AMALGAMATED DIRECT PRODUCTS OF TOPOLOGICAL GROUPS

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1. Introduction

In 1950, B.H. Neumann and Hanna Neumann [8] introduced the notion of amalgamated direct product of groups. In this paper we introduce the analogous notion of amalgamated direct product of topological groups and discuss its properties.

The basic existence theorem we prove says that if A and B are any topological groups and C is a common subgroup of A and B then the amalgamated direct product $A \times_C B$ exists if and only if C is central in A and B. If $A \times_C B$ exists then the underlying group structure of $A \times_C B$ is the amalgamated direct product of the underlying groups of A and B. Of course, if $C = \{e\}$ then the amalgamated direct product of A and B is the usual direct product $A \times B$ of the two topological groups. Amongst the permanence properties of the amalgamated direct product is the following one: If A and B are Lie groups and C is a closed central subgroup of A and B, then $A \times_C B$ is a Lie group.

In attempting to introduce the notion of amalgamated direct product of topological groups, the first task is to find the "correct" definition. To do this we examine the algebraic amalgamated direct product, and notice that it has a certain extension property. This extension property appears to be important since the main (known) results about amalgamated direct products of groups seem to fall into place when this property is used.

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This extension property has an obvious analogue for topological groups, and significantly the usual direct product of topological groups has this analogous property. So this property is the key to defining amalgamated direct products of topological groups.

In [4] amalgamated direct products of topological groups play a vital role in proving that the free product of any two Hausdorff topological groups with any closed central subgroup amalgamated is Hausdorff.

We content ourselves in this paper with one application. We show that every compact connected Hausdorff group G is an amalgamated directed product of its commutator subgroup and the identity component of the centre of G. [From this, one can readily deduce the observation of Karl Heinrich Hofmann and the second author that every continuous homomorphism from the commutator subgroup of a compact connected Hausdorff group G into any unitary group $U(n)$ can be extended to a continuous homomorphism of G into $U(n)$.]

2. An Extension Property

If G is a group then the centre of G will be denoted by $Z(G)$. If A is a subgroup of G, then the centralizer of A in G will be denoted by $C_G(A)$.

Definition [1,7,8]. A group G is said to be the *direct product of its subgroups* A *and* B *with amalgamated subgroup* C if

(i) G is generated by $A \cup B$, (ii) $A \cap B = C$, and (iii) $B \subseteq C_G(A)$. It is denoted by $A \times_C B$.

Proposition 1. *Let* $G = A \times_C B$ *be the direct product of its subgroups* A *and* B *with amalgamated subgroup* C. *If* ϕ_1 *and* ϕ_2 *are any homomorphisms of* A *and* B, *respectively, into any group* H *such that* $\phi_1|C = \phi_2|C$ *and* $\phi_2(B) \subseteq C_H(\phi_1(A))$, *then there exists a homomorphism* $\Phi : A \times_C B \to H$ *such that* $\Phi|A = \phi_1$ *and* $\Phi|B = \phi_2$.

Proof. Since G is generated by $A \cup B$ and $B \subseteq C_G(A)$, each element g in G can be written $g = ab$, where $a \in A$ and $b \in B$. We define $\Phi : G \to H$ by $\Phi(ab) = \phi_1(a)\,\phi_2(b)$. Firstly we must verify that Φ is a well-defined function. Suppose $a_1b_1 = a_2b_2$, where a_1 and a_2 are in A, and b_1 and b_2 are in B. Then $a_2^{-1}a_1 = b_2b_1^{-1}$. So $a_2^{-1}a_1 = b_2b_1^{-1}$ is in $C = A \cap B$. Thus $\phi_1(a_2^{-1}a_1) = \phi_2(b_2b_1^{-1})$; that is, $(\phi_1(a_2))^{-1}\,\phi_1(a_1) = \phi_2(b_2)(\phi_2(b_1))^{-1}$, which implies that $\phi_1(a_1)\,\phi_2(b_1) = \phi_1(a_2)\,\phi_2(b_2)$. So $\Phi(a_1b_1) = \Phi(a_2b_2)$ and Φ is indeed a well-defined function.

