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EMBEDDINGS IN CONTRACTIBLE OR COMPACT OBJECTS

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1. Introduction. It is well known that any completely regular space X can be embedded in a contractible space, since X can be embedded in its Stone-Čech compactification $\beta(X)$ which, in turn, is a subset of a product of copies of the unit interval. However, this embedding is not closed; the metrics which exist on $\beta(X)$ do not seem naturally related to any initially given on X; and $\beta(X\times Y) \neq \beta(X)\times\beta(Y)$, so that $\beta(X)$ does not inherit algebraic structures, such as that of a group, from X (1).

There is another embedding of any Hausdorff space X in a contractible space EX discussed by Segal in [11]. However, it is difficult to prove topological properties of EX, such as being Hausdorff or metric, and the equation $E(X\times Y) = EX\times EY$, while true in the category of Hausdorff k-spaces, is not known to be true in the usual topological category.

Segal states in [11] that EX can be identified with a set of step functions $[0, 1[\to X$, but remarks that from this point of view the topology seems obscure. Now, a topology on this set X^* of step functions has been defined for the case where X is a Hausdorff topological group by Hartman and Mycielski [7], who find that X^* with this topology is a path-connected and locally path-connected topological group containing X as a closed subgroup.

In this paper we modify the Hartman-Mycielski definition to give a topology on X^* for any space X, and we prove that X^* is a contractible and locally contractible space containing X. We also construct a natural isomorphism $(X\times Y)^* \simeq X^*\times Y^*$, which explains why X^* inherits an algebraic structure from X. If X is Hausdorff, metric or completely regular, so also is X^*; and we prove that Segal's bijection $EX \to X^*$ is continuous.

In Section 3 we use the embedding $H \to H^*$ to give a short proof that the Graev free topological group $F(X)$ on a functionally separable space X is functionally separable, and hence Hausdorff. The only property

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(1) Another contractible embedding is $X\to CX$, where $CX = (X\times I)/(X\times\{0\})$ is the cone on X; but again $C(X\times Y) \neq CX\times CY$.

of $F(X)$, apart from the definition, that is used is that X generates $F(X)$ algebraically.

The main object of Section 4 is to generalize this last result to the free topological groupoid $F(\Gamma)$ on a topological graph Γ. The notion of a topological groupoid was developed by Ehresmann in a series of papers for applications to differential topology and differential geometry. More recently, Brown and Hardy have considered topological groupoids as natural generalizations both of topological spaces and of topological groups; they have developed some of the basic theory [3], [4], and applications to topological groups have been given by them [3] and by Nickolas [9], while applications to algebraic topology have been given by Hardy and Puppe [6]. This explains the interest in generalizing properties of topological spaces and topological groups to topological groupoids. The main result of Section 4 (Theorem 4) has been proved by different methods in [5] – however, the proof here does not rely, as does that of [5], on any explicit construction of the topology on $F(\Gamma)$.

We would like to thank Graeme Segal for giving us details of constructions outlined in [11].

2. A contractible embedding. Let X be any topological space. By X^* we mean the set of continuous-from-the-right step functions $[0, 1[\to X$, i.e., functions f for which there is a partition $0 < a_1 < a_2 < \ldots < a_k < 1$ of $[0, 1[$ such that f is constant on $[a_i, a_{i+1}[$ for $i = 0, 1, \ldots, k$ (here $a_0 = 0$, and $a_{k+1} = 1$).

A topology is defined on X^* if $N = N(a, b, V, \varepsilon)$ is a sub-basic neighbourhood of $f \in X^*$ whenever

(i) $0 \leqslant a < b < 1$, f is constant on $[a, b[$, V is a neighbourhood of $f(a)$, and $\varepsilon > 0$;

(ii) $h \in N$ means that $|\{t \in [a, b[: h(t) \notin V\}| < \varepsilon$, where $|\cdot|$ denotes Lebesgue measure.

Remark. This definition is a modification of the definition of a topology on X^* given by Hartman and Mycielski in [7] for the case where X is a topological group, and it is not hard to see that in such a case the two definitions give the same topology (2). The main result sketched in [7] is that X^* is path-connected and locally path-connected. Our main theorem is

THEOREM 1. *For any space X, the space X^* is contractible and locally contractible.*

Proof. Let $g \in X^*$. Define φ: $X^* \times [0, 1] \to X^*$ by

$$\varphi(f, r)(t) = \begin{cases} g(t), & 0 \leqslant t < r, \\ f(t), & r \leqslant t < 1. \end{cases}$$

(2) When X is metric, another definition of X^* using all Lebesgue measurable functions $[0,1] \to X$ is given in [8].

