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Varieties of topological groups

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We define a variety $\underline{\underline{V}}$ of topological groups to be a family of topological groups closed under cartesian products, quotient groups and subgroups. The class of groups, which with some topology appear in $\underline{\underline{V}}$, is a variety of groups. A topological group $F(X, \underline{\underline{V}})$ with generating space X is a free topological group of $\underline{\underline{V}}$ on X if any continuous mapping of X into any G in $\underline{\underline{V}}$ can be extended to a continuous homomorphism of $F(X, \underline{\underline{V}})$ into G. Algebraically $F(X, \underline{\underline{V}})$ is the free group on the set X of the underlying algebraic variety.

If $\underline{\underline{V}}$ is the variety of all (all abelian) topological groups then $F(X, \underline{\underline{V}})$ is the free (free abelian) topological group defined by Markov. We generalize many of Markov's results; in particular, a necessary and sufficient condition for $F(X, \underline{\underline{V}})$ to exist is found and uniqueness is proved. Further, if $\underline{\underline{V}}$ is non-indiscrete, $F(X, \underline{\underline{V}})$ is disconnected. Any topological group Y in $\underline{\underline{V}}$ is a quotient group of $F(Y, \underline{\underline{V}})$.

A full variety $\underline{\underline{V}}$ is one which contains every topological group algebraically isomorphic to a group in $\underline{\underline{V}}$. If X is any Tychonoff space then $F(X, \underline{\underline{V}})$ exists, is Hausdorff and has X as a closed subset. Algebraically fully invariant subgroups of $F(X, \underline{\underline{V}})$ are closed. A topological group $F_M(X)$ is moderately free if its topology is maximal (with respect to X) and it is algebraically relatively free on X. We see that $F(X, \underline{\underline{V}})$ is moderately free on X. If $F_M(X)$ is in any variety $\underline{\underline{W}}$, then it is a quotient group of $F(X, \underline{\underline{W}})$. Further, $F_M(X)$ is a free topological group of the variety it generates. We prove that finitely generated subgroups of $F_M(X)$ have the discrete topology. If X

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is compact Hausdorff, then we show $F_M(X)$ is a normal space. In fact we explicitly determine the topology of $F_M(X)$ in this case.

A β-variety $\underline{\underline{V}}$ is one for which $F(X, \underline{\underline{V}})$ exists and is Hausdorff for each compact Hausdorff space X. For any β-variety $\underline{\underline{V}}$ and Tychonoff space X, $F(X, \underline{\underline{V}})$ exists, is Hausdorff and has X as a closed subset. We find a necessary and sufficient condition for a variety to be a β-variety. Whilst every full variety is a β-variety we show that the converse is false.

We prove that the family of all topological varieties with a given underlying algebraic variety is not a set. In fact the family of all β-varieties with a given underlying algebraic variety is not a set. We also prove the striking result that a variety generated by a family of topological groups of bounded cardinal is not full.

Finally, the varieties $\underline{\underline{V}}(R)$ and $\underline{\underline{V}}(T)$ generated by the additive group of reals and the circle group respectively, each with its usual topology, are examined. Each is shown to be a β-variety but not a full variety. Further, it is shown that a locally compact Hausdorff abelian group is in $\underline{\underline{V}}(T)$ if and only if it is compact. Thus R is not in $\underline{\underline{V}}(T)$ and consequently $\underline{\underline{V}}(R)$ properly contains $\underline{\underline{V}}(T)$.

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