Clearly $\Phi|A = \phi_1$ and $\Phi|B = \phi_2$, so it only remains to check that Φ is a homomorphism. Let $g = ab$ and $g_1 = a_1b_1$ be any elements of G, where $a \in A$, $a_1 \in A$, $b \in B$ and $b_1 \in B$. Now using the fact that $B \subseteq C_G(A)$, we see that $gg_1 = (ab)(a_1b_1) = (aa_1)(bb_1)$. So $\Phi(gg_1) = \phi_1(aa_1)\,\phi_2(bb_1) = \phi_1(a)\,\phi_1(a_1)\,\phi_2(b)\,\phi_2(b_1)$. Observing that $\phi_2(B) \subseteq C_H(\phi_1(A))$, we see that $\Phi(gg_1) = (\phi_1(a)\,\phi_2(b))(\phi_1(a_1)\,\phi_2(b_1))$ $= \Phi(g)\,\Phi(g_1)$ and so Φ is a homomorphism, as required.

Remark. Observe that if, in the above definition, $C = \{e\}$, the trivial group, then $A \times_C B$ is just the direct product $A \times B$ of the groups A and B.

Corollary 1. *Let $A \times B$ be the direct product of any groups A and B. If ϕ_1 and ϕ_2 are any homomorphisms of A and B, respectively, into any group H such that the elements of $\phi_1(A)$ commute with the elements of $\phi_2(B)$, then there exists a homomorphism $\Phi : A \times B \to H$, such that $\Phi|A = \phi_1$ and $\Phi|B = \phi_2$.*

Remark. If $G_{p.}$ is the category of all groups and group homomorphisms, then $A \times B$ is of course the product in this category, of A and B. Thus we might be a little surprised by Corollary 1 above which says that $A \times B$ has a coproduct-like property.

Corollary 2. *Let $A \times_C B$ be the direct product of groups A and B with amalgamated subgroup C. Then there is a canonical homomorphism f of $A \times B$ onto $A \times_C B$ having kernel $\{(c,c^{-1}) : c \in C\}$.*

Proof. If ϕ_1 and ϕ_2 are the identity maps on A and B respectively, then Corollary 1 above says that $f : A \times B \to A \times_C B$, given by $f((a,b)) = \phi_1(a)\,\phi_2(b) = ab$ where $a \in A$ and $b \in B$, is a homomorphism. As $A \times_C B$ is generated by $A \cup B$ and $f(A \times B)$ contains A and B, f is into.

If $f((a,b)) = e$, then $\phi_1(a)\,\phi_2(b) = e$. So $\phi_1(a) = (\phi_2(b))^{-1}$. But $\phi_1(a) \in A$ and $(\phi_2(b))^{-1} \in B$. Therefore $\phi_1(a) = (\phi_2(b))^{-1} \in A \cap B = C \subseteq A \times_C B$. As ϕ_1 and ϕ_2 are identity maps on A and B respectively, we deduce that $a \in C \subseteq A$ and $b \in C \subseteq B$. As ϕ_1 and ϕ_2 are identity maps on C we see that if $a = c \in C \subseteq A$ then $b = c^{-1} \in C \subseteq B$.

So the kernel of f is contained in $\{(c,c^{-1}) : c \in C\}$. What's more, every element in $\{(c,c^{-1}) : c \in C\}$ is clearly in the kernel of f.

Remark. As yet we have not commented on the existence or uniqueness of the amalgamated direct product. Corollary 2 above gives the clue to existence.

Corollary 3. *Let A and B be groups having a common subgroup C. Then $A \times_C B$ exists if and only if $C \subseteq Z(A)$ and $C \subseteq Z(B)$.*

Proof. Property (iii) of the definition of amalgamated direct product clearly implies that $C \subseteq Z(A)$ and $C \subseteq Z(B)$ are necessary conditions for $A \times_C B$ to exist.

Now assume that C is a central subgroup of A and B. Consider the direct product $A \times B$ and let $K = \{(c,c^{-1}) : c \in C\}$. Then using the centrality of C, we see that K is a normal subgroup of $A \times B$.

Let f be the canonical homomorphism of $A \times B$ onto the quotient group $(A \times B)/K = G$. Clearly f is one-to-one on A and on B. So G has subgroups $f(A)$ and $f(B)$ isomorphic to A and B, respectively, such that (i) G is generated by $f(A) \cup f(B)$, (ii) $f(A) \cap f(B) = f(C)$, which is isomorphic to C, and (iii) $f(B) \subseteq C_G(f(A))$. Thus G is $f(A) \times_{f(C)} f(B)$, and so $A \times_C B$ exists.

