WHEN IS A FUNCTION THAT SATISFIES THE CAUCHY–RIEMANN EQUATIONS ANALYTIC?

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1. The Looman–Menchoff theorem—An extension of Goursat's theorem. It is well known[1] that a complex-valued function $f = u + iv$, defined and analytic on a domain D in the complex plane satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D. The standard textbooks, such as those authored by Ahlfors, Cartan, Churchill, Jameson, Knopp, Sansone and Gerretson, avoid answering the question as to whether or not the converse holds. Most instead offer the following partial converse due to Goursat **[13]**.

THEOREM 1. *If $f = u + iv$, defined on a domain D, is such that*

(i) $\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ *exist everywhere in* D,

(ii) u, v *satisfy the Cauchy–Riemann equations everywhere in* D, *and if further*

(iii) f *is continuous in* D,

(iv) $\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ *are continuous in* D,

then f is analytic in D.

This is a substantially revised version of an article by the present authors and S. A. R. Disney that appeared in the Gazette of the Australian Mathematical Society 2 (3) (1975), 67–81. S. A. R. Disney's name does not appear above only because he preferred it that way.

[1] This is a rare instance of a well-known result that is indeed well known.

The remaining standard texts offer the stronger result:

THEOREM 2. *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *u, v, as functions of two real variables, are differentiable everywhere in D,*
then f is analytic in D.

Recently the authors began a search to discover precisely what is known regarding the converse. The only modern book we were able to find tnat addresses itself to this problem is Derrick **[8]**. He points out that far weaker conditions than those of Theorem 2 are known to imply analyticity but that the Cauchy–Riemann equations themselves do not imply analyticity! Indeed, the function f given by

$$f(z) = \begin{cases} \exp(-z^{-4}) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0, \end{cases}$$

first noticed by Looman [**20**, 107], (see also [**39**, 70]), is readily seen to satisfy the Cauchy–Riemann equations everywhere, but, as $f(z)/z \to \infty$ as $z \to 0$ with $\arg z = \pi/4$, fails to be analytic at the origin. Observe that f must have an essential singularity at 0 otherwise $\partial f/\partial x$ could not exist there.

Derrick **[8]** suggests that the 'best' result in this direction appears to be

THEOREM 3. (Looman–Menchoff) *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *f is continuous in D,*
then f is analytic in D.

Menchoff's proof (see [**34**, 199] and [**24**, 9]), based on the concepts of Lebesgue integration and Baire category is, according to Saks **[36]** "... undoubtedly one of the most elegant and unexpected applications of the modern theory of real functions to the elementary problems of an entirely classical aspect." A proof of the theorem is given in the appendix.

A word of caution. The naive local version of the Looman–Menchoff theorem is: if a function is continuous at z_0 and satisfies the Cauchy–Riemann equations there, it is complex-differentiable at z_0. This assertion is false! For example [**8**, 15], the function

$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0, \end{cases}$$

is continuous everywhere, satisfies the Cauchy–Riemann equations at 0, but is not complex-differentiable at the origin. To the best of our knowledge the strongest result in this direction is the standard one: if $f = u + iv$ is such that (i) u, v are differentiable at z_0, (ii) u, v satisfy the Cauchy–Riemann equations at z_0, then f is (complex) differentiable at z_0. See [**17**, 35].

Although Looman and Menchoff clearly did improve on Goursat's theorem others have obtained still more subtle results.

2. Extensions of the theorems of Green, Morera and Goursat. The earliest contribution to the problem appears to be that of Paul Montel who, in a 1913 note in the *Comptes Rendus*, asserted the

THEOREM 4. *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *f is bounded in D,*
then f is analytic in D.

