$, $h \in H$. Indeed it is a free group on the set $\{g^{-1}h^{-1}gh : g \in G, h \in H\} \setminus \{e\}$. (See [17] and [13].) We denote the set $\{g^{-1}h^{-1}gh : g \in G, h \in H\}$ by $[G, H]$.

## § 2. Results

**Theorem 1.** Let $G$ and $H$ be any topological groups with $G \neq \{e\}$ and $H \neq \{e\}$. Then $G * H$ is metrizable if and only if $G$ and $H$, and hence also $G * H$, have the discrete topology.

**Proof.** If $G$ and $H$ are discrete, then $G * H$ is discrete and so is metrizable.

Conversely, assume $G * H$ is metrizable but not discrete. Then the cartesian subgroup $gp[G, H]$, with the induced topology $\tau$ is metrizable. Let $\tau_1$ be the topology induced on $[G, H]$ by $\tau$. As the map $(G * H) \times (G * H) \to G * H$ given by $(x, y) \to x^{-1} y^{-1} xy$, $x \in G * H$, $y \in G * H$ is continuous, the restriction of this map to $G \times H$ is continuous; that is,

$$G \times H \xrightarrow{\theta} ([G, H], \tau_1), \text{ where } \theta(g, h) = g^{-1} h^{-1} gh$$

is continuous. The kernel of the latter map is $G \times \{e\} \cup \{e\} \times H$, so there is an induced continuous map $\phi$ from the smash product $G \wedge H \to ([G, H], \tau_1)$. (Recall that the smash product $G \wedge H$ is the quotient topological space obtained by identifying all the elements of the set $G \times \{e\} \cup \{e\} \times H$. Then $\phi$ extends to a continuous homomorphism $\phi$ from the free topological group $F(G \wedge H)$ on $G \wedge H$ into $(gp[G, H], \tau)$.

Graev [4] proves that if $G$ and $H$ are Hausdorff topological groups then $G * H$ is Hausdorff. He does this by putting a Hausdorff topology $\tau_2$ on the set $[G, H]$, extending this topology to a locally invariant topology $\tau_3$ on $gp[G, H]$, and then topologizing $G * H$ as the product $G \times H \times (gp[G, H], \tau_3)$. (That this gives a topological group topology is not easy to show.)

Graev's topology, $\tau_3$, is described in [3], [16], [10] and [4] and is clearly the finest locally invariant topology on $gp[G, H]$, which will induce the topology $\tau_2$ on $[G, H]$. From this it follows that if we factor out the commutator subgroup then the resultant group $(gp[G, H], \tau_3)/\delta(gp[G, H])$ is the free abelian topological group $FA([G, H], \tau_2)$ on the space $([G, H], \tau_2)$.

Returning to Graev's proof that $G * H$ is Hausdorff, it is completed by observing that the topology of $G * H$ must be finer than that of $G \times H \times (gp[G, H], \tau_3)$, and so is Hausdorff. In particular, then, $(gp[G, H], \tau)$ has a finer topology than $(gp[G, H], \tau_3)$. So we have a sequence

$$F(G \wedge H) \xrightarrow{\Phi} (gp[G, H], \tau) \xrightarrow{\Gamma} (gp[G, H], \tau_3)$$

where $\Phi$ and $\Gamma$ are continuous algebraic isomorphisms.

Now factor out the commutator subgroup of each term, and note that $F(G \wedge H)/\delta(F(G \wedge H)) = FA(G \wedge H)$. So we have

$$FA(G \wedge H) \xrightarrow{\Phi_1} (gp[G, H], \tau)/\delta(gp[G, H]) \xrightarrow{\Gamma_1} FA([G, H], \tau_2)$$

where $\Phi_1$ and $\Gamma_1$ are the induced continuous algebraic isomorphisms.

Now $G \wedge H$ and $([G, H], \tau_2)$ each contain a copy of $G$ and $\Gamma_1 \Phi_1$ maps the copy in $G \wedge H$ onto the copy in $([G, H], \tau_2)$. As $G * H$ was assumed to be non-discrete, then either $G$ or $H$ is non-discrete. Without loss of generality, assume it is $G$. So $G$ is non-discrete and metrizable and so contains a convergent sequence $\{g_n\}_{n=1}^{\infty}$ such that $g_n \to e$ and $g_n \neq e$, for any $n$. Let $S$ be the set $(\bigcup_{n=1}^{\infty} \{g_n\}) \cup \{e\}$. Then $S$ is a compact subspace of $G$. By Theorem 1.10 of [15] the subgroup $A$ of $FA(G \wedge H)$ generated by $S$ is topologically isomorphic to $FA(S)$, and the subgroup $B$ of $FA([G, H], \tau_2)$ generated by $S$ is also topologically isomorphic to $FA(S)$. As $\Gamma_1 \Phi_1 (A) = B$ it follows that $\Phi_1 (A)$ is topologically isomorphic to $FA(S)$.

We have assumed that $G * H$ is metrizable so its subgroup $(gp[G, H], \tau)$ is metrizable. There the quotient group $(gp[G, H], \tau)/\delta(gp[G, H])$ is metrizable, since it is Hausdorff. (It is Hausdorff since $\Gamma_1$ is a continuous one-to-one map of it into the Hausdorff space $FA([G, H], \tau_2)$.) Thus its subgroup $\Phi_1 (A)$ is metrizable. But $\Phi_1 (A)$ is topologically isomorphic to $FA(S)$, and so is a $k_\omega$-space (Theorem B) $A$ $k_\omega$-space which is metrizable is locally compact [2]. So the free abelian group $\Phi_1 (A)$ has a locally compact Hausdorff group topology. By Dudley [1] this implies that the topology is discrete – which is a contradiction, since the topology of $S$ is not discrete. Hence $G * H$ is not metrizable and non-discrete.

As an extension of the above theorem we have

**Theorem 2.** Let $A$ be a common closed central subgroup of topological groups $G$ and $H$ with $A \neq G$ and $A \neq H$. Then the amalgamated free product $G *_A H$ is metrizable if and only if $A$ is an open subgroup of $G$ and of $H$, and both $G$ and $H$ are metrizable.

**Proof.** If $G$ and $H$ are metrizable and $A$ is open in $G$ and in $H$, then by Proposition 4 of [11], $G *_A H$ is homeomorphic to $(G \times_A H) \times D$, where $D$ is a discrete group and $G \times_A H$ is the amalgamated direct product of $G$ and $H$. Then, as $G \times_A H$ is a Hausdorff quotient of $G \times H$, [9], it is metrizable. Hence $G *_A H$ is metrizable.

Conversely, assume that $G *_A H$ is metrizable. Then replacing the Graev construction of the topology of $G * H$ by the Khan-Morris construction [10], an argument analogous to that in Theorem 1 show that the quotient groups $G/A$ and $H/A$ are both discrete.

This implies that $A$ is open in both $G$ and $H$. Of course $G *_A H$ metrizable also implies that $G$ and $H$ are metrizable, and the proof is complete.

**Remark.** It should be noted that Theorem 2 generalizes Theorem 7 of [11], the proof however is quite different. Theorem 7 of [11] says that unless $A$ is open in $G$ and $H$, $G *_A H$ is not a complete metric group. This is proved by showing that if it were, then a certain free group would be a complete metric group – which is impossible unless it is discrete [1]. However as free groups do admit metrizable group topologies the same argument could not have been used to show that $G *_A H$ is not metrizable.

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