ON THE HAUSDORFFNESS OF FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH NORMAL AMALGAMATION

Eli Katz and Sidney A. Morris

(received 27 August, 1981)

The main problem in the theory of amalgamated free products of topological groups is ascertaining whether the free product of two Hausdorff topological groups, G and H, with a closed subgroup, A, amalgamated, $G*_AH$, is necessarily Hausdorff. Katz and Morris [3,4,5] have given affirmative answers when G and H are k_ω-groups and A is a compact subgroup, a normal subgroup or the product of a compact subgroup and a normal subgroup. Ordman [12] gave an affirmative answer for some locally invariant groups. The only papers dealing with amalgamated free products of arbitrary Hausdorff groups G and H are Khan and Morris [7,8] where Hausdorffness is proved when A is a central subgroup of G and H.

In this note we reduce the problem of Hausdorffness to that of existence. We show that if A is a normal subgroup of Hausdorff groups G and H, and $G*_AH$ exists then $G*_AH$ is Hausdorff. This result is then applied to give an easy proof of the Khan-Morris [7] result mentioned above. Another consequence is that if A is a closed normal subgroup of a Hausdorff group G, then $G*_AG$ is Hausdorff.

Definition. Let A be a common subgroup of topological groups G and H. The topological group $G*_AH$ is said to be the *free product of the topological groups G and H with amalgamated subgroup A if*

(i) G and H are topological subgroups of $G*_AH$,

(ii) $G \cup H$ generates $G*_AH$ algebraically, and

(iii) every pair ϕ_1,ϕ_2 of continuous homomorphisms of G and H, respectively, into any topological group D, which agree on A, extend to a continuous homomorphism Φ of $G*_AH$ into D.

Math. Chronicle 12(1983) 113-117.

Remarks. (i) We note that basic category theory does not imply that the topological amalgamated free product of topological groups G and H exists, because the definition requires that G and H be topological subgroups of $G*_A H$.

(ii) If $G*_A H$ exists then the underlying group is the algebraic amalgamated free product of the underlying groups. (The standard reference for algebraic amalgamated free products is [9].)

(iii) If A is the group with one element, then $G*_A H$ is simply $G*H$, the free product of the topological groups G and H. [1,2,11,13].

The fundamental theorem of free products of topological groups is:

Theorem (Graev [1]). *If G and H are Hausdorff topological groups then $G*H$ exists and is Hausdorff.*

We now state our result.

Theorem. *Let A be closed normal subgroup of Hausdorff topological groups G and H. If $G*_A H$ exists then it is Hausdorff. Also A is a closed subgroup of $G*_A H$.*

Proof. Let ϕ_1 be the canonical continuous homomorphism of G onto G/A and ϕ_2 the canonical continuous homomorphism of H onto H/A. As $G/A*H/A$ exists and contains G/A and H/A, ϕ_1 and ϕ_2 can be considered as maps of G and H, respectively, into $G/A*H/A$. Thus there exists a continuous homomorphism Φ of $G*_A H$ into $G/A*H/A$. By Graev's Theorem above, $G/A*H/A$ is Hausdorff and therefore the kernel of Φ is a closed subgroup of $G*_A H$. But the kernel is A. As A is Hausdorff it is a T_1-space and therefore the identity element is closed in A and hence also in $G*_A H$. (An element of $G*_A H$ is representable in the form ghk where g is in G, h is in H and k is a product of commutators $[g,a]$ or $[h,a]$.) Thus $G*_A H$ is a T_1-space and consequently a Hausdorff group (Proposition 3 of [10].)

So we have now reduced the question of Hausdorffness of $G*_A H$ to that of existence. The following Lemma gives a necessary and sufficient condition.

Lemma. *Let A be a subgroup of topological groups G and H. Then $G*_A H$ exists if and only if there is a topological group F and continuous homomorphisms $\theta : G \to F$ and $\psi : H \to F$ such that θ and ψ agree on A, and $\theta : G \to \theta(G)$ and $\psi : H \to \psi(H)$ are topological group isomorphisms.*

Proof. It is well-known that the category of topological groups has co-products and so $G*H$ exists. Let δ be the canonical homomorphism of $G*H$ onto the algebraic amalgamated free product of the underlying groups of G and H. We denote this by $G \amalg_A H$. Let τ be the quotient topology on $G \amalg_A H$ under the map δ. Then $(G \amalg_A H, \tau)$, as a quotient of a topological group, is itself a topological group. It is readily seen this topological group satisfies conditions (ii) and (iii) of the definition of $G*_A H$. We now show that it also satisfies (i).