Corollary 4. *Let A_1, A_2, B_1 and B_2 be groups such that C_1 is a central subgroup of A_1 and B_1 and C_2 is a central subgroup of A_2 and B_2. If $\phi_1 : A_1 \to A_2$ and $\phi_2 : B_1 \to B_2$ are surjective homomorphisms such that $\phi_1|C_1 = \phi_2|C_1$ then there exists a homomorphism Φ of $G_1 = A_1 \times_{C_1} B_1$ onto $G_2 = A_2 \times_{C_2} B_2$ such that $\Phi|A_1 = \phi_1$ and $\Phi|B_1 = \phi_2$. In particular, if ϕ_1 and ϕ_2 are isomorphisms and $\phi_1(C_1) = C_2$, then Φ is an isomorphism.*

Proof. That a homomorphism $\Phi : G_1 \to G_2$ such that $\Phi|A_1 = \phi_1$ and $\Phi|B_1 = \phi_2$ exists follows from Proposition 1. The surjectivity of Φ is a consequence of the fact that $\Phi(G_1) \supseteq A_2 \cup B_2$ which generates G_2.

If ϕ_1 and ϕ_2 are isomorphisms then we have homomorphisms $\Phi_1 : G_1 \to G_2$ such that $\Phi_1|A_1 = \phi_1$ and $\Phi_1|B_1 = \phi_2$, and $\Phi_2 : G_2 \to G_1$ such that $\Phi_2|A_2 = \phi_1^{-1}$ and $\Phi_2|B_2 = \phi_2^{-1}$. Clearly $\Phi_1 = \Phi_2^{-1}$ and so Φ_1 is an isomorphism of G_1 onto G_2.

Remark. Corollary 4 gives us the uniqueness of $A \times_C B$. As another consequence of Corollary 4 we have

Corollary 5. *If C is a central subgroup of A and B, then there is a canonical homomorphism Φ of $A \times_C B$ onto $A/C \times B/C$ having kernel C.*

Proof. If $\phi_1 : A \to A/C$ and $\phi_2 : B \to B/C$ are the canonical homomorphisms, then using Corollary 4 with $C_2 = \{e\}$, we have that there is a homomorphism Φ of $A \times_C B$ onto $A/C \times B/C$. To see that the kernel of Φ is C, observe that, in the notation of Corollary 2,

$$A \times B \xrightarrow{\quad f \quad} A \times_C B \xrightarrow{\quad \Phi \quad} A/C \times B/C$$

is such that Φf is the canonical map of $A \times B$ onto $A/C \times B/C$. The kernel of Φf is $C \times C$ and so the kernel of Φ must be $f(C \times C) = C$.

We can now easily derive a result of Chehata and Shawky [2].

Corollary 6. *Let C be a central subgroup of A and B, S_A a complete set of coset representatives of C in A and S_B a complete set of coset representatives of C in B, such that the representative of C is the identity element in each case. Then each $g \in A \times_C B$ can be written uniquely in the form $g = abc$, where $a \in S_A$, $b \in S_B$ and $c \in C$.*

Proof. If $g \in A \times_C B$, then clearly $g = a_1 b_1$, where $a_1 \in A$ and $b_1 \in B$. So $g = a_1 b_1 = (ac_1)(bc_2)$, where $a \in S_A$, $b \in S_B$, $c_1 \in C$, $c_2 \in C$. Thus $g = ab(c_1 c_2) = abc$, where $c = c_1 c_2 \in C$.

Now suppose that $g = abc = a'b'c'$, where $a \in S_A$, $a' \in S_A$, $b \in S_B$, $b' \in S_B$, $c \in C$ and $c' \in C$. Corollary 5 yields a homomorphism $\Phi : A \times_C B \to A/C \times B/C$ and we see that $\Phi(abc) = (Ca,Cb) = (Ca',Cb') = \Phi(a'b'c')$. Thus $Ca = Ca'$ and $Cb = Cb'$. Observing that a and a' are in S_A, we deduce that $a = a'$. Similarly we have that $b = b'$. Hence $abc = a'b'c' = abc'$, which implies that $c = c'$. So g does have a unique representation in the required form.

3. Amalgamated Direct Products of Topological Groups

Definition. Let A and B be (not necessarily Hausdorff) topological groups. Then the topological group G is said to be the *free product of A and B* if it has the properties:

(i) A and B are topological subgroups of G,

(ii) G is generated algebraically by $A \cup B$,

and (iii) if ϕ_1 and ϕ_2 are continuous homomorphisms of A and B, respectively, into any topological group H, there exists a continuous homomorphism $\Phi : G \to H$ such that $\Phi|A = \phi_1$ and $\Phi|B = \phi_2$.

