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Locally Compact Group Topologies on an Algebraic Free Product of Groups*

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It is proved that any homomorphism from a locally compact group G into a free products of groups with the discrete topology is either continuous, or trivial in the sense that the image of G lies wholly in a conjugate of one of the factors of the free product. Hence, any locally compact group topology on a (nontrivial) free product of groups is discrete.

1. INTRODUCTION

In [1], Dudley proved the striking fact that any homomorphism from a locally compact group into a free group with the discrete topology is continuous. Our aim is to generalize this result to the case where the range group is an arbitrary free product of groups. In fact, we show that any homomorphism from a locally compact group G into a free product of groups with the discrete topology is either continuous, or trivial in the sense that the image of G lies wholly in a conjugate of one of the factors of the free product.

Dudley proved his results by means of the introduction of "norms" into certain groups, free groups and free abelian groups in particular, and the exploitation of their special properties. His method, however, is not immediately applicable to the problem we consider, since a free product of groups is not generally normable. Our proof instead depends heavily on the Kurosh subgroup theorem for free products and a theorem of Iwasawa [6] on the structure of connected locally compact groups.

An immediate consequence of our result is that a (nontrivial) free product of groups does not admit any locally compact group topology other than the

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discrete one. Thus, in particular, the coproduct in the category of topological groups of a (nontrivial) family of topological groups is locally compact if and only if it is discrete (c.f. [1], Theorem 3]).

2. Preliminaries

In this section, we state a version of Kurosh's subgroup theorem (c.f. [3, 5, 8, 9]) and record three facts about subgroups of free products that we shall need.

THEOREM. *If $H = \prod_{i\in I}^{*} H_i$ is a free product of groups, for some index set I, and S is a subgroup of H, then S is itself a free product with decomposition $S = F^* \prod_{j\in J}^{*} (w_j^{-1}S_jw_j)$, where F is a free group, J is some index set, each w_j is an element of H, and each S_j is a subgroup of one of the H_i, $i \in I$.*

(1) If a subgroup S of H is divisible then S is a conjugate in H of a subgroup of H_i, for some $i \in I$.

(2) If a subgroup S of H is abelian then either S is a conjugate in H of a subgroup of H_i, for some $i \in I$, or S is isomorphic to the infinite cyclic group Z.

(3) If the elements of a subgroup S of H have bounded length (with respect to the H_i), then S is a conjugate in H of a subgroup of H_i, for some $i \in I$.

Finally, note that all topological groups considered here are Hausdorff.

3. Results

We begin with a lemma that is a special case of Dudley's results; however, we include a proof here that depends on the known algebraic structure of compact abelian groups and so is in keeping with our other proofs.

LEMMA. *If ϕ is any homomorphism of a compact group G into the infinite cyclic group Z, then $\phi(G) = \{0\}$.*

Proof. Suppose that ϕ is nontrivial. Then, $\phi(G)$ is isomorphic to Z, and so we can assume that ϕ is surjective.

If G is abelian, then since Z is projective, G is algebraically isomorphic to $K \times Z$, where K is the kernel of ϕ. But, since G is compact, it would then follow that Z is algebraically compact, which is false. (For information on algebraically compact groups, see [2, 7].) Hence, ϕ is trivial.

Now, let G be any compact group. For each $x \in G$, the closure of the group generated by x is a compact abelian group and its image under ϕ is, therefore, trivial. In particular, $\phi(x) = 0$, and since this holds for each $x \in G$, ϕ is trivial, as required.

THEOREM 1. *If ϕ is any homomorphism from a compact group G into a free product $H = \prod_{i \in I}^{*} H_i$, then $\phi(G) \leqslant x^{-1}H_j x$, for some $x \in H$ and $j \in I$.*