Recall that a function f on D is said to be locally bounded if it is bounded in some neighbourhood of each point of D. Now analyticity in D means analyticity in some neighbourhood of each point of D, whence condition (iii) can be replaced by

(iii)′ *f is locally bounded in D.*

As every continuous function is locally bounded it follows that Theorem 4 with (iii) replaced by (iii)′ implies Theorem 3. Although this result appears to be quite strong, observe that condition (i) implies the separate continuity of f which in turn implies its measurability, **[26]**. (In fact f is necessarily of Baire class I).

Montel neither proved this result in his note **[27]** nor did he publish a proof elsewhere. In spite of this, it was stated as a theorem in Menchoff's monograph **[24]**—one in a series edited by Montel. Montel did, however, indicate how the proof is an "immediate application" of a strengthened version of the following classical result on exact differentials. (Here, as elsewhere, the term integrable means Lebesgue integrable and all integrals are Lebesgue integrals. However, as every bounded Riemann integrable function is Lebesgue integrable, with the exception of Theorems 8, 9 and 10 and Question 1 at the end of §3, Lebesgue may be replaced by Riemann throughout.)

THEOREM 5. *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) *$\partial P/\partial y$, $\partial Q/\partial x$ exist everywhere in K,*

(ii) *$\partial P/\partial y = \partial Q/\partial x$ everywhere in K,*

and if further

(iii) *P, Q are continuous in K,*

(iv) *$\partial P/\partial y$, $\partial Q/\partial x$ are continuous in K,*

then

$$\int_C Pdx + Qdy = 0.$$

Of course this is a special case of the following version of Green's theorem—a proof of which may be constructed by making technical adjustments to that on page 289 of **[1]**. Parenthetically we remark that although condition (iv) below implies (ii), (ii) is included as it is clearly necessary.

THEOREM 6. *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) *$\partial P/\partial y$, $\partial Q/\partial x$ exist everywhere in K,*

(ii) *$\partial Q/\partial x - \partial P/\partial y$ is integrable in K,*

and if further

(iii) *P, Q are continuous in K,*

(iv) *$\partial P/\partial y$, $\partial Q/\partial x$ are continuous in K,*

then (Green's formula)

$$\int_C Pdx + Qdy = \int\int_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

In 1923 Looman **[20]**, by weakening the hypotheses in Theorem 5, offered a proof of Theorem 3. Unfortunately the proof was found to contain a serious gap centring around a surprising fact. Even though $\partial P/\partial x$, $\partial Q/\partial y$ do not occur in Green's formula, an example of Fesq **[11]** reveals that some assumptions regarding them must be made if one wishes to relax condition (iii) and still have a valid statement. Indeed, even if the right-hand side of the formula is zero, the statement is false without such assumptions. It was Tolstoff **[41]** who first realized this. Unfortunately this error appears in the papers of both Montel and Looman (see also **[46]**), and was not corrected until Menchoff's paper **[23**, 9] appeared. (See also **[34**, 199].)

Tolstoff **[40]** was the first to prove Montel's theorem. Implicit in his work is the observation that whenever one has a Green-type theorem (see Theorem 6) and a Morera-type theorem (see Theorem 7) one obtains a Goursat-type theorem (Theorem 1). For example, let us see how the classical Goursat theorem follows from the classical Green theorem (more accurately, from its corollary—Theorem 5—on exact differentials) and the classical Morera theorem.

THEOREM 7. (Morera, cf. [**35**, 120]) *If f, defined on a domain D, is such that*
(i) *f is continuous in D,*
(ii) $\int_{\partial R} f(z)dz = 0$ *for each rectangle R (*) in D,*
then f is analytic in D.

Proof (Of Goursat's theorem). For any rectangle R

$$\int_{\partial R} f(z)dz = \int_{\partial R} (u+iv)dz = \int_{\partial R} udx - vdy + i\int_{\partial R} vdx + udy = 0$$

by Theorem 5 and the Cauchy–Riemann equations. Hence by Morera's theorem, f is analytic in D.

