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Connected and Locally Connected Closed Subgroups of Products of Locally Compact Abelian Groups

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ABSTRACT. It is shown that any connected or locally connected closed subgroup of a product of locally compact abelian groups is topologically isomorphic to a product of locally compact abelian groups.

This paper is one in a series (Morris, 1972; Hunt et al, 1975; Brown et al, 1975) which attempts to expose the structure of closed subgroups of infinite products of locally compact abelian groups. In particular we mention the following Theorem (Brown et al, 1975):

THEOREM A

Let G be a closed subgroup of a product $\prod_{i\varepsilon I} R_i \times K$, where each R_i is an isomorphic copy of the topological group of real numbers, K is a compact abelian group and I is an index set. If G is connected or I is countable, then G is topologically isomorphic to a product $\prod_{j\varepsilon J} R_j \prod_{\ell\varepsilon L} Z_\ell \times K_1$, where K_1 is a compact group, each Z_ℓ is an isomorphic copy of the discrete group of integers, and J and L are some index sets. If G is connected, L = ϕ; if I is countable, J and L are countable.

It should be noted that Theorem A would be false if the assumption that G is connected or I is countable were omitted. Indeed Leptin, 1955 showed that there is closed subgroup of an uncountable product of copies of Z which is not topologically isomorphic to a product of locally compact abelian groups.

We prove the following theorem:

THEOREM.