Then $\varphi(f, 0) = f$, and $\varphi(f, 1) = g$. So to prove that X^* is contractible it is sufficient to prove that φ is continuous.

Let $(f, r) \in X^* \times [0, 1]$, and let $N_\varepsilon = N(a, b, V, \varepsilon)$ be a sub-basic neighbourhood of $\varphi(f, r)$. Then $\varphi(f, r)$ is constant on $[a, b[$. We distinguish two cases: $r \leqslant a$ and $r > a$.

Case 1. $r \leqslant a$. Then f itself is constant on $[a, b[$, and V is a neighbourhood of $f(a) = \varphi(f, r)(a)$. Hence $N_\delta = N(a, b, V, \delta)$ is a neighbourhood of f for any $\delta > 0$.

If $r < a$, then

$$\varphi(N_\varepsilon \times [0, a[) \subset N_\varepsilon.$$

If $r = a$, then

$$\varphi(N_{\varepsilon/2} \times]a - \varepsilon/2, a + \varepsilon/2[) \subset N_\varepsilon.$$

Indeed, if $h \in N_{\varepsilon/2}$ and $a - \varepsilon/2 < s < a + \varepsilon/2$, then $\varphi(h, s)$ differs on $[a, b[$ from h by a set of Lebesgue measure less than $\varepsilon/2$, while the Lebesgue measure of $\{t \in [a, b[: h(t) \notin V\}$ is also less than $\varepsilon/2$.

Case 2. $r > a$. Since $\varphi(f, r)$ is constant on $[a, b[$, V is now a neighbourhood of $g(a)$.

If $r < b$, then $f([r, b[) = \{g(a)\}$. Hence $M = N(r, b, V, \varepsilon/2)$ is a neighbourhood of f and

$$\varphi(M \times]r - \varepsilon/2, r + \varepsilon/2[) \subset N_\varepsilon.$$

If $r = b$, then $]b - \varepsilon, b + \varepsilon[$ is a neighbourhood of r, and

$$\varphi(X^* \times]b - \varepsilon, b + \varepsilon[) \subset N_\varepsilon.$$

If $r > b$, then $]b, 1[$ is a neighbourhood of r, and

$$\varphi(X^* \times]b, 1[) \subset N_\varepsilon.$$

This completes the proof that φ is continuous, and so that X^* is contractible.

It is easy to check that if M is any sub-basic neighbourhood of g, then $\varphi(M \times \{r\}) \subset M$ for $r \in [0, 1]$. It follows that $\varphi(M \times \{r\}) \subset M$ for $r \in [0, 1]$ and any basic neighbourhood M of g. Hence each basic neighbourhood of g is contractible, and so X^* is locally contractible.

There is a canonical map $i: X \to X^*$ such that, for $x \in X$, $i(x)$ is the constant step function with value x. For the case where X is a Hausdorff topological group, it is asserted in [7] that i is a closed embedding. We prove

PROPOSITION 1. *The map $i: X \to X^*$ is an embedding. If X is Hausdorff, then i is a closed embedding.*

Proof. Let $x \in X$, and let $N = N(a, b, V, \varepsilon)$ be a sub-basic neighbourhood of $i(x)$. Then $i^{-1}(N) = V$, whence i is continuous, and $i(V) = N \cap i(X)$, so i is an embedding.

Suppose now that X is Hausdorff, and $f \notin i(X)$. Then f is not constant, and so takes at least two distinct values. Suppose that f is constant on $[a, b[$ with value x, and f is constant on $[c, d[$ with value y, where $x \neq y$. Let U and V be disjoint neighbourhoods of x and y, respectively, and let

$$M = N\big(a, b, U, \tfrac{1}{2}(b-a)\big) \quad \text{and} \quad N = N\big(c, d, V, \tfrac{1}{2}(d-c)\big).$$

Then $M \cap N$ is a neighbourhood of f which does not contain a constant function.

PROPOSITION 2. *If* $\varphi\colon X \to Y$ *is continuous, then the induced map* $\varphi^*\colon X^* \to Y^*$, $f \mapsto \varphi \circ f$, *is continuous. Further, if* φ *is an embedding, so also is* φ^*.

