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FREE PRODUCTS OF k_ω-TOPOLOGICAL GROUPS WITH NORMAL AMALGAMATION

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It is shown in this paper that if A is a closed normal subgroup of k_ω-topological groups G and H, then the free product of G and H with A amalgamated, $G *_A H$, exists, is Hausdorff and indeed a k_ω-group.

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amalgamated free product	k_ω-space
topological group	k_ω-group

1. Introduction and preliminaries

In [1] Khan and Morris showed that if G and H are any Hausdorff topological groups and A is a closed central subgroup of G and H, then $G *_A H$ exists and is Hausdorff. In this paper we are able to significantly weaken the condition of centrality, however the price we pay for this is that the topological groups are assumed to be k_ω-groups. (Recall that the class of k_ω-spaces includes, for example, all connected locally compact groups, all compact Hausdorff spaces and all countable CW-complexes). We show that if A is a closed normal subgroup of k_ω-groups G and H, then $G *_A H$ exists, is Hausdorff – indeed a k_ω-group – contains G and H as closed topological subgroups and has as its algebraic structure the amalgamated free product of the underlying groups.

The standard reference for amalgamated free products of groups is Magnus, Karrass and Solitar [2]. For completeness we include some definitions here.

Definition. Let A be a common subgroup of groups G and H. The group $G *_A H$ is said to be the *free product of G and H with amalgamated subgroup A* if

(i) G and H are subgroups of $G *_A H$.

(ii) $G \cup H$ generates $G *_A H$ algebraically.

(iii) Every pair ϕ_1, ϕ_2 of homomorphisms of G and H, respectively, into any group D which agree on A, extend to a homomorphism Φ of $G *_A H$ into D.

Definition. Let A be a common subgroup of topological groups G and H. The topological group $G *_A H$ is said to be the *free product of the topological groups G and H with amalgamated subgroup A* if

(i) G and H are topological subgroups of $G *_A H$.

(ii) $G \cup H$ generates $G *_A H$ algebraically.

(iii) Every pair ϕ_1, ϕ_2 of continuous homomorphisms of G and H, respectively, into any topological group D, which agree on A, extend to a continuous homomorphism of $G *_A H$ into D.

Observe that we use the symbol $G *_A H$ for both the algebraic and topological amalgamated free product, however which we mean should always be clear from the context.

We note that basic category theory ('general abstract nonsense') does not imply that the topological amalgamated free product of topological groups G and H exists, because we choose to have in our definition that G and H are topological subgroups of $G *_A H$.

Notation. We denote the embedding map of A in G by g and the embedding map of A in H by h. Again, without fear of confusion, an element denoted by g, g_i, g'_i and so on will always belong to G. Similarly h, h_i and so on belong to H and a, a_i and so on belong to A.

A Hausdorff topological space A is said to be a *k_ω-space* if $Z = \bigcup_{n=1}^{\infty} Z_n$ where each Z_n is compact, $Z_n \subseteq Z_{n+1}$ and a subset A of Z is closed if and only if $A \cap Z_n$ is compact for all n. We refer to $Z = \bigcup Z_n$ as a *k_ω-decomposition.*

Lemma. *Let A be a closed common subgroup of k_ω-groups G and H. Then G and H have k_ω-decompositions $G = \bigcup G_n$ and $H = \bigcup H_n$ such that*

(i) *$G_n = G_n^{-1}$ and $H_n = H_n^{-1}$,*

(ii) *$H_nH_m \subseteq H_{n+m}$ and $G_nG_m \subseteq G_{n+m}$,*

(iii) *$A \cap G_n \subseteq H_{n+1}$ and $A \cap H_n \subseteq G_{n+1}$*

for each positive integer n.

Proof. Let $G = \bigcup G'_n$ and $H = \bigcup H'_n$ be the given k_ω-decompositions of G and H. Put $G_1 = G'_1 \cap (G'_1)^{-1}$ and $H_1 = H'_1 \cap (H'_1)^{-1}$. Now suppose that $G_1, \ldots, G_n$ and $H_1, \ldots, H_n$ have been defined to satisfy (i), (ii) and (iii) and $G_i \subseteq G_{i+1}$, $H_i \subseteq H_{i+1}$, $i = 1, 2, \ldots, n-1$.