It is denoted by $A * B$.

Remarks. It is well-known, see for example [5, Theorem 3.2], that for any topological groups A and B, $A * B$ exists. What's more, it is unique up to isomorphism. The underlying group structure of $A * B$ is simply that of the (algebraic) free product of the underlying groups of A and B. Further, it is easily seen, using property (iii) of the definition, that $A * B$ has the finest group topology on the underlying group which will induce the given topologies on A and B.

Proposition 2. *Let A and B be any topological groups. Then the natural homomorphism θ of the free product $A * B$ onto the direct product $A \times B$ of A and B with the product topology is an open continuous mapping.*

Proof. Let θ_1 and θ_2 be the identity maps on A and B respectively. Then the definition of the free product implies that there exists a continuous homomorphism θ of $A * B$ onto $A \times B$ which extends θ_1 and θ_2.

As $A * B$ is a topological group, the multiplication map $(A * B) \times (A * B) \to A * B$, $(g,h) \to gh$, where $g \in A * B$ and $h \in A * B$, is continuous. So its restriction $\delta : A \times B \to A * B$, given by $\delta((a,b)) = ab$, is also continuous. So we have

$$A \times B \xrightarrow{\ \delta\ } A * B \xrightarrow{\ \theta\ } A \times B$$

and $\theta\delta$ is the identity map. So $\theta|\delta(A \times B)$ is an open map (indeed a homeomorphism). By the Lemma stated below, then, $\theta : A * B \to A \times B$ is an open map, as required.

Lemma [6, p.24 Exercise 6(i)]. *Let* ϕ *be a homomorphism of a topological group* G *into a topological group* H . *If* X *is a non-empty subset of* G *such that the restriction* $\phi : X \to H$ *is an open map, then* $\phi : G \to H$ *is also an open map.*

Proposition 3. *Let* A *and* B *be any topological groups. The product topology of* $A \times B$ *is the finest group topology on the underlying group which will induce the given topologies on* A *and* B .

Proof. Suppose that there exists a group topology τ on the underlying group of $A \times B$ which is strictly finer than the product topology. Then the natural homomorphism $\theta : A * B \to (A \times B,\tau)$, being an extension of the continuous homomorphisms θ_1 and θ_2 of Proposition 2, would still be continuous. But this is a contradiction to θ being a quotient mapping of $A * B$ onto $A \times B$ with the product topology. Thus the stated result is true.

Proposition 4. *Let* $A \times B$ *be the direct product with the product topology of any topological groups* A *and* B . *If* ϕ_1 *and* ϕ_2 *are continuous homomorphisms of* A *and* B , *respectively, into any topological group* H *such that the elements of* $\phi_1(A)$ *commute with the elements of* $\phi_2(B)$, *then there exists a continuous homomorphism* $\Phi : A \times B \to H$ *such that* $\Phi|A = \phi_1$ *and* $\Phi|B = \phi_2$.

Proof. By Corollary 1 of Proposition 1 there exists a homomorphism $\Phi : A \times B \to H$ such that $\Phi|A = \phi_1$ and $\Phi|B = \phi_2$. So we have to show only that Φ is continuous. If Φ is not continuous then we can put a group topology τ_1 on $A \times B$ by saying a subset O of $A \times B$ is in τ_1 if and only if it is the inverse image of an open set in H. We can then form the union of this topology τ_1 and the product topology to produce a group topology τ on $A \times B$ which is strictly finer than the product topology. Further, since ϕ_1 and ϕ_2 are continuous, τ induces the given topologies on A and B. This contradicts Proposition 3. Hence $\Phi : A \times B \to H$, where $A \times B$ has the product topology, is continuous, and we have the required result.

Definition. A topological group G is said to be the *(topological) direct product of its topological subgroups* A *and* B *with amalgamated subgroup* C if it has the properties:

(i) G is generated algebraically by $A \cup B$,

(ii) $A \cap B = C$

(iii) $B \subseteq C_G(A)$,

and (iv) if ϕ_1 and ϕ_2 are any continuous homomorphisms of A and B, respectively, into any topological group H such that $\phi_1|C = \phi_2|C$ and $\phi_2(B) \subseteq C_H(\phi_1(A))$, then there exists a continuous homomorphism $\Phi : G \to H$ such that $\Phi|A = \phi_1$ and $\Phi|B = \phi_2$.