Proof. Let $f \in X^*$, and let $N = N(a, b, V, \varepsilon)$ be a sub-basic neighbourhood of $g = \varphi \circ f$. Let $0 = a_0 < a_1 < \ldots < a_k < a_{k+1} = 1$ be a partition of $[0, 1]$ such that f is constant on each $[a_i, a_{i+1}[$. Let $[a_l, a_{l+r}[$ be the smallest interval of the a's containing $[a, b[$. Then g is constant on $[a_l, a_{l+r}[$. For each $i = 0, 1, \ldots, r-1$, choose a neighbourhood U_i of $f(a_{l+i})$ such that $\varphi(U_i) \subset V$. Then the intersection M of the sets $N\big(a_{l+i}, a_{l+i+1}, U_i, \varepsilon/(r+1)\big)$ for $i = 0, 1, \ldots, r-1$ is a neighbourhood of f such that $\varphi^*(M) \subset N$.

Suppose now that φ is an embedding. Let $f \in X^*$ and let $N = N(a, b, U, \varepsilon)$ be a sub-basic neighbourhood of f. Then $\varphi(U) = V \cap \varphi(X)$ for some neighbourhood V of $\varphi f(a)$, and

$$\varphi^*(N) = \varphi^*(X^*) \cap N(a, b, V, \varepsilon).$$

PROPOSITION 3. *If X is Hausdorff, so also is* X^*.

Proof. Let f and g be distinct elements of X^*. Then there is a $t \in [0, 1[$ such that $f(t) \neq g(t)$. Let $a, b \in [0, 1[$ be such that $a \leqslant t < b$, and f and g are both constant on $[a, b[$. Let U and V be disjoint neighbourhoods of $f(t)$ and $g(t)$, respectively, and let $\varepsilon = \frac{1}{2}(b-a)$. Then the sets $N(a, b, U, \varepsilon)$ and $N(a, b, V, \varepsilon)$ are disjoint neighbourhoods of f and g, respectively.

PROPOSITION 4. *The natural map* $\varphi\colon (X \times Y)^* \to X^* \times Y^*$ *is a homeomorphism.*

Proof. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projections. By Proposition 2, p^* and q^* are continuous, and so is $\varphi = (p^*, q^*)$.

On the other hand, φ has an inverse ψ, since if $f\colon [0, 1[\to X$ and $g\colon [0, 1[\to Y$ are step functions, so also is $(f, g)\colon [0, 1[\to X \times Y$.

We now prove that ψ is continuous.

Let $(f, g) \in (X \times Y)^*$, and let $N = N(a, b, U \times V, \varepsilon)$ be a sub-basic neighbourhood of (f, g). Then $N_1 = N(a, b, U, \varepsilon/2)$ and $N_2 = N(a, b, V, \varepsilon/2)$ are sub-basic neighbourhoods of f and g, respectively, such that $\psi(N_1 \times N_2) \subset N$.

Corollary 1. *The functor $X \mapsto X^*$ commutes with finite limits.*

This follows from Propositions 2 and 4.

Corollary 2. *If X has the structure of a topological semigroup, monoid, group, or groupoid, so also does X^*.*

Proof. The first three cases follow easily from Proposition 4. For the case of X being a topological groupoid we also need the part of Proposition 2 dealing with embeddings, from which we infer that if D is the domain of composition of the groupoid X, then D^* is the domain of composition of X^*.

Remark. The previous results show how to recover and strengthen the results of [7]. We now show the relation of X^* to a space EX described by Segal in [11].

For any space X, there is a simplicial space TX such that $(TX)_n = X^{n+1}$ (the $(n+1)$-fold Cartesian product), with face and degeneracy operators given by

$$\partial_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n),$$

$$s_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_i, x_i, \ldots, x_n).$$

The realization $|TX|$ of TX is the space EX. Explicitly, EX is the quotient of $\prod_{n\geqslant 0} \Delta^n \times X^{n+1}$ by the relation $(\varphi^* t, x) \sim (t, \varphi x)$ for $t \in \Delta^m$, $x \in X^{n+1}$ and a simplicial operator φ. Let Δ^n be the set of points $(t_1, t_2, \ldots, t_n)$ in $\boldsymbol{R}^n$ such that $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant 1$. Then a continuous map

$$\lambda_n\colon \Delta^n \times X^{n+1} \to X^*$$

is defined by letting $\lambda_n\big((t_1, t_2, \ldots, t_n), (x_0, x_1, \ldots, x_n)\big)$ be the step function with value x_i on $[t_i, t_{i+1}[$ $(t_0 = 0, t_{k+1} = 1)$. Further, the maps λ_n respect the identifications giving EX, and define a continuous bijection $\lambda\colon EX \to X^*$. (This definition of λ is due to Segal (private communication).)

