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ON MIXED PARTIAL DERIVATIVES[1]

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It is well-known that if f is a real-valued function of two real variables then, under suitable conditions, its two mixed second order partial derivatives D_2D_1f and D_1D_2f are equal. (Here D_1f and D_2f denote the partial derivatives of f with respect to its first and second arguments, respectively.) In this note we record some lesser-known but interesting conditions which imply that $D_2D_1f = D_1D_2f$. We begin, however, by stating some well-known results.

1. CLASSICAL RESULTS

In 1740 Euler [5] (see also Clairaut [2]) published a proof that $D_1D_2f = D_2D_1f$, for all f. Of course his proof was not valid. Subsequently a number of mathematicians have considered the problem and proved theorems such as Theorems 1 and 2 below using the Mean Value Theorem. Theorem 1 can be found in [7] while Theorem 2 is a special case of symmetry properties of higher order Fréchet derivatives proved for example in [3].

THEOREM 1. *Let f be a real valued function of two real variables such that*

(i) D_1f exists in some neighbourhood of (a,b)

(ii) D_2D_1f exists in some deleted neighbourhood of (a,b)

and (iii) $L = \lim_{(x,y)\to(a,b)} D_2D_1f(x,y)$ exists.

Then $D_2D_1f(a,b)$ exists and equals L. If in addition

(iv) $D_2f(x,b)$ exists for all x in some neighbourhood of a,

then $D_1D_2f(a,b)$ exists and equals $D_2D_1f(a,b)$.

COROLLARY 1. *If f is a real-valued function of two real variables such that*

(i) D_1f, D_2f and D_2D_1f exist in some neighbourhood of (a,b)

and (ii) D_2D_1f is (jointly) continuous at (a,b)

then $D_1D_2f(a,b)$ exists and equals $D_2D_1f(a,b)$.

1. This was written in response to a question in the December 1975 issue of this Gazette.

COROLLARY 2. *If f is a real-valued function of two real variables such that*

(i) D_2D_1f *is (jointly) continuous at* (a,b)

and (ii) $D_1D_2f(a,b)$ *exists*

then $D_1D_2f(a,b) = D_2D_1f(a,b)$.

THEOREM 2. *If f is a real-valued function of two real variables such that*

(i) D_1f *and* D_2f *exist in some neighbourhood of* (a,b)

and (ii) D_1f *and* D_2f *are differentiable at* (a,b)

then $D_1D_2f(a,b) = D_2D_1f(a,b)$.

Corollary 1 and Theorem 2, which are proved in many analysis textbooks, are commonly associated with the names of Schwarz (cf.[9]) and Young (cf.[14], respectively. (see also [6].)

2. TOLSTOV'S RESULTS

In 1949 G.P. Tolstov published two papers on this subject. In the first [12] the following negative result is proved:

There exist functions f_1 and f_2 defined on a rectangle R which possess (jointly) continuous first order partial derivatives throughout R and are such that

(i) $D_1D_2f_1$ and $D_2D_1f_1$ exist throughout R but $D_2D_1f_1$ and $D_1D_2f_1$ are unequal at all points of a set of positive measure

and (ii) $D_1D_2f_2$ and $D_2D_1f_2$ exist almost everywhere in R but $D_1D_2f_2$ and $D_2D_1f_2$ are unequal almost everywhere in R.

The following positive result announced by Tolstov in [11] is proved in his second paper [13]. (Indeed, he proves a marginally stronger result than Theorem 3, below.)

THEOREM 3.(Tolstov). *Let f be a real-valued function of two variables such that*

(i) D_1^2f, D_1D_2f, D_2D_1f *and* D_2^2f *exist throughout some domain* D. *(D_1^2f denotes the second partial derivative of f with respect to the first variable.)*

Then $D_1D_2f = D_2D_1f$ *almost everywhere in* D. *Further if also*

(ii) $D_1D_2f - D_2D_1f$ *is separately continuous in* D *(that is, continuous in each variable separately)* then $\qquad D_1D_2f = D_2D_1f$ *throughout* D.

Since a function possessing both first order partial derivatives is automatically separately continuous, an easy induction establishes the next theorem.