The topological amalgamated direct product is also denoted by $A \times_C B$.

Remarks. The first and most important question that presents itself is: Does $A \times_C B$ exist for all topological groups A, B and C such that $C \le Z(A)$ and $C \le Z(B)$? Shortly we shall answer this in the affirmative. In the meantime observe that if A and B are any topological groups such that $A \cap B = C = \{e\}$, the trivial group, then $A \times_C B$ exists and is the topological direct product. (This follows from Proposition 4 and the Remark after Proposition 1).

We note that if for topological groups A and B with $A \cap B = C$, $A \times_C B$ exists, then the underlying group of $A \times_C B$ is the direct product of the underlying groups of A and B with C amalgamated. (This follows immediately from the definition of topological amalgamated direct product).

Using an analogous method to that used in the proof of Corollary 4 of Proposition 1 we can show that if the topological amalgamated direct product $A \times_C B$ exists, then it is unique up to topological group isomorphism.

Using property (iv) of the definition of topological amalgamated direct product we see that if $A \times_C B$ exists then it has the finest group topology on the underlying group which will induce the given topologies on A and B. Indeed this is a characterization of the topological amalgamated direct product.

Let A and B be topological groups with $A \cap B = C$ lying in the centre of A and B. So the algebraic direct product of A and B with C amalgamated exists. In order to prove that the topological direct product of A and B with C amalgamated exists, it suffices to show that the algebraic amalgamated direct product $A \times_C B$ admits one group topology which induces the given topologies on A and B. (The topology of the topological amalgamated direct product is then the union of all such topologies).

Theorem 1. *Let A and B be topological groups with a common topological subgroup C. Then the topological direct product $A \times_C B$ of A and B with C amalgamated exists if and only if C is a central subgroup of A and B.*

Proof. Property (iii) of the definition of topological amalgamated direct product implies that $C \subseteq Z(A)$ and $C \subseteq Z(B)$ are necessary conditions for $A \times_C B$ to exist.

Now assume that C is central in A and B. Then by Corollary 3 of Proposition 1, the algebraic direct product $A \times_C B$ of the underlying groups of A and B with C amalgamated exists. Let $A \times B$ be the topological direct product of the topological groups A and B. By Corollary 2 of Proposition 1 there is a homomorphism f of the topological direct product $A \times B$ onto the algebraic amalgamated direct product $A \times_C B$ such that f maps A and B identically onto themselves, and the kernel K of f is $\{(c,c^{-1}) : c \in C\}$. Let τ be the quotient topology on $A \times_C B$ under this map f. Then $(A \times_C B,\tau)$ is a topological group. By the above Remarks, in order to show that the required topological amalgamated direct product exists, it suffices to show that τ induces the given topologies τ_1 and τ_2 on A and B, respectively.

Let τ induce the topologies τ_3 and τ_4 on A and B, respectively. Observing that $f : A \times B \to (A \times_C B,\tau)$ is an open map, that the kernel K of f lies in $A \times C$ and that $f(A \times C) = A$, we have that $f : A \times C \to (A,\tau_3)$ is an open map. More precisely $f : (A,\tau_1) \times C \to (A,\tau_3)$ is an open continuous surjective map. However let us consider the natural continuous homomorphism $f : (A,\tau_1) \times C \to (A,\tau_1)$. As the restriction of f to $(A,\tau_1) \times \{e\}$ yields an open map (indeed a homeomorphism): $(A,\tau_1) \times \{e\} \to (A,\tau_1)$, the Lemma stated earlier implies that $f : (A,\tau_1) \times C \to (A,\tau_1)$ is an open continuous map. This is impossible unless $\tau_1 = \tau_3$. Similarly $\tau_2 = \tau_4$ and so τ induces the given topologies on A and B. Thus we have the required result.

Corollary 1. *Let $A \times_C B$ be the topological direct product of topological groups A and B with amalgamated subgroup C. Then the canonical homomorphism f of the topological direct product $A \times B$ onto $A \times_C B$ is an open continuous homomorphism having kernel $\{(c,c^{-1}) : c \in C\}$.*

Proof. The only part of the above statement which has not been proved yet is the openness of f. Using an argument like that in Proposition 4 we see that if f were not open then $A \times_C B$ would admit a group topology finer than the quotient topology of f which would induce the given topologies on A and B. This topology on $A \times_C B$ would give rise to a group topology on the underlying group of $A \times B$ which would be finer than the product topology but which would induce the given topologies on A and B. However this contradicts Proposition 3 and so f is an open map.