Proof. First, let us suppose that ϕ is onto and that there exist two nontrivial groups H_{i_1} and H_{i_2} in $\{H_i : i \in I\}$. Choose nontrivial elements $h_1 \in H_{i_1}$ and $h_2 \in H_{i_2}$. Let $g \in G$ be such that $\phi(g) = h_1 h_2$. If we now set $A = \overline{gp(\{g\})}$, the closure in G of the subgroup generated by g, we see that A is a compact abelian group and, therefore, $\phi(A)$ is an abelian subgroup of H containing $gp(\{h_1 h_2\})$. Then, as noted in Section 2, we have either (i) $\phi(A)$ is algebraically isomorphic to Z, or (ii) $\phi(A) \leqslant y^{-1}H_j y$ (for some $y \in H$ and $j \in I$). Case (i) is shown to be impossible by the Lemma. If Case (ii) holds, then the elements of $\phi(A)$ have bounded length, which is nonsense since $gp(\{h_1 h_2\}) \leqslant \phi(A)$. Thus, if ϕ is onto, there can in fact be at most one nontrivial group in $\{H_i : i \in I\}$ and the statement of the theorem clearly holds.

If ϕ is not onto, the Kurosh theorem nevertheless assures us that $\phi(G)$, being a subgroup of $\prod_{i \in I}^{*} H_i$, is a free product $F * \prod_{j \in J}^{*} (w_j^{-1} S_j w_j)$, where F is a free group, S_j is a subgroup of some H_i, and each w_j is an element of H. By the argument of the first paragraph, this free product can have only one factor; that is, either (a) $\phi(G)$ is algebraically isomorphic to F, or (b) $\phi(G) \leqslant w_j^{-1} H w_j$, for some $i \in I$ and $w_j \in H$. If (a) holds, then since F is itself a free product of copies of Z, a further application of the argument in the first paragraph yields that $\phi(G)$ is algebraically to Z. This possibility is ruled out by the lemma, and so (b) holds, which proves the theorem.

THEOREM 2. *If ϕ is any homomorphism from a connected locally compact group G into a free product $H = \prod_{i \in I}^{*} H_i$, then $\phi(G) \leqslant x^{-1}H_j x$, for some $x \in H$ and $j \in I$.*

Proof. Since G is a connected locally compact group, Iwasawa's structure theorem [6, Sect. 4.13 of 10] says that G has a compact connected subgroup K and subgroups $R_1, R_2, \ldots, R_n$, for some positive integer n, where each R_j is topologically isomorphic to the additive group of reals with its usual topology, such that each $g \in G$ can be decomposed in the form: $g = r_1 r_2 \cdots r_n k$, with $k \in K$ and $r_j \in R_j$, for each j. Each R_j is divisible and so, as noted in Section 2, $\phi(R_j)$ is contained in some conjugate $w_j^{-1} H_j w_j$ in $\prod_{i \in I}^{*} H_i$, $j = 1, \ldots, n$. Also, by the previous theorem, $\phi(K) \subseteq w_i^{-1} H_i w_i$, for some $i \in I$. But, since each $g \in G$ has a representation $g = r_1 \cdots r_n k$, the lengths of the elements of $\phi(G)$ are, therefore, bounded. Thus, by the remarks in Section 2, $\phi(G) \leqslant x^{-1} H_l x$, for some $x \in H$ and $l \in I$.

We now present our main Theorem.

THEOREM 3. *Let ϕ be any homomorphism from a locally compact group G into a free product $H = \prod_{i\in I}^{*} H_i$. If H is given the discrete topology, then (at least) one of the following holds:*

(1) *ϕ is continuous,*
(2) *$\phi(G) \leqslant x^{-1}H_j x$, for some $j \in I$ and $x \in H$.*

Proof. Let C be the connected component of the identity in G. Then, C is a closed, hence, locally compact normal subgroup of G. Our proof is divided into two cases depending on whether $\phi(C)$ is trivial or not.

First, suppose that $\phi(C)$ is not equal to the identity, e, in H. The previous theorem tells us that $\phi(C) \leqslant x^{-1}H_j x$, for some $j \in I$ and $x \in H$. More precisely, $\phi(C) = x^{-1}Sx$, for some subgroup S of H_j. Since C is normal in G, we see that $x\phi(C)\,x^{-1} = S$ is a normal subgroup of $x\phi(G)\,x^{-1}$. Thus, for any word $w \in x\phi(G)\,x^{-1}$, $w^{-1}Sw \subseteq S \subseteq H_j$. This implies that $w \in H_j$; that is, that $x\phi(G)\,x^{-1} \leqslant H_j$. Hence, $\phi(G) \leqslant x^{-1}H_j x$, as required.