The moral of this proof is clear. If one can reduce the conditions involved in Morera's theorem and those involved in Green's theorem one can obtain a strengthened version of Goursat's theorem. For instance, one can readily deduce the Looman–Menchoff theorem (Theorem 3) from the classical Morera theorem and the following extension of Green's theorem due to Paul J. Cohen [**7**]—the same Cohen of Continuum Hypothesis fame.

THEOREM 8. *Let R be a closed rectangle. If P, Q are real-valued functions of two variables on R such that*
(i) *$\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ exist everywhere in R,*
(ii) *$\partial Q/\partial x - \partial P/\partial y$ is integrable in R,*
and if further
(iii) *P, Q are continuous in R,*
then

$$\int_{\partial R} Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

Such a proof would, however, reverse the chronological order of things. In fact Cohen's ideas were inspired by the proof of the Looman–Menchoff theorem.

It was via the implication Morera + Green $\Rightarrow$ Goursat that Tolstoff [**40**] proved Montel's theorem. First, he proved a strong version of Morera (with continuous replaced by integrable, locally bounded and separately continuous); next, he proved a strong version of Theorem 5, and finally he combined these as in the above proof of Goursat's theorem, to yield Montel's theorem.

3. More technical results. We saw in §1 that a function that satisfies the Cauchy–Riemann equations everywhere need not be analytic. With this in mind one cannot fail to be impressed by the distributional result.

THEOREM 9. *Suppose f is locally integrable on D and, as a distribution, satisfies the Cauchy–Riemann equations. Then f agrees almost everywhere with a function analytic in D.*

Proof. The proof proceeds by a smoothing argument. We present only an outline as full details may be readily found in [**49**, 117]. Let k be a 'bump' function, viz, k is a C^∞ function defined on the complex plane, $k \geqq 0$; $\int\int k(z)dxdy = 1$; and k vanishes outside the open unit disc. For $\varepsilon > 0$ put $k_\varepsilon(z) = \varepsilon^{-2}k(z/\varepsilon)$. Then the convolution $f_\varepsilon = f * k_\varepsilon$ given by

(*) Throughout, all rectangles are assumed to have their sides parallel to the co-ordinate axes.

$$f_\varepsilon(z)=\int\int f(z-\zeta)k_\varepsilon(\zeta)d\xi d\eta$$

is a C^∞ function of z for those z distant greater than ε from the boundary of D. Moreover [47, 118], $f_\varepsilon \to f$ a.e. in D as $\varepsilon \to 0$.

If g is a (differentiable) function the Cauchy–Riemann equations may be written as $\partial g/\partial \bar{z}=0$, where $\partial/\partial\bar{z}$ is the formal partial differential operator

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Hence, to assert of f that as a distribution it satisfies these equations means that for each test function ϕ,

$$\int\int f(\zeta)\frac{\partial\phi}{\partial\bar{\zeta}}d\xi d\eta=0.$$

Given this, the following equations are readily verified,

$$\frac{\partial f_\varepsilon}{\partial\bar{z}}=\frac{\partial}{\partial\bar{z}}\int\int f(\zeta)k_\varepsilon(z-\zeta)d\xi d\eta=\int\int f(\zeta)\frac{\partial}{\partial\bar{z}}k_\varepsilon(z-\zeta)d\xi d\eta$$

$$=-\int\int f(\zeta)\frac{\partial}{\partial\bar{\zeta}}k_\varepsilon(z-\zeta)d\xi d\eta=0 \text{ as } k_\varepsilon \text{ is a test function.}$$

Thus f_ε is a C^∞ function satisfying the Cauchy–Riemann equations and hence, by Goursat's theorem, is analytic. But, by Cauchy's integral formula, for almost all $z\in D$, and for appropriate contours γ,

$$f(z)=\lim_{\varepsilon\to 0}f_\varepsilon(z)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_\gamma\frac{f_\varepsilon(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{\zeta-z}d\zeta,$$

so that f agrees a.e. with a function analytic in D.