Now observe that the maps θ and ψ extend to a continuous homomorphism $\Phi : G*H \to F$ and Φ factors through $G \amalg_A H$ to give a continuous homomorphism $\gamma : (G \amalg_A H, \tau) \to F$ such that $\gamma\delta = \Phi$. But $\Phi : G \to \Phi(G)$ is a topological group isomorphism, since $\Phi|G = \theta$. Thus $\delta : G \to \delta(G)$ is a topological group isomorphism. Similarly $\delta : H \to \delta(H)$ is a topological group isomorphism. Thus $(G \amalg_A H, \tau)$ also satisfies condition (i) and so is $G*_A H$.

We can now deduce the Khan-Morris result [7] for central amalgamations.

Corollary 1. *If A is a closed central subgroup of Hausdorff topological groups G and H then $G*_A H$ exists and is Hausdorff.*

Proof. The result immediately follows from the Theorem and the Lemma by putting F equal to the direct product of the topological groups G and H with A amalgamated. (See [6].)

Corollary 2. *If* G *is a subgroup of a Hausdorff group* H *and* A *is a closed normal subgroup of* G *and* H*, then* $G *_A H$ *exists and is Hausdorff.*

Proof. Apply the Theorem and Lemma with $F = H$.

Corollary 3. *Let* A *be a closed normal subgroup of Hausdorff groups* G *and* H*, such that* $G *_A H$ *exists. Then* $(G *_A H) *_A G$ *exists and is Hausdorff.*

Proof. This follows from Corollary 2 by replacing H there by $G *_A H$, and observing that if A is a closed normal subgroup of G and H then it is also a closed normal subgroup of $G *_A H$.

Remark. Of course under the conditions of Corollary 3 we can similarly show that

$$(\ldots(((G *_A H) *_A K_1) *_A K_2) *_A \ldots) *_A K_n$$

where each $K_i = G$ or H, exists and is Hausdorff.

Corollary 4. *If* A *is a closed normal subgroup of a Hausdorff group* G *then* $G *_A G$ *exists and is Hausdorff.*

Remark. Of course under the conditions of Corollary 4, we can show that $(\ldots((G *_A G) *_A G) *_A \ldots) *_A G$ exists and is Hausdorff.

REFERENCES

1. M.I. Graev, *On free products of topological groups*, Izv. Akad. Nauk. SSSR Ser Mat 14 (1950), 343-354.

2. E. Katz, *Free products in the category of k_ω-groups*, Pacific J. Math. 59 (1975), 493-495.

3. Elyahu Katz and Sidney A. Morris, *Free products of topological groups with amalgamation*, submitted.

4. Elyahu Katz and Sidney A. Morris, *Free products of topological groups with amalgamation II*, submitted.

5. Eli Katz and Sidney A. Morris, *Free products of k_ω-topological groups with normal amalgamation*, submitted.

6. M.S. Khan and Sidney A. Morris, *Amalgamation direct products of topological groups*, Math. Chronicle, 11 (1982), 49-65.

7. M.S. Khan and Sidney A. Morris, *Free products of topological groups with central amalgamation*, Trans. Amer. Math. Soc., to appear.

8. M.S. Khan and Sidney A. Morris, *Free Products of topological groups with central amalgamation II*, Trans. Amer. Math. Soc., to appear.

9. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (Dover Publ. Inc., New York, 1976).

10. Sidney A. Morris, *Pontryagin duality and the structure of locally compact abelian groups*, London Math. Soc. Lecture Note Series No. 29, (Cambridge Univ. Press, 1977).

11. Sidney A. Morris, Edward T. Ordman and H.B. Thompson, *The topology of free products of topological groups*, Proc. Second Internat. Conf. Theory of Groups, Canberra 1973, Lecture Notes in Math. 372 (1974), 504-515.

12. Edward T. Ordman, *Free products of topological groups with equal uniformities I*, Colloq. Math. 31 (1974), 37-43.

13. Edward T. Ordman, *Free products of topological groups which are k_ω-spaces*, Trans. Amer. Math. Soc. 191 (1974), 61-73.

La Trobe University
Cleveland State University