Arch. Math., Vol. 40, 459—463 (1983)

0003-889X/83/4005-0459 $ 2.50/0

On the average distance property of compact connected metric spaces

By

SIDNEY A. MORRIS and PETER NICKOLAS

1. Introduction. Stadje [2] proved the following surprising result:

Theorem A. *Let (X, d) be a compact connected metric space. Then there is a uniquely determined constant $a(X, d)$ with the following property: For each positive integer n, and for all $x_1, \ldots, x_n \in X$, there exists $y \in X$ for which*

$$\frac{1}{n} \sum_{i=1}^{n} d(x_i, y) = a(X, d).$$

Indeed Stadje proved that the same result holds for any compact connected space X, with d replaced by any continuous symmetric function $f\colon X \times X \to \mathbb{R}$.

Stadje remarks that "this property of compact connected metric spaces is non-trivial even in the simplest examples", and that he does not know a direct geometrical proof of it. (In 3, we give a simple proof that for $X = [0, 1]$, with the Euclidean metric d, $a(X, d)$ exists and equals $\frac{1}{2}$.) The expression for $a(X, d)$ given by Stadje is

$$a(X, d) = \sup_{\mu} \inf_{\nu} \int_X \int_X d(x, y)\, \mu(dx)\, \nu(dy),$$

when μ and ν run over all Borel probability measures on X. Clearly this formula does not readily allow one to calculate $a(X, d)$ for concrete examples.

In this paper, we give a formula for $a(X, d)$ for a family of "nice" spaces, which allows $a(X, d)$ to be evaluated in a routine manner; this family includes all spheres S^n and all compact connected metric groups.

2. The main result. We shall refer to the number $a(X, d)$ of Theorem A as the *Stadje number* of (X, d).

Theorem 1. *Let (X, d) be a compact connected metric space. If there exists a Borel probability measure μ_0 on X such that the integral $\int_X d(x, y)\, \mu_0(dx)$ is independent of the choice of y in X, then the Stadje number $a(X, d)$ is equal to $\int_X d(x, y)\, \mu_0(dx)$ for any y.*

Proof. Let e be any fixed element of X, and let $\nu \in M^1(X)$, the set of Borel probability measures on X. Then

$$\begin{aligned}\int_X\int_X d(x,y)\,\nu(dy)\,\mu_0(dx) &= \int_X\int_X d(x,y)\,\mu_0(dx)\,\nu(dy) \quad \text{(by Fubini's Theorem)}\\ &= \int_X\int_X d(x,e)\,\mu_0(dx)\,\nu(dy)\\ &\quad \left(\text{as } \int_X d(x,y)\,\mu_0(dx) \text{ is independent of } y\right)\\ &= \left(\int_X d(x,e)\,\mu_0(dx)\right)\int_X \nu(dy)\\ &= \int_X d(x,e)\,\mu_0(dx).\end{aligned}$$

So we have, for any $\nu \in M^1(X)$,

$$\int_X\int_X d(x,y)\,\nu(dy)\,\mu_0(dx) = \int_X\int_X d(x,y)\,\mu_0(dx)\,\nu(dy) = \int_X d(x,e)\,\mu_0(dx). \tag{1}$$

Therefore, for any $\mu \in M^1(X)$,

$$\begin{aligned}\int_X d(x,e)\,\mu_0(dx) &= \int_X\int_X d(x,y)\,\mu(dx)\,\mu_0(dy)\\ &\geqq \inf_{\nu\in M^1(X)} \int_X\int_X d(x,y)\,\mu(dx)\,\nu(dy).\end{aligned}$$

Hence

$$\sup_{\mu\in M^1(X)}\ \inf_{\nu\in M^1(X)} \int_X\int_X d(x,y)\,\mu(dx)\,\nu(dy) \leqq \int_X d(x,e)\,\mu_0(dx).$$

But by (1),

$$\begin{aligned}\int_X d(x,e)\,\mu_0(dx) &= \inf_{\nu}\int_X\int_X d(x,y)\,\mu_0(dx)\,\nu(dy)\\ &\leqq \sup_{\mu}\inf_{\nu}\int_X\int_X d(x,y)\,\mu(dx)\,\nu(dy),\end{aligned}$$

and so

$$a(X,d) = \sup_{\mu}\inf_{\nu}\int_X\int_X d(x,y)\,\mu(dx)\,\nu(dy) = \int_X d(x,e)\,\mu_0(dx).$$

This completes the proof.

In order to show that Theorem 1 can be applied both to spheres and to topological groups we use the following lemma.