The space EX is contractible, since it is the space $B\bar{X}$, where $\bar{X}$ is as on p. 107 of [10]. There is also a map $j\colon X \to EX$, namely the composite $X \to \Delta^0 \times X \to EX$, and it can be proved that if X is Hausdorff, then j is a closed embedding.

Since EX is defined as a quotient space, it cannot be expected that $E(X \times Y)$ is homeomorphic to $EX \times EY$ (although we have no counter-example). However, this equation does hold in the category of k-Hausdorff k-spaces (when $(TX)_n = X^{n+1}$ is the weak product). Notice also that, by Proposition 3, if X is Hausdorff, then so also is X^*, and hence EX is Hausdorff. This seems harder to prove directly.

We conclude this section by giving two other useful properties preserved under $X \mapsto X^*$.

PROPOSITION 5. *A bounded metric on X induces a metric on X^* compatible with its topology, and $i: X \to X^*$ is an isometry.*

Proof. Let ϱ be a bounded metric on X. Then the formula

$$d(f, g) = \int_0^1 \varrho\big(f(t), g(t)\big)\,dt$$

defines a metric on X^* such that $i: (X, \varrho) \to (X^*, d)$ is an isometry. We now verify that the metric d induces the given topology on X^*.

Let $N = N(a, b, V, \varepsilon)$ be a sub-basic neighbourhood of $f \in X^*$, where V is a ϱ-neighbourhood of $f(a)$ of radius r in X. Then the d-neighbourhood of f of radius $r\varepsilon$ is contained in N. Indeed, if

$$d(g, f) < r\varepsilon \quad \text{and} \quad A = \{t: \varrho\big(f(t), g(t)\big) > r\},$$

then $d(f, g) > r|A|$, and so $|A| < \varepsilon$, whence $g \in N$.

Conversely, let W be a d-neighbourhood of f of radius δ. Suppose that $0 = a_0 < a_1 < \ldots < a_{k+1} = 1$, and f is constant on $[a_i, a_{i+1}[$ for $i = 0, 1, \ldots, k$. Let $N_i = N(a_i, a_{i+1}, W_i, \delta')$, where W_i is the ϱ-neighbourhood of $f(a_i)$ of radius $\delta/2$. We assert that

$$N_0 \cap N_1 \cap \ldots \cap N_k \subset W.$$

To prove this, let g be contained in this intersection, let

$$A = \{t \in [0, 1): \varrho\big(f(t), g(t)\big) > \delta/2\} \quad \text{and} \quad B = [0, 1[\setminus A,$$

and let ϱ be bounded by m. Then

$$d(f, g) = \int_A \varrho\big(f(t), g(t)\big)\,dt + \int_B \varrho\big(f(t), g(t)\big)\,dt < m(k+1)\delta' + \delta/2$$

which is less than δ for sufficiently small δ'. (Our proof requires the metric ϱ to be bounded, although the necessity of this is not stated in the corresponding result in [7] (3).)

PROPOSITION 6. *If X is completely regular, so also is X^*.*

The proof of this proposition is similar to the preceding one, since a space is completely regular if and only if its topology is given by a family of bounded pseudo-metrics.

3. The Graev free topological group. Let X be a space with base point e. The *Graev free topological group* on X is a continuous map $i: X \to F(X)$ of X into a topological group $F(X)$, where $i(e)$ is the identity of $F(X)$, and i is universal for continuous maps from X to topological groups taking e to the identity. For further background to the following theorem, see [12] (and the review of [12] in Mathematical Reviews).

(3) An example where ϱ is unbounded and the metric topology on X^* is not the above topology is given in [1].

THEOREM 2. *Let X be a functionally separable space with base point e. Then $F(X)$, the Graev free topological group on X, is Hausdorff.*

Proof. Note that for general spaces the condition of functionally separable is stronger than Hausdorff, but for topological groups (indeed, for completely regular spaces) the conditions are equivalent.