Theorem 9 is a particular instance of the general regularity theorem: any distributional solution of a hypo-elliptic system of partial differential equations (of which the Cauchy–Riemann system is a paradigm example) is in fact a C^∞ function. Moreover, the above proof-technique of "smoothing" is one of the key tools needed to prove the general result. See [16, 101].

From Theorem 9 the following result of Rademacher [29] may be deduced.

THEOREM 10. *If $f=u+iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist a.e. in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations a.e. in D,*
and if further
(iii) *f is locally integrable in D,*
(iv) *f is separately absolutely continuous in D,*
(v) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are locally integrable in D,*
then f is analytic in D.

Proof. Note first the redundancy: (iv) implies (i). (iv) together with (v) suffice to ensure the validity of integration by parts:

$$\int\int f(z)\frac{\partial\phi}{\partial\bar{z}}dxdy=-\int\int\frac{\partial f}{\partial\bar{z}}\phi(z)dxdy,$$

for all text functions ϕ. This equation shows that the classical and distributional derivatives of f 'coincide'. Thus by Theorem 9, f agrees almost everywhere with a function g analytic in D. In fact, $f=g$ everywhere in D. To see this, suppose to the contrary that for some $z_0=x_0+iy_0\in D$,

$f(z_0) \neq g(z_0)$. Then, as f is separately continuous, f and g must disagree at all points on some line-segment through z_0 parallel to the x-axis. For each $x_0 + iy$ on this segment they must similarly disagree at all points on some line-segment through $x_0 + iy$ parallel to the y-axis. The union of all these line segments is not of measure zero. Q.e.D.

Aside from its lack of elegance, due to (iii), (iv) and (v), Theorem 10 does not appear to be particularly strong. Nonetheless it is not contained in any of the others. On the credit side, however, the weakening of (i) and (ii) from 'everywhere' to 'almost everywhere' suggests the possibility of weakening Looman–Menchoff to, say,

If $f = u + iv$, defined on a domain D is such that

(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist a.e. in D,

(ii) u, v satisfy the Cauchy–Riemann equations a.e. in D,

and if further

(iii) f is continuous in D,

then f is analytic in D.

This conjecture is seriously false. For a counter-example, due to Urysohn, [**48**, 464], first construct a planar Cantor-set as follows. From the closed unit square delete $([0,1] \times (\frac{1}{3}, \frac{2}{3})) \cup ((\frac{1}{3}, \frac{2}{3}) \times [0,1])$ thus leaving a set K_1 consisting of four closed squares. Continue deleting in the usual way so that at the nth stage we are left with a set K_n consisting of 4^n closed squares whose centres we denote by $z_{n,k}(k = 1, 2, \cdots, 4^n)$. Then $K = \bigcap_n K_n$ is a (totally disconnected) set of planar measure zero. Next put

$$f(z) = \lim_{n \to \infty} 4^{-n} \sum_{k=1}^{4^n} (z - z_{n,k})^{-1};$$

then f is everywhere continuous, analytic off K (and thus satisfies, (i), (ii), (iii) above), yet fails to have an analytic continuation throughout K! The difficulty lies in weakening the condition that the partial derivatives exist everywhere. That some such weakening is possible is illustrated by the version of Looman–Menchoff appearing in Saks [**34**, 199]—a proof of which appears in the Appendix to this paper.

THEOREM 11. *If $f = u + iv$, defined on a domain D, is such that*

(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D, except on a countable set,

(ii) u, v satisfy the Cauchy–Riemann equations a.e. in D,

and if further

(iii) f is continuous in D,

then f is analytic in D.

One may weaken still further the conditions on the partial derivatives of f; results along these lines have been obtained by Cafiero [**6**] and Fesq [**11**], the latter deriving the following.

THEOREM 12. *If $f = u + iv$, defined on a domain D, is such that*

(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D, except on a countable union of sets of finite one-dimensional Hausdorff measure,

(ii) u, v satisfy the Cauchy–Riemann equations a.e. in D,

and if further

(iii) f is locally bounded in D,

(iv) f is separately continuous in D,

then f is analytic in D.