Lemma. *Let (X, d) be a compact metric space and let $\mu_0 \in M^1(X)$. If for each pair of points y and e in X there exists an isometry $T\colon X \to X$ such that $T(y) = e$ and $\mu_0(A) = \mu_0(T(A))$ for all Borel sets A, then the hypothesis of Theorem 1 is satisfied; that is, $\int_X d(x,y)\,\mu_0(dx)$ is independent of y.*

Proof. For fixed $y, e \in X$, let T be an isometry as above. Then

$$\begin{aligned}\int_X d(x,y)\,\mu_0(dx) &= \int_X d(T(x),T(y))\,\mu_0(dx) && \text{(as } T \text{ is an isometry)}\\ &= \int_X d(T(x),e)\,\mu_0(dx) && (\text{as } T(y) = e)\\ &= \int_X d(x,e)\,\mu_0(dT^{-1}x) && \text{(as } T \text{ is measurable)}\\ &= \int_X d(x,e)\,\mu_0(dx) && (\text{as } \mu_0 \text{ is invariant under } T),\end{aligned}$$

and the lemma is proved.

Consider the n-sphere, S^n, with metric d inherited from the Euclidean metric on $\mathbb{R}^{n+1}$, and with μ_0 the normalisation of the usual n-dimensional measure on S^n. Then for each y and e in S^n there exists an isometry T as in the above lemma, namely, a suitable rotation. Hence we obtain

Theorem 2. *Let S^n, d, and μ_0 be as above. Then* $a(S^n, d) = \int_{S^n} d(x, y)\, \mu_0(dx)$ *for any* $y \in S^n$.

If (G, d) is any compact connected metric topological group, where d is left-invariant, and if μ_0 is the normalized Haar measure on G, then the required isometries T may be taken to be left translations by elements of G. (Recall that d is called left-invariant if $d(gg_1, gg_2) = d(g_1, g_2)$ for all $g, g_1, g_2 \in G$. Further, every metrizable topological group may be topologized by a left-invariant metric [1, Theorem 8.3].)

Thus we obtain

Theorem 3. *Let G, d, and μ_0 be as above. Then* $a(G, d) = \int_G d(x, y)\, \mu_0(dx)$ *for any* $y \in G$.

3. Examples. Before proceeding to apply the theorems of 2, we give a direct proof that $a(X, d)$ exists when $X = [0, 1]$ and d is the usual metric. Once it is known that $a(X, d)$ exists it is, as Stadje remarks, trivial to show that it must be $\frac{1}{2}$ (in the notation of Theorem A take $n = 2$, $x_1 = 0$, $x_2 = 1$).

If $x_1, \ldots, x_n$ are arbitrary points in $[0, 1]$ consider the continuous function

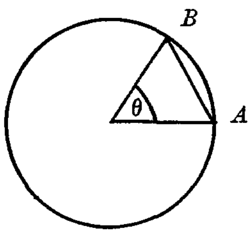
$$h(t) = \frac{1}{n} \sum_{i=1}^{n} |x_i - t|, \quad t \in [0, 1].$$

Clearly

$$h(0) = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \text{while} \quad h(1) = 1 - \frac{1}{n} \sum_{i=1}^{n} x_i = 1 - h(0),$$

so that either $h(0) \leqq \frac{1}{2} \leqq h(1)$ or $h(1) \leqq \frac{1}{2} \leqq h(0)$. The Intermediate Value Theorem then implies that there exists a $y \in [0, 1]$ for which $h(y) = \frac{1}{2}$. Thus $a(X, d)$ exists and equals $\frac{1}{2}$.

The evaluation of $a(S^n, d)$ for any n is clearly, by Theorem 2, reduced to a routine calculation. The simplest such calculation is for S^1, and is given below.



$B = (\frac{1}{2} \cos\theta, \frac{1}{2} \sin\theta)$,
$A = (\frac{1}{2}, 0)$.

For convenience we choose S^1 to be the circle in $\mathbb{R}^2$ of radius $\frac{1}{2}$ (so the diameter is 1) and centre the origin.

Clearly $d(A, B) = \sin(\frac{1}{2}\theta)$, so

$$a(S^1, d) = \int_{S^1} d(x, A)\, \mu_0(dx) = \int_0^{2\pi} (\sin(\tfrac{1}{2}\theta)) \left(\frac{1}{2\pi}\, d\theta\right) = \frac{2}{\pi}.$$

Remarks. (i) It was observed by Stadje that by normalising d so that its maximum value is 1, as we have done in the above example, $\frac{1}{2} \leqq a(X, d) < 1$ always.

(ii) We note that the metric d_1 on S^1 determined by arc length (normalised as above) induces the same topology on S^1 as d, but that $a(S^1, d_1) = \frac{1}{2}$.