To prove that $F(X)$ is Hausdorff, it suffices to construct for every non-identity element a of $F(X)$ a continuous homomorphism θ from $F(X)$ to a Hausdorff topological group G such that $\theta(a)$ is not the identity of G.

It is standard that $F(X)$ is generated by the set $X\setminus\{e\}$ (in fact, $F(X)$ is algebraically free on $X\setminus\{e\}$). So there are distinct elements $y_1, y_2, \ldots, y_r$ of $X\setminus\{e\}$ such that a belongs to the group generated by $Y = \{y_1, y_2, \ldots, y_r\}$. Let H be the free group on Y with the discrete topology, and let $G = H^*$, so that G is a path-connected, Hausdorff topological group.

Since X is functionally separable, it is easily seen that for each $i = 1, 2, \ldots, r$ there is a continuous function g_i: $X \to [0, 1]$ such that $g_i(y_i) = 1$, and $g_i(e) = g_i(y_j) = 0$ for $i \neq j$. Since G is path-connected, there is, for each $i = 1, 2, \ldots, r$, a path λ_i: $[0, 1] \to G$ such that $\lambda_i(0) = e$ and $\lambda_i(1) = y_i$. Let g: $X \to G$ be the product in G of the maps $\lambda_1 \circ g_1$, $\lambda_2 \circ g_2, \ldots, \lambda_r \circ g_r$. Then $g(y_i) = y_i$ for $i = 1, 2, \ldots, r$, and $g(e) = e$. Also g is continuous, and so extends to a continuous morphism θ: $F(X) \to G$. Then $\theta(a)$ is not the identity of G (since the inclusion $Y \to F(X)$ extends to a homomorphism $H \to F(X)$ mapping $\theta(a)$ to a). This completes the proof.

4. Topological graphs and free topological groupoids. Let X be any functionally separable space with base point e. Then there is a continuous base point preserving injection of X into a compact Hausdorff topological group P. The proof of this is simple: the Stone-Čech compactification gives a continuous injection from X to a subspace of a product of copies I_a of the unit interval, and each I_a is embeddable as a subspace of the circle group T_a; this gives a map i: $X \to P$, where P is the product of the T_a, and it is easy to change i so that it maps e to e.

The purpose of Theorem 3 is to generalize this result to a form useful in the proof of a part of Theorem 4. The space with base point we replace by a *topological graph* Γ (see [3]), which consists of a space Γ of "arrows" and of "objects" $\mathrm{Ob}(\Gamma)$ together with maps ∂', ∂: $\Gamma \to \mathrm{Ob}(\Gamma)$ and u: $\mathrm{Ob}(\Gamma) \to \Gamma$ such that $\partial' u = \partial u = 1$. In particular, any space with base point e is a topological graph with space of arrows X, space of objects $\{e\}$ and the inclusion u: $\{e\} \to X$.

We replace topological groups by topological groupoids (see [3] and [4]). In particular, we shall need the tree groupoid [3] determined by a space Y, which has the arrow space $Y \times Y$, the object space Y, ∂' and ∂ are the first and second projections, u is the diagonal, and the composition is $(y, z)\cdot(x, y) = (x, z)$. This groupoid is denoted here by $Y \mathbin{\pmb{\times}} Y$.

THEOREM 3. *Let Γ be a functionally separable topological graph. Then there is a continuous injection j from Γ to a compact, functionally separable groupoid, and j is an embedding if Γ is completely regular and Hausdorff.*

Proof. As shown in Proposition 1 of [5], there is a continuous injection $\gamma\colon \Gamma \to \Delta$, where Δ is the Stone-Čech compactification of Γ.

Since Δ is Hausdorff, its space I of identities is closed in Δ. Since Δ is also compact, and hence completely regular, the space X, obtained from Δ by identifying I with a single point e, is also completely regular. Therefore, there is an embedding g of (X, e) into (P, e), where P is a compact Hausdorff group. Let $p\colon \Delta \to X$ be the identification map. Then $h = gp$ is a topological graph morphism of Δ into P.

Let $Y = \mathrm{Ob}(\Delta)$. Then Y is compact and Hausdorff. Let $k\colon \Delta \to Y \mathbin{\pmb{\times}} Y$ be the topological graph morphism (∂', ∂) of Δ to the tree topological groupoid $Y \mathbin{\pmb{\times}} Y$. Then k is injective on objects. Hence so also is $f = (h, k)$, which maps Δ to the product groupoid $P \times (Y \mathbin{\pmb{\times}} Y)$. But h is injective on each set $\Gamma(y, y')$ for $y, y' \in Y$, so it follows that f is injective. Since Δ, P, and Y are compact and Hausdorff, f is an embedding.