As A is closed, $A \cap G_n$ is compact. Therefore $A \cap G_n \subseteq H'_m$ for some $m \geqslant n+1$. So we define

$$H_{n+1} = H_n \cdot H_n \cup (H'_m \cap (H'_m)^{-1}).$$

Similarly observe that as $A \cap H_n$ is compact, there exists $k \geq n+1$ such that $A \cap H_n \subseteq G'_k$. So we define

$$G_{n+1} = G_n \cdot G_n \cup (G'_k \cap (G'_k)^{-1}).$$

Clearly $G_1, \ldots, G_{n+1}$ and $H_1, \ldots, H_{n+1}$ satisfy conditions (i), (ii) and (iii). Thus we can recursively define G_n and H_n for all n. It is easily seen, then, that $G = \bigcup G_n$ and $H = \bigcup H_n$ are k_ω-decompositions of G and H. $\square$

Let A be a common subgroup of groups G and H, $G * H$ the free product of G and H, and $G *_A H$ the amalgamated free product. Further, let Γ be the canonical homomorphism of $G * H$ onto $G *_A H$.

It is readily seen that the kernel K of Γ is given by:

$$K = \text{the normal subgroup generated by } \{h(a)g(a)^{-1} \colon a \in A\}.$$

If G and H are k_ω-groups with k_ω-decompositions $G = \bigcup G_n$ and $H = \bigcup H_n$, then we let $X = \bigcup_{n=1}^{\infty} X_n$ where

$$X_n = \{uvu^{-1} \colon u \in (G_n \cup H_n)^n,\ v = g(a)h(a)^{-1} \text{ or } v = h(a)g(a)^{-1} \text{ and } g(a) \in G_n \text{ and } h(a) \in H_n\}$$

and $Y_n = (X_n)^n$.

Obviously each X_n and Y_n is compact and $K = \bigcup_{n=1}^{\infty} Y_n$.

2. The main result

Theorem. *Let A be a closed normal subgroup of k_ω-groups G and H. Then $G *_A H$ exists and is a k_ω-group.*

Observe that Γ is a homomorphism of $G * H$ onto the algebraic amalgamated free product of the underlying groups of G and H. Proposition 1 tells us that if we give this group the quotient topology under this canonical homomorphism then it is Hausdorff and hence a k_ω-group. (Proposition 1 requires several lemmas which occupy most of this paper.) The proof of the theorem is then completed using Proposition 2. Proposition 2 says that if the quotient topology mentioned above is Hausdorff then it must contain G and H as closed topological subgroups. Thus this group with the quotient topology is the amalgamated free product of G and H, namely $G *_A H$.

Proposition 1. *Let A be a closed normal subgroup of k_ω-groups G and H. Let Γ be the canonical homomorphism of the free product $G * H$ onto the algebraic amalgamated free product $G *_A H$ of the underlying groups of G and H. Then this group with the quotient topology under Γ is Hausdorff, and hence a k_ω-group.*

The proof of this proposition is delayed to Section 3.

Proposition 2. *Let A, G, H, $G *_A H$, and Γ be as in Proposition 1. If $G *_A H$, with the quotient topology is a k_ω-group then $\Gamma : G \to G *_A H$ and $\Gamma : H \to G *_A H$ are closed embeddings; that is, Γ is a topological group isomorphism of G and H onto their images, and $\Gamma(G)$ and $\Gamma(H)$ are closed, in the quotient topology on $G *_A H$.*