(iii) Once it is known that $a(S^1, d)$ exists, an elementary argument shows that $a(S^1, d)$ is $\int_0^{2\pi} (\sin(\frac{1}{2}\theta)) \left(\frac{1}{2\pi}\, d\theta\right)$ and hence is $\frac{2}{\pi}$. We argue as follows: For any n, let $x_1, \ldots, x_n$ be points uniformly distributed in S^1. Then

$$a(S^1, d) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_n)$$

for some point y_n, which by symmetry may be taken to satisfy

$$d(y_n, A) \leqq \sin\frac{\pi}{2n} < \frac{\pi}{2n}$$

(where $A = (\frac{1}{2}, 0)$ as above). Therefore

$$\left| a(S^1, d) - \frac{1}{n} \sum_{i=1}^{n} d(x_i, A) \right| < \frac{\pi}{2n},$$

and hence

$$a(S^1, d) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(x_i, A),$$

and this limit is clearly equal to the desired integral.

(iv) While for each n we can write down an integral whose value is $a(S^n, d)$, which can be approximated to any desired accuracy, we do not know whether, for $n > 1$, these numbers have a "nice" representation as is the case for $a(S^1, d)$, which equals $2/\pi$.

(v) Finally, we note that our method does not appear to extend to a wider class of spaces than the one we have discussed. For example, we do not know $a(X, d)$ even for such a pleasant space as an ellipse.

Acknowledgement. The research for this paper was done while the second author was a Visitor in the Department of Pure Mathematics at La Trobe University. He wishes to record his gratitude for the hospitality extended.

Added in proof (2. 2. 1983). Since this paper was written there has been considerable activity on this topic which is now known as "Numerical Geometry".

(1) Professor Jan Mycielski has drawn our attention to the paper "The rendezvous value of a metric space" by O. Gross which appeared in Ann. of Math. Studies **52**, 49–53 (1964). Gross proved the metric space case of Stadje's Theorem (Theorem A).

(2) Two further papers on this topic have appeared: J. Strantzen "An average distance result in Euclidean n-space". Bull. Austral. Math. Soc. **26**, 321–330 (1982); and D. Yost "Average distances in compact connected spaces". Bull. Austral. Math. Soc. **26**, 331–342 (1982).

Strantzen points out that Stadje's proposition that says $a(X, d) \leqq \frac{1}{2}\sqrt{5 - 2\sqrt{3}}\, D(X, d)$, where d is the Euclidean metric and X is any compact convex subset of $\mathbb{R}^n$, is wrong. Strantzen gives the correct bound. This has also been pointed out to us by E. Szekeres and G. Szekeres who independently confirmed Strantzen's bound and the falsity of Stadje's bound.

(3) Professor Karl H. Hofmann has informed us (private communication) that the examples of spheres S^n and topological groups given in 3 of our paper can be generalized as follows:

Theorem. *Let G be a compact metrizable group and H a closed subgroup of G. Let μ_0 be a measure of the space G/H of left cosets of H that is invariant under translations of G. Further let d be a quotient metric on G/H which is invariant under translations of G. Then $a(G/H, d) = \int_{G/H} d(x, H)\, \mu_0(dx)$.*

By putting $H = \{1\}$ we obtain all compact connected metric topological groups. By putting $G = SO(n)$ and $H = SO(n-1)$, we obtain the spheres.

(4) Dr. Graham Elton (private communication) has extended Stadje's Theorem and our Theorem 1 above as follows:

Theorem. *Let X be any compact connected Hausdorff space. Let $f\colon X \times X \to \mathbb{R}$ be any continuous symmetric function. Then, given any regular Borel probability measure μ on X, there is a point $y \in X$ such that $a(X, f) = \int_X f(x, y)\, \mu(dx)$.*

(5) With help from Dr. Graham Elton and Dr. Arthur Jones, we can answer the question raised in our Remarks (iv): Let S^n be a sphere of radius $\frac{1}{2}$ in the Euclidean space $(\mathbb{R}^{n+1}, d)$. Then

$$a(S^n, d) = \frac{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}{\sqrt{\pi}\,\Gamma\left(\frac{2n+1}{2}\right)}$$

where Γ is the gamma function.

So, for example, $a(S^2, d) = \frac{2}{3}$; $a(S^3, d) = 32/(15\pi)$.

(6) A survey of all the known contributions to this topic including unpublished work of Miss Joan Cleary, Dr. Graham Elton, Dr. Sidney Morris and Dr. David Wilson appears in "Numerical geometry" by Joan Cleary (Honours project, La Trobe University 1982). Another survey entitled "Numerical geometry — numbers for shapes" is being prepared for publication by Joan Cleary and Sidney Morris.

References

[1] EDWIN HEWITT and KENNETH A. ROSS, Abstract harmonic analysis. Vol. 1. Berlin 1963.

[2] WOLFGANG STADJE, A property of compact connected spaces. Arch. Math. **36**, 275–280 (1981)

Eingegangen am 8. 1. 1982

Anschrift der Autoren:

Sidney A. Morris
Department of Pure Mathematics
La Trobe University
Bundoora 3083
Victoria
Australia

Peter Nickolas
Department of Mathematics
University of Queensland
St. Lucia 4067
Queensland
Australia