Finally, if Γ is completely regular and Hausdorff, then $\gamma\colon \Gamma \to \Delta$ is an embedding, and so also is $fi\colon \Gamma \to P \times (Y \mathbin{\pmb{\times}} Y)$.

Remark. In the proof of Theorem 3 we can also embed Y in a product

$$Q = \prod_{\beta \in B} T_\beta$$

of copies of the circle group. Taking $R = P \times Q$, we can then obtain a continuous graph injection $\Gamma \to R \times (R \mathbin{\pmb{\times}} R)$, where R is a product of copies of the circle group.

The free topological groupoid $F(\Gamma)$ on a topological graph Γ is defined in [3], p. 433. It can be constructed as a universal topological groupoid $U_\sigma(\Gamma^{(2)})$ as in the proof of Proposition 3 of [3], the existence of such a universal topological groupoid having been proved in [4].

It is proved in Theorem 1 of [3] that if G is a topological groupoid, $\sigma\colon \mathrm{Ob}(G) \to Y$ is continuous, and G, Y are Hausdorff k_ω-spaces, then the universal topological groupoid $U_\sigma(G)$ is Hausdorff. The proof uses an explicit construction of the topology on $U_\sigma(G)$. This result is used by Hardy and Morris [5] to prove that $F(\Gamma)$ is functionally separable if Γ is functionally separable. We shall give a different proof of this fact, and one which does not depend on any construction of the topology on $F(\Gamma)$.

We shall need to assume the standard and easily proved facts that $F(\Gamma)$ is algebraically the free groupoid on Γ, and that $\mathrm{Ob}(F(\Gamma)) = \mathrm{Ob}(\Gamma)$ topologically. (These follow from Propositions 5 and 9 of [4].)

THEOREM 4 (Hardy and Morris). *Let Γ be a functionally separable topological graph. Then $F(\Gamma)$, the free topological groupoid on Γ, is functionally separable. Further, if Γ is completely regular and Hausdorff, then the canonical injection $i\colon \Gamma \to F(\Gamma)$ is a topological embedding.*

Proof. Let a and b be distinct arrows in $F(\Gamma)$. It suffices to show that there is a topological groupoid morphism θ from $F(\Gamma)$ to a functionally separable topological groupoid such that $\theta(a) \neq \theta(b)$.

If a and b are both identities, say $a = 1_x$ and $b = 1_y$, where $x, y \in \mathrm{Ob}(\Gamma)$, then there is a continuous function $f\colon \mathrm{Ob}(\Gamma) \to \boldsymbol{R}$ such that $f(x) \neq f(y)$, and f extends to a morphism of topological graphs

$$f'\colon \Gamma \to \boldsymbol{R}\times\boldsymbol{R}.$$

Since $\boldsymbol{R}\times\boldsymbol{R}$ is a topological groupoid, f' extends to a morphism

$$\theta\colon F(\Gamma) \to \boldsymbol{R}\times\boldsymbol{R}$$

of topological groupoids. Clearly, $\theta(a) \neq \theta(b)$.

Suppose now that a is not an identity. Since Γ is functionally separable, there are a completely regular topological graph Γ' and a continuous bijection $j\colon \Gamma \to \Gamma'$. Then the induced map $F(j)\colon F(\Gamma) \to F(\Gamma')$ is also a continuous bijection. Let X be the space with base point obtained from Γ' by identifying the identities of Γ' with a single point, and let $p\colon \Gamma' \to X$ be the projection. Then X is functionally separable, and, by Theorem 2, so also is $F(X)$. Further, the composite $\theta = F(p)\circ F(j)$ satisfies $\theta(a) \neq \theta(b)$ (this follows from standard algebraic facts on universal groupoids [2]). This completes the proof that $F(\Gamma)$ is functionally separable.

Suppose now that Γ is completely regular and Hausdorff. By Theorem 3 there is a topological graph embedding $f\colon \Gamma \to G$, where G is a compact Hausdorff groupoid. Then f extends to a morphism $f^*\colon F(\Gamma) \to G$ of topological groupoids such that $f^* i = f$. Since f is an embedding, so also is i.

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