Recall ([**14**, 53] and [**12**]), that the one-dimensional Hausdorff measure of a set E in the plane is defined to be

$$\sup_{\varepsilon > 0} \inf \left\{ \sum_1^\infty \operatorname{diam}(E_n) : E = \bigcup_1^\infty E_n, \operatorname{diam}(E_n) < \varepsilon \right\}.$$

Any contour of finite length has finite one-dimensional Hausdorff measure, whereas a (non-trivial) rectangle has infinite one-dimensional Hausdorff measure.

Condition (i) dates back to Besicovitch **[3]** who proved that if a function defined on a simply-connected domain D is continuous everywhere and (complex) differentiable everywhere, except on a countable union of sets of finite linear measure, then in fact if is (complex) differentiable everywhere in D.

We conclude this section with two questions which, as far as we know, have not been answered.

QUESTION 1. *Suppose $f = u + iv$, defined on a domain D, is such that*

(i) *$\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ exist everywhere in D,*

(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D,*

and suppose further that

(iii) *f is locally integrable in D.*

Does it follow that f is analytic in D?

QUESTION 2. *Suppose $f = u + iv$, defined on a domain D, is such that u, v satisfy the Cauchy–Riemann equations everywhere in D. What is the structure of the set E of points of non-analyticity of f?*

Regarding Question 2, Trokhimchuk [**43**, 109] shows that E is (closed and) totally disconnected, so that f is necessarily analytic on a dense open subset of D. However, Pelling (private communication) has shown that E can have positive measure.

4. Odds and ends. The only 'odd' worth mentioning is the characterization (due to Heffter **[15]**) in terms of Cauchy–Riemann 'difference' equations, of those continuous functions that are analytic. His result is that a continuous function $u + iv$ on a domain D is analytic in D iff for each rectangle $R = [a, b] \times [c, d]$ in D there are points (x, y), $(x', y') \in R$ for which

$$\frac{u(b, y) - u(a, y)}{b - a} = \frac{v(x, d) - v(x, c)}{d - c},$$

$$\frac{u(x', d) - u(x', c)}{d - c} = -\frac{v(b, y') - v(a, y')}{b - a}.$$

For even odder odds see **[43]**.

As for 'ends', there are a number of papers dealing with the weakening of the conditions in Green's and Morera's theorems; see **[5]**, **[11]**, **[29]**, **[32]**, **[35]** and **[37]**. Further, the extension from rectangles to more general regions is dealt with in **[25]**, **[28]**, **[31]** and **[45]**. For a readable account of a fascinating and surprising extension of Morera's theorem consult Zalcman **[47]** and **[49]**. *Inter alia* he shows how relatively weak versions of Green lead to surprisingly strong versions of Morera. Whilst on the topic of Morera mention must be made of a remarkable example of Vitushkin [**12**, 95]. He constructs a compact, totally-disconnected planar set E of $(1 + \varepsilon)$-dimensional Hausdorff measure zero for each $\varepsilon > 0$, and a (non-constant) function f continuous on the extended complex plane, analytic on the complement of E (and thus without an analytic continuation to E), such that for any closed contour γ disjoint from E, $\int_\gamma f(z)dz = 0$!

Appendix. *A Proof of the Looman–Menchoff Theorem.* The proof rests heavily on two lemmas each of which is a variation on the 'fundamental theorem of calculus' theme. The first, due to Menchoff **[24]**, allows one to estimate certain naturally arising contour integrals. The proof of this lemma is entirely elementary and may be found in [**34**, 198]. The second lemma is well known in measure theory and may be found in [**33**, 166] or [**9**, 214].