Proof. Clearly $G *_A H$ with the quotient topology has k_ω-decomposition $G *_A H = \bigcup_n \Gamma((G_n \cup H_n)^n)$. To show that g is an embedding and $\Gamma(G)$ is closed, it suffices to verify that for each n, there exists an m such that

$$\Gamma(G) \cap \Gamma((G_n \cup H_n)^n) \subseteq \Gamma(G_m).$$

But this follows from the easily checked containment

$$\Gamma(G) \cap \Gamma((G_n \cup H_n)^n) \subseteq \Gamma((G_{n^2}) \cup (G_{n^2} \cap A)^n). \qquad \square$$

Proof of the Theorem. We claim that the algebraic amalgamated free product with the quotient topology under Γ is the free product of the topological groups G and H with the subgroup A amalgamated. Property (i) of the definition follows from Proposition 2. Property (ii) is obviously true. Property (iii) follows from the fact that the topological group we are considering is a quotient topological group of the free product $G * H$, and $G * H$ has the universal property.

3. Proofs

Lemma 1. *If $g \in G_n$, $h \in H_n$, $v \in Y_n$, $h(a) \in H_n$ and $\varepsilon = \pm 1$, then for $n \geqslant 1$*

$$h(a)v[g,h]^\varepsilon = [g,h]^\varepsilon h(a_1)v_1$$

where $h(a_1) \in H_{20n}$ and $v_1 \in Y_{20n}$.

Proof. Consider $\varepsilon = +1$, first.

$$\begin{aligned}
h(a)v[g,h] &= [g,h]([g,h]^{-1}h(a)[g,h])([g,h]^{-1}v[g,h])\\
&= [g,h](h^{-1}g^{-1}hgh(a)g^{-1}h^{-1}gh)v_2 && \text{where } v_2 \in Y_{n+4}\\
&= [g,h](h^{-1}g^{-1}h(gg(a)g^{-1})h^{-1}gh)(h^{-1}g^{-1}hgg(a)^{-1}h(a)g^{-1}h^{-1}gh)v_2\\
&= [g,h](h^{-1}g^{-1}hg(a_2)h^{-1}gh)v_3v_2 && \text{where } v_3 \in X_{n+5} \text{ and } g(a_2) \in G_{3n+1}\\
&= [g,h](h^{-1}g^{-1}(hh(a_2)h^{-1}gh)(h^{-1}g^{-1}hh(a_2)^{-1}g(a_2)h^{-1}gh)v_3v_2\\
&= [g,h](h^{-1}g^{-1}h(a_3)gh)v_4v_3v_2 && \text{where } v_4 \in X_{3n+5} \text{ and } h(a_3) \in H_{5n+2}\\
&= [g,h](h^{-1}(g^{-1}g(a_3)g)h)(h^{-1}g^{-1}g(a_3)gh)v_4v_3v_2\\
&= [g,h](h^{-1}g(a_4)h)v_5v_4v_3v_2 && \text{where } v_5 \in X_{5n+5} \text{ and } g(a_4) \in G_{7n+3}\\
&= [g,h](h^{-1}h(a_4)h)(h^{-1}h(a_4)^{-1}g(a_4)h)v_5v_4v_3v_2
\end{aligned}$$

$= [g, h]h(a_1)v_6v_5v_4v_3v_2$ where $v_6 \in X_{7n+6}$ and $h(a_1) \in H_{9n+4} \subseteq H_{20n}$

$= [g, h]h(a_1)v_1$ where $v_1 \in Y_{7n+6} \subseteq Y_{20n}$.

The case with $\varepsilon = -1$ is proved similarly. □

Lemma 2. *If $g \in G_n$, $g(a_1) \in G_n$, $h \in H_n$, $h(a_2) \in H_n$, and $\varepsilon = \pm 1$, then for $n \geqslant 1$*

$$[gg(a_1), hh(a_2)]^\varepsilon = [g, h]^\varepsilon h(a_3)v$$

where $h(a_3) \in H_{200n}$ and $v \in Y_{200n}$.