Corollary 2. *Let A and B be Hausdorff topological groups with C a common central subgroup of A and B. If C is a closed subgroup of A or B, then $A \times_C B$ is a Hausdorff topological group.*

Proof. Without loss of generality, assume that C is closed in A. By Corollary 1 of Theorem 1 it suffices to show that $K = \{(c,c^{-1}) : c \in C\}$ is a closed subset of the topological direct product $A \times B$. Let $(c_\alpha, c_\alpha^{-1})_{\alpha \in I}$ be a set in K converging to (a,b). Thus $(c_\alpha)_{\alpha \in I}$ converges to a and $(c_\alpha^{-1})_{\alpha \in I}$ converges to b. As C is closed in A, $a = c \in C$. Thus $(c_\alpha^{-1})_{\alpha \in I}$ converges to c^{-1}. But as B is Hausdorff $(c_\alpha^{-1})_{\alpha \in I}$ cannot converge to two distinct points b and c^{-1}. Thus $b = c^{-1}$. So $(a,b) = (c,c^{-1}) \in K$. Hence K is closed in $A \times B$, as required.

Remarks. In the above proof, the Hausdorffness of A appears not to be used, but in fact it is a consequence of the properties which were used; namely that C is closed in A, and B is Hausdorff.

On the other hand one might have expected that it would be necessary to have C closed in A and B. However this is not so, for example if $\mathbb{R}$ is the additive group of reals with its usual topology and $\mathbb{Q}$ is the dense topological subgroup of $\mathbb{R}$ consisting of the rational numbers, then the topological amalgamated direct product $\mathbb{R} \times_{\mathbb{Q}} \mathbb{Q}$ is $\mathbb{R}$ itself, which is certainly Hausdorff.

Corollary 3. *Let A and B be topological groups with a common central subgroup C.*

(i) *If A and B are locally compact groups, then $A \times_C B$ is a locally compact group.*

(ii) *If A and B are Lie groups and C is closed in A or B then $A \times_C B$ is a Lie group.*

(iii) *If A and B are locally invariant groups then $A \times_C B$ is locally invariant.*

(iv) *If A and B are compact groups then $A \times_C B$ is compact.*

(v) *If A and B are connected groups then $A \times_C B$ is connected.*

(vi) *If A and B are pathconnected groups then $A \times_C B$ is pathconnected.*

(vii) *If A and B are locally connected groups then $A \times_C B$ is locally connected.*

(viii) *If A and B are σ-compact groups then $A \times_C B$ is σ-compact.*

(ix) *If A and B are k_ω-spaces and C is closed in A or B then $A \times_C B$ is a k_ω-space.*

(x) *If A and B are Banach spaces and C is a closed vector subspace of A and B, then $A \times_C B$ admits a Banach space structure.*

Proof. All of these results follow from Corollaries 1 and 2 of Theorem 1.

Corollary 4. *Let* A_1, A_2, B_1 *and* B_2 *be topological groups such that* C_1 *is a central subgroup of* A_1 *and* B_1 *, and* C_2 *is a central subgroup of* A_2 *and* B_2 *. Let* $G_1 = A_1 \times_{C_1} B_1$ *and* $G_2 = A_2 \times_{C_2} B_2$ *. If* $\phi_1 : A_1 \to A_2$ *and* $\phi_2 : B_1 \to B_2$ *are continuous homomorphisms such that* $\phi_1 | C_1 = \phi_2 | C_1$ *then there exists a continuous homomorphism* $\Phi : G_1 \to G_2$ *such that* $\Phi | A_1 = \phi_1$ *and* $\Phi | B_1 = \phi_2$ *. Further* (i) *if* ϕ_1 *and* ϕ_2 *are one-to-one and* $\phi_1(C_1) = \phi_1(A_1) \cap \phi_2(B_1)$ *, then* Φ *is one-to-one;* (ii) *if* ϕ_1 *and* ϕ_2 *are surjective, then* Φ *is surjective;* (iii) *if* ϕ_1 *and* ϕ_2 *are algebraic isomorphisms and* $\phi_1(C_1) = C_2$ *, then* Φ *is an algebraic isomorphism;* (iv) *if* $\phi_1 : A_1 \to \phi_1(A_1)$ *and* $\phi_2 : B_1 \to \phi_2(B_1)$ *are open mappings and* $\phi_1(C_1) = C_2$ *, then* $\Phi : A_1 \times_{C_1} B_1 \to \Phi(A_1 \times_{C_1} B_1)$ *is an open mapping;* (v) *if* ϕ_1 *and* ϕ_2 *are topological group embeddings and* $\phi_1(C_1) = C_2$*, then* Φ *is a topological group embeddings;* (vi) *if* ϕ_1 *and* ϕ_2 *are topological isomorphisms and* $\phi_1(C_1) = C_2$ *, then* Φ *is a topological isomorphism.*