LEMMA 1. *Let R be a closed rectangle and $\phi: R \to \mathbf{R}$ a function both of whose partial derivatives $\partial\phi / \partial x$, $\partial\phi / \partial y$ exist everywhere in R. Let N be a constant and $E \subset R$ a closed non-empty set such that*

$$|\phi(\xi, y)-\phi(x, y)| \leqq N|\xi - x|,$$
$$|\phi(x, \eta)-\phi(x, y)| \leqq N|\eta - y|,$$

whenever $(x, y) \in E$, $(\xi, y) \in R$, $(x, \eta) \in R$. *Suppose* $[a, b] \times [\alpha, \beta]$ *is the smallest rectangle in* R *containing* E, *then*

$$\left|\int_a^b [\phi(x,\beta)-\phi(x,\alpha)]dx - \int\int_E \frac{\partial \phi}{\partial y} dxdy\right| \leqq 5N.m(R-E)$$

$$\left|\int_\alpha^\beta [\phi(b,y)-\phi(a,y)]dy - \int\int_E \frac{\partial \phi}{\partial x} dxdy\right| \leqq 5N.m(R-E),$$

where m *denotes* (*planar*) *Lebesgue measure.*

In order to state the second lemma we need the notion of the derivative of a set function. Suppose λ is a (complex) Borel measure on an open set K in $\boldsymbol{R}^2$, that $z \in K$ and c is a complex number. If for every sequence of Borel sets (B_n) that 'shrink nicely' to z (cf. [**33**, 163]),

$$\lambda(B_n)/m(B_n) \rightarrow c \quad \text{as } n \rightarrow \infty,$$

λ is said to be differentiable (with respect to m) at z. c is called the derivative of λ at z and is denoted by $(d\lambda/dm)(z)$.

LEMMA 2. *Let* λ *be a complex Borel measure on the open set* $K \subset \boldsymbol{R}^2$. *Then*
(i) $d\lambda/dm$ *exists a.e. in* K *and belongs to* $L^1(K)$.
If further λ *is absolutely continuous with respect to* m *then*
(ii) $\lambda(B) = \int_B \frac{d\lambda}{dm}(z)dm(z)$ *for every Borel set* B, *and*
(iii) $d\lambda/dm$ *coincides a.e. with the Radon–Nikodým derivative of* λ *with respect to* m.

We are now ready to present the

Proof. (Of Looman–Menchoff). By way of contradiction suppose that the (closed) set $E \subset D$ of points at which f fails to be analytic is non-empty. For each natural number n put

$$E_n = \{z \in D: |f(z+h)-f(z)|/|h| \leqq n \quad \text{and} \quad |f(z+ih)-f(z)|/|h| \leqq n,$$
$$\text{for real } h \text{ with } 0<|h| \leqq 1/n\}.$$

As f is continuous, each E_n is closed and as both first-order partial derivatives of f exist throughout D, the (E_n) cover D. By the Baire category theorem [**33**, 103], applied to the complete metric space E, there are a natural number N and an open rectangle K whose closure lies in D, such that $\emptyset \neq E \cap K \subset E_N$.

For each (closed) rectangle R in K define the (complex-valued) set function λ by the contour integral

$$\lambda(R) = \int_{\partial R} f(z)dz.$$

If R, R' are rectangles in K put

$$\lambda(R \cup R') = \int_{\partial(R \cup R')} f(z)dz,$$

and if R is a rectangle in K put

$$\lambda(K-R) = \lambda(K) - \lambda(R).$$

λ so defined is an additive set function on the Boolean algebra $\mathcal{R}$ generated by the rectangles in K. By the Hahn extension theorem [**9**, 136, Corollary 9], λ has a (unique) extension to a measure (also denoted by λ) on the σ-algebra generated by $\mathcal{R}$. It is clear that this σ-algebra is precisely the collection of all Borel sets of K and hence λ is a (complex) Borel measure on K.