Proof. Consider $\varepsilon = +1$, first.

$$\begin{aligned}
&[gg(a_1), hh(a_2)] \\
&\quad = g(a_1)^{-1}g^{-1}h(a_2)^{-1}h^{-1}gg(a_1)hh(a_2) \\
&\quad = g(a_1)^{-1}g^{-1}h(a_2)^{-1}g(g^{-1}h^{-1}gh)h^{-1}g(a_1)hh(a_2) \\
&\quad = g(a_1)^{-1}g^{-1}h(a_2)^{-1}g[g, h]h^{-1}g(a_1)hh(a_2) \\
&\quad = (g(a_1)^{-1}g^{-1}g(a_2)^{-1}g)(g^{-1}g(a_2)h(a_2)^{-1}g) \\
&\qquad \cdot [g, h](h^{-1}g(a_1)h(a_1)^{-1}h)(h^{-1})hh(a_2)) \\
&\quad = (g(a_1)^{-1}g(a_4))v_1[g, h]v_2(h(a_5)h(a_2)) \\
&\qquad \text{where } v_1, v_2 \in X_{n+1},\ g(a_4) \in G_{3n+1} \text{ and } h(a_5) \in H_{3n+1} \\
&\quad = g(a_6)v_1[g, h]v_2h(a_7) \quad \text{where } g(a_6) \in G_{4n+1},\ h(a_7) \in H_{4n+4} \\
&\quad = h(a_6)(h(a_6)^{-1}g(a_6))v_1[g, h]v_2h(a_7) \\
&\quad = h(a_6)v_3v_1[g, h]v_2h(a_7) \qquad \text{where } v_3 \in X_{4n+2} \\
&\quad = h(a_6)v_4[g, h]v_2h(a_7) \qquad \text{where } v_4 \in Y_{4n+2} \\
&\quad = [g, h]h(a_8)v_5v_2h(a_7) \qquad \text{by Lemma 1} \\
&\qquad \text{where } v_5 \in Y_{20(4n+2)} \text{ and } h(a_8) \in H_{20(4n+2)} \\
&\quad = [g, h]h(a_8)h(a_7)(h(a_7)^{-1}v_5h(a_7))(h(a_7)^{-1}v_2h(a_7)) \\
&\quad = [g, h]h(a_3)v \quad \text{where } v \in Y_{20(4n+2)+4n+2} \subseteq Y_{200n} \\
&\qquad\qquad \text{and } h(a_3) \in H_{20(4n+2)+4n+1} \subseteq Y_{200n}.
\end{aligned}$$

The case $\varepsilon = -1$ is proved similarly. □

Lemma 3. *Let $h(a_i) \in H_n$, $v_i \in Y_n$, $g_i \in G_n$, $h_i \in H_n$, $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$, $n \geqslant 1$. Then*

$$\begin{aligned}
&([g_1, h_1]^{\varepsilon_1}h(a_1)v_1)([g_2, h_2]^{\varepsilon_2}h(a_2)v_2) \cdots ([g_n, h_n]^{\varepsilon_n}h(a_n)v_n) \\
&\quad = [g_1, h_1]^{\varepsilon_1}[g_2, h_2]^{\varepsilon_2} \cdots [g_n, h_n]^{\varepsilon_n}h(a)v
\end{aligned}$$

where $h(a) \in H_m$ and $v \in Y_m$, where $m = 20^n(n+1)^n$.

Proof. We shall prove by induction that

$$(*)\qquad \begin{aligned}&([g_1,h_1]^{\varepsilon_1}h(a_1)v_1)([g_2,h_2]^{\varepsilon_2}h(a_2)v_2)\cdots([g_k,h_k]^{\varepsilon_k}h(a_k)v_k)\\ &\quad=[g_1,h_1]^{\varepsilon_1}\cdots[g_k,h_k]^{\varepsilon_k}h(a)v \quad \text{for } k\leq n\end{aligned}$$

where $h(a)\in H_{20^k(n+1)^k}$ and $v\in Y_{20^k(n+1)^k}$.