Proof. The only part which requires further comment is (iv). This follows by considering the commutative diagram

$$\begin{array}{ccc} A_1 \times B_1 & \xrightarrow{\ p_1\ } & A_1 \times_{C_1} B_1 \\ \Big\downarrow \theta & & \Big\downarrow \Phi \\ A_2 \times B_2 & \xrightarrow[\ p_2\ ]{} & A_2 \times_{C_2} B_2 \end{array}$$

where all the maps are canonical homomorphisms.

The openness of $\Phi : A_1 \times_{C_1} B_1 \to \Phi(A_1 \times_{C_1} B_1)$ follows from the fact that p_1 and p_2 are open continuous homomorphisms, $\theta : A_1 \times B_1 \to \theta(A_1 \times B_1)$ is an open (continuous) homomorphism, and the kernel of p_2 is a subset of $\theta(A_1 \times B_1)$.

Corollary 5. *If C is a central subgroup of the topological groups A and B, then there are canonical open continuous homomorphisms p_1 and p_2 of the direct product of A and B with C amalgamated, $A \times_C B$, onto A/C and B/C, respectively.*

Proof. Clearly there are canonical surjective continuous homomorphisms $p_1 : A \times_C B \to A/C$ and $p_2 : A \times_C B \to B/C$. Observing that the restriction of p_1 to A yields $p_1 : A \to A/C$ which is an open map, the Lemma stated earlier shows that $p_1 : A \times_C B \to A/C$ is also an open map. Similarly p_2 is an open map.

Remark. Observe that in the case $C = \{e\}$ the maps p_1 and p_2 of Corollary 5 of Theorem 1 are just the canonical projections of $A \times B$ onto A and B , respectively.

4. An Application to Compact Groups

Theorem 2. *Let G be a compact connected Hausdorff group and let $Z_0(G)$ and G' denote the identity component of the centre of G and the commutator subgroup of G, respectively. Then G is the topological direct product of G' and $Z_0(G)$ with the subgroup $G' \cap Z_0(G)$ amalgamated.*

Proof. Firstly we note that $Z_0(G)$ is closed and central and G' is compact. It follows from [3,6.59] or [9,6.5.6] that in a compact connected group G each element, x , say, can be expressed as $x = yz$, for some $y \in G'$ and $z \in Z_0(G)$. Clearly $Z_0(G)$ lies in the centralizer of G' in G . Put $A = G' \cap Z_0(G)$. Then, by definition, G is algebraically the direct product of G' and $Z_0(G)$ with A amalgamated. Now consider the map $g : G \times G \to G$

defined by $g(x,y) = xy$. Then g is continuous as G is a topological group. Therefore the restricted map $g^* : G' \times Z_0(G) \to G$ is also continuous. As $G = G' \,.\, Z_0(G)$, the map g^* is also surjective. Further, since $Z_0(G)$ is central, it is routine to verify that g^* is a homomorphism. The compactness of G, G' and $Z_0(G)$ implies that g^* is also an open mapping. Hence G is topologically isomorphic to the quotient group $(G' \times Z_0(G))/\text{kernel } (g^*)$. But

$$\begin{aligned} \text{kernel } (g^*) &= \{(y,z) \in G' \times Z_0(G) : yz = e\} \\ &= \{(y,z) \in G' \times Z_0(G) : y = z^{-1}\} \\ &= \{(y,y^{-1}) : y \in G' \text{ and } y^{-1} \in Z_0(G)\} \\ &= \{y,y^{-1}) : y \in A\} . \end{aligned}$$

But by Corollary 1 of Theorem 1 this says that G is the topological direct product of G' and $Z_0(G)$ with A amalgamated.

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