Our aim now is to show that λ is identically zero, as once this is established $\lambda(R)=0$ for each rectangle R in K, whence, by Morera's theorem f is analytic throughout K. As this conclusion contradicts the assumed non-emptiness of E (in K) the proof will then be complete. To show that λ is the zero measure it suffices to show that (a) λ is absolutely continuous with respect to m, and (b) the derivative $d\lambda/dm$ vanishes almost everywhere in K (cf. Lemma 2 (ii)).

The basic tool neeeded to prove both (a) and (b) is the estimate (*) below. To establish it let R be any rectangle in K meeting E and with side lengths $\leqq 1/N$. If J denotes the smallest rectangle containing $R \cap E$ it follows from the definition of E_N, and Lemma 1 when used in conjunction with the Cauchy–Riemann equations, that $|\lambda(J)| \leqq 20N.m(R-E)$. But by the Cauchy–Goursat theorem, λ vanishes on any rectangle not meeting E, so that this inequality may be written as

$$(*) \qquad |\lambda(R)| \leqq 20N.m(R-E).$$

Armed with (*) we can verify (a) above. Toward this end let F be a subset of K of (planar) Lebesgue measure zero. As such, given any $\varepsilon > 0$ there is a sequence (R_n) of rectangles in K which cover F and which are such that $\Sigma_n m(R_n) < \varepsilon/20N$. By subdividing if necessary we may assume further that the side-lengths of all R_n are $\leqq 1/N$ and that all R_n meet E. Then it follows that $|\lambda(F)| \leqq \Sigma_n |\lambda(R_n)| \leqq 20N.\Sigma_n m(R_n - E) \leqq 20N.\Sigma_n m(R_n) < \varepsilon$ so that $\lambda(F)=0$ and (a) is established.

As for (b), if $z \in K - E$, by taking sufficiently small rectangles in the definition of the derivative of λ and invoking Cauchy–Goursat again, we see that $(d\lambda/dm)(z) = 0$. As for points in E note first that for any $z \in K$, if $R \subset K$ is a rectangle containing z, meeting E, and of side length $\leqq 1/N$, by (*),

$$(**) \qquad \frac{|\lambda(R)|}{m(R)} \leqq 20N.\frac{m(R-E)}{m(R)} = 20N.\frac{m(CE \cap R)}{m(R)},$$

CE being the complement of E in K. Denote by m_{CE} the restriction of m to CE so that for any Borel set B,

$$m_{CE}(B) = m(CE \cap B) = \int_B \chi_{CE}(z)\,dxdy,$$

χ_{CE} being the characteristic function of CE. By Lemma 2 (iii) we have, for almost all $z \in K$

$$\frac{dm_{CE}}{dm}(z) = \chi_{CE}(z),$$

so that, for almost all $z \in E$, $(dm_{CE}/dm)(z) = 0$. Thus if (R_n) is any sequence of rectangles as above that 'shrink nicely' to $z \in E$, for almost all such z

$$0 = \frac{dm_{CE}}{dm}(z) = \lim_{n\to\infty} \frac{m_{CE}(R_n)}{m(R_n)} = \lim_{n\to\infty} \frac{m(CE \cap R_n)}{m(R_n)},$$

so that by (**) $\lambda(R_n)/m(R_n) \to 0$. Hence, $d\lambda/dm(z) = 0$ for a.a. $z \in E$, as was required.

As Lemma 1 holds if the partial derivatives are assumed to exist only on the complement of a countable set [**34**, 198], the above proof actually proves the slightly stronger Theorem 11.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

SCHAUDER BASES

P. R. HALMOS

Euclidean spaces have three basic properties: they are (1) vector spaces, with (2) a concept of length, and (3) a concept of angle. Banach spaces, whose intensive study began in the 1930's, constitute a generalization that retains requirements (1) and (2) but does not insist on (3). (Precisely: a Banach space is a real or complex vector space endowed with a norm, with respect to which, as a metric space, it is complete.)

Generalizations can turn out to be too general (as wise hindsight sometimes shows); they can lead