Clearly this proposition is true for $k=1$. Now assume that it is true for $k=r<n$ and consider

$$\begin{aligned}
&([g_1,h_1]^{\varepsilon_1}h(a_1)v_1)\cdots([g_r,h_r]^{\varepsilon_r}h(a_r)v_r)([g_{r+1},h_{r+1}]^{\varepsilon_{r+1}}h(a_{r+1})v_{r+1})\\
&\quad=([g_1,h_1]^{\varepsilon_1}\cdots[g_r,h_r]^{\varepsilon_r}h(a')v')([g_{r+1},h_{r+1}]^{\varepsilon_{r+1}}h(a_{r+1})v_{r+1}\\
&\qquad\text{where } v'\in Y_{20^r(n+1)^r} \text{ and } h(a')\in H_{20^r(n+1)^r}\\
&\quad=([g_1,h_1]^{\varepsilon_1}\cdots[g_{r+1},h_{r+1}]^{\varepsilon_{r+1}})h(a'')v''h(a_{r+1})v_{r+1} \qquad \text{by Lemma 1}\\
&\qquad\text{where } v''\in Y_{20^{r+1}(n+1)^r} \text{ and } h(a'')\in H_{20^{r+1}(n+1)^r}\\
&\quad=[g_1,h_1]^{\varepsilon_1}\cdots[g_{r+1},h_{r+1}]^{\varepsilon_{r+1}}h(a'')h(a_{r+1})(h(a_{r+1})^{-1}v''h(a_{r+1}))v_{r+1}\\
&\quad=[g_1,h_1]^{\varepsilon_1}\cdots[g_{r+1},h_{r+1}]^{\varepsilon_{r+1}}h(a''\cdot a_{r+1})v'''v_{r+1}\\
&\qquad\text{where } v'''\in Y_{20^{r+1}(n+1)^r+1}\\
&\quad=[g_1,h_1]^{\varepsilon_1}\cdots[g_{r+1},h_{r+1}]^{\varepsilon_{r+1}}h(a)v\\
&\qquad\text{where } a\in H_{20^{r+1}(n+1)^{r+1}} \text{ and } v\in Y_{20^{r+1}(n+1)^{r+1}}.
\end{aligned}$$

So the proposition $(*)$ is true, and the lemma is proved. □

We choose a set S_G of coset representatives of A and G such that if $g\in S_G$ and $g\notin G_n$, then $Ag\cap G_n=\emptyset$; that is, the representative of each coset is chosen to lie in the smallest G_n of any element in that coset. A set S_H of coset representatives of A in H is similarly chosen.

Lemma 4. *Let $w=gh[g_1g(a_1),h_1h(a_1')]^{\varepsilon_1}\cdots[g_ng(a_n),h_nh(a_n')]^{\varepsilon_n}$ be an element in the kernel of the canonical homomorphism $G*H\to G*_AH$, where $g\in G_n$, $h\in H_n$, $g_i\in S_G\cap G_n$, $h_i\in S_H\cap H_n$, $h(a_i)\in H_n$, $h(a_i')\in H_n$, $\varepsilon_i=\pm1$, for $i=1,\ldots,n$. Then $w\in Y_r$, where $r=20^{200n}(200n+1)^{200n}+1$.*

Proof. By Lemma 2,

$$\begin{aligned}
w&=gh([g_1,h_1]^{\varepsilon_1}h(a_1'')v_1)\cdots([g_n,h_n]^{\varepsilon_n}h(a_n'')v_n)\\
&\qquad\text{where } v_i\in Y_{200n} \text{ and } h(a_i'')\in H_{200n}\\
&=gh[g_1,h_1]^{\varepsilon_1}\cdots[g_n,h_n]^{\varepsilon_n}h(a)v \qquad \text{by Lemma 3}\\
&\qquad\text{where } h(a)\in H_{r-1} \text{ and } v\in Y_{r-1}.
\end{aligned}$$

Now consider the sequence of groups

$$G * H \overset{\Gamma}{\to} G *_A H \overset{\Phi}{\to} G/A * H/A \overset{\Psi}{\to} G/A \times H/A$$

where all the maps are canonical homomorphisms.