THEOREM 4. (Tolstov). *Let* f *be a real-valued function of two real variables possessing all possible partial derivatives of order* $\leq m$ *in a domain* D. *Then for every mixed partial derivative of order* $< m$, *the order of differentiation is immaterial, while for derivatives of order* m *this is true almost everywhere in* D.

While Tolstov's proof of Theorem 3 is too technically complicated to give here, we will attempt to give some insight by including a proof which requires more restrictive hypotheses.

It is perhaps surprising that Theorem 3 (like Theorem 2) requires the existence of D_1^2f and D_2^2f. Their appearance here should be compared with the situation for the improved versions of Green's Theorem given in [4], which require the existence of partial derivatives not occurring in Green's formula. In particular we recall Cafiero's version of Green's Theorem (see Theorem 12 of [4]).

THEOREM 5. (Cafiero). *Let* P *and* Q *be real-valued functions of two real variables defined on a domain* D *such that*

(i) D_1P, D_2P, D_1Q *and* D_2Q *exist throughout* D,

(ii) $D_1Q - D_2P$ *is Lebesgue integrable in* D

and (iii) P *and* Q *are locally bounded on* D.

Then
$$\int_{\partial R} P(x,y)dx + Q(x,y)dy = \iint_R [D_1Q(x,y) - D_2P(x,y)]dxdy$$
for every rectangle R *in* D.

Theorem 5 allows us to prove the following special case of Theorem 3.

PROPOSITION. *Let* f *be a real-valued function of two real variables such that*

(i) D_1^2f, D_1D_2f, D_2D_1f *and* D_2^2f *exist throughout some domain* D,

(ii) D_1f *and* D_2f *are locally bounded in* D

and (iii) $D_1D_2f - D_2D_1f$ *is Lebesgue integrable in* D.

Then $\qquad D_1D_2f = D_2D_1f$ *almost everywhere in* D.

Proof. Let R be a rectangle in D with vertices (a,b), (x,b), (x,y) and (a,y). By Cafiero's Theorem,

$$\iint_R (D_1D_2f(s,t) - D_2D_1f(s,t))dsdt$$

$$= \int_{\partial R} (D_1f(s,t)ds + D_2f(s,t)dt)$$

$$= f(x,b) - f(a,b) + f(a,y) - f(x,y) + f(x,y) - f(x,b) + f(a,b) - f(a,y)$$

$$= 0.$$

Hence by Fubini's Theorem, the repeated integral

$$F(x,y) = \int_b^y [\int_a^x D_1D_2f(s,t) - D_2D_1f(s,t)ds]dt$$

exists and equals zero. Now using the fact that every Lebesgue integrable function is equal almost everywhere to the derivative of its indefinite integral (see [8,p.48] we obtain

$$D_1D_2F(x,y) = D_1D_2f(x,y) - D_2D_1f(x,y)$$

almost everywhere in R. Since the left hand side is zero, the required result follows.

3. REMARKS and EXAMPLES

Theorem 3 is formally stronger than the global analogue of Theorem 2 in the sense that any function which satisfies the global analogue of Theorem 2 must, by Theorem 2, satisfy the conditions of Theorem 3. However, consideration of the functions f_3, f_4 and f_5 defined below shows that, with the exception just noted, none of Theorems 1, 2 or 3 implies (either of) the others.

$$f_3(x,y) = \begin{cases} \dfrac{x^m y^m}{x^{2m}+y^{2m}} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

where m is any integer ≥ 2.

$f_4(x,y) = \int_x^y g(t)dt,$ where g is a continuous nowhere differentiable function (see [10]).

$$f_5(x,y) = \begin{cases} (x-y)^4 \sin((x-y)^{-1}) & if\ x \neq y \\ 0 & if\ x = y. \end{cases}$$

Moreover, using a result proved in [1,p.27] we could replace f_5 by a function which satisfies the conditions of Theorem 2 at every point but whose second order partial derivatives all fail almost everywhere to be even separately continuous.

Finally we confess that none of the sufficient conditions given here for local equality of the mixed partial derivatives are by any means necessary. Each first order partial derivative of the function f_6 defined by

$$f_6(x,y) = xyg(x)g(y)$$

where g is a nowhere differentiable function, exists only on an axis but $D_1D_2f_6(0,0) = D_2D_1f_6(0,0)$.

The authors are not aware of any best possible result.

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