Second, suppose that $\phi(C) = \{e\}$. It is clear that $D = G/C$ is a locally compact (Hausdorff) totally disconnected group and that ϕ induces a homomorphism $\psi\colon D \to H$ such that $\phi = \psi \circ p$, where p is the natural quotient homomorphism of G onto D. If ψ is continuous, the continuity of p ensures that ϕ is also continuous. If $\psi(D) \leqslant x^{-1}H_j x$, the identity $\phi = \psi \circ p$ ensures that $\phi(G) \leqslant x^{-1}H_j x$. Hence, we need only show that the theorem holds for $\psi\colon D \to \prod_{i\in I}^{*} H_i$.

By [4, Theorem 7.7] every neighborhood of the identity in D contains a compact open subgroup K. If for any such K, $\psi(K) = \{e\}$ it is clear that the kernel of ψ is open in D and hence, ψ is continuous, and the theorem is proved. Suppose, on the other hand, that no compact open subgroup K of D has the property that $\psi(K) = \{e\}$. If K_1 and K_2 are two such subgroups, Theorem 1 shows that $\psi(K_1) \leqslant x_1^{-1}H_{i_1}x_1$ and $\psi(K_2) \leqslant x_2^{-1}H_{i_2}x_2$, for some x_1, x_2, H_{i_1} and H_{i_2}. Now, $K_1 \cap K_2$ is again a compact open subgroup of D, but $\psi(K_1 \cap K_2) \leqslant \psi(K_1) \cap \psi(K_2) \leqslant x_1^{-1}H_{i_1}x_1 \cap x^{-1}H_{i_2}x_2$, and by assumption, $\psi(K_1 \cap K_2) \neq \{e\}$. Therefore, the intersection $x_1^{-1}H_{i_1}x_1 \cap x_2^{-1}H_{i_2}x_2 \neq \{e\}$ and thus, $H_{i_1} = H_{i_2}$ and $x_1^{-1}H_{i_1}x_1 = x^{-2}H_{i_2}x_2$. Hence, $\psi(K) \subseteq x_1^{-1}H_{i_1}x_1$, for every compact open subgroup K of D. Fixing such a subgroup K, we have $x_1\psi(K)\,x_1^{-1} = S$, for some subgroup S of H_{i_1}. Let w be any element of $\psi(D)$ and choose $d \in D$ such that $w = \psi(d)$. Now, $d^{-1}Kd$ is again a compact open subgroup of D and so $\psi(d^{-1}Kd) \leqslant x_1^{-1}H_{i_1}x_1$. But then we have

$$\begin{aligned} H_{i_1} &\geqslant x_1\psi(d^{-1}Kd)\,x_1^{-1}, \\ &= x_1w^{-1}\psi(K)\,wx_1^{-1}, \\ &= x_1w^{-1}x_1^{-1}Sx_1wx_1^{-1}, \end{aligned}$$

which, since $S \leqslant H_{i_1}$, is possible only if $x_1 w^{-1} x_1^{-1} \in H_{i_1}$. However, this implies that $w \in x_1^{-1} H_{i_1} x_1$, and so $\psi(D) \leqslant x_1^{-1} H_{i_1} x_1$, which completes the proof.

COROLLARY 1. *A free product $\prod_{i\in I}^{*} H_i$ of at least two nontrivial groups does not admit any locally compact group topology other than the discrete one.*

COROLLARY 2. *If ϕ is any homomorphism from a locally compact group into a free group with the discrete topology then ϕ is continuous. Hence, a free group. does not admit any locally compact group topology other than the discrete one.*

Remarks. (1) In [11] the authors asked: What are the locally compact subgroups of $T \amalg T$, where T denotes the circle group and $T \amalg T$ denotes the coproduct in the category of topological groups of two copies of T? It is now clear that if G is a locally compact subgroup of $A \amalg B$, for any topological groups A and B, G is either discrete or topologically isomorphic to a subgroup of A or B. Hence, the locally compact subgroups of $T \amalg T$ are either discrete groups or topologically isomorphic to T.

(2) It should be noted that our lemma is still valid if we replace Z by an arbitrary free abelian group. It then follows easily by our methods that any homomorphism from a locally compact group into a free abelian group with the discrete topology is continuous. This is also a special case of Dudley's results.

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