Firstly observe that $\Psi\Phi\Gamma(w) = 1$, and so $g \in g(A)$ and $h \in h(A)$; that is, $g = g(a)$ and $h = h(a')$, for some a and a' in A. Now

$$\begin{aligned}\Phi\Gamma(w) = 1 &= \Phi\Gamma([g_1, h_1]^{\varepsilon_1} \cdots [g_n, h_n]^{\varepsilon_n})\\ &= \Phi\Gamma([g_1, h_1])^{\varepsilon_1} \cdots \Phi\Gamma([g_n, h_n])^{\varepsilon_n}.\end{aligned}$$

However each $\Phi([g_i, h_i])$ lies in the cartesian subgroup $[G/A, H/A]$ of $G/A * H/A$, which subgroup is defined to be the kernel of Ψ. Indeed the cartesian subgroup is a free group with free basis $\{c^{-1}d^{-1}cd : c \in G/A \text{ and } d \in H/A\}\backslash\{1\}$ [2, p. 412]. Thus $\Phi\Gamma([g_1, h_1])^{\varepsilon_1} \cdots \Phi\Gamma([g_n, h_n])^{\varepsilon_n}$ can equal 1 only because of trivial cancellations, such as $\Phi\Gamma([g_i, h_i])^{\varepsilon_i} = \Phi\Gamma([g_{i+1}, h_{i+1}])^{-\varepsilon_{i+1}}$. Hence $[g_1, h_1]^{\varepsilon_1} \cdots [g_n, h_n]^{\varepsilon_n} = 1$, by the same trivial cancellations, since $\Phi\Gamma([g_i, h_i]^{\varepsilon_i}) = \Phi\Gamma([g_{i+1}, h_{i+1}])^{-\varepsilon_{i+1}}$ implies $[g_i, h_i]^{\varepsilon_i} = [g_{i+1}, h_{i+1}]^{-\varepsilon_{i+1}}$ keeping in mind that each coset has a unique representative. So

$$w = g(a)h(a')h(a)v = g(a)h(a'')v.$$

Finally observe that $\Gamma(w) = 1 = \Gamma(g(a)h(a''))$. But this can equal 1 only if $a = (a'')^{-1}$. So

$$\begin{aligned} w &= g(a)h(a^{-1})v \\ &= v_1 \cdot v \qquad \text{where } v_1 \in X_{r-1} \\ &= v'' \in Y_r. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Proposition 1. In order to see that $G *_A H$ with the quotient topology under Γ is Hausdorff, and hence a k_ω-group, it suffices to verify that the kernel K of Γ is closed in $G * H$.

In [3] it is shown that $G * H$ has the following k_ω-decomposition:

$$G * H = \bigcup_n T_n,$$

where $T_n = \{ghk : g \in G_n, h \in H_n, k = [g_1, h_1]^{\varepsilon_1} \cdots [g_m, h_m]^{\varepsilon_m}, \varepsilon_i = \pm 1,$ each $g_i \in G_n, h_i \in H_n$, and $m \leq n\}$.

To show that K is closed in this k_ω-space, it suffices to prove that for each n, there exists an s such that $K \cap T_n = Y_s \cap T_n$. Indeed it is enough to verify that $K \cap T_n \subseteq Y_s$.

So let $w \in T_n \cap K$

$$w = gh[g_1', h_1']^{\varepsilon_1} \cdots [g_m', h_m']^{\varepsilon_m}$$

and $g \in G_n$, $h \in H_n$, each $g_i' \in G_n$, $h_i' \in H_n$ and $m \leq n$.

Each $g_1' = g_i g(a_i)$, where $g_i \in S_G$, the set of coset representatives defined just before Lemma 4. Then $g_i \in G_n$ and so $g(a_i) \in G_{2n}$.

Similarly $h_i' = h_i g(a_i')$, where $h_i \in S_H \cap H_n$ and $h(a_i') \in H_{2n}$.

Now applying Lemma 4 the result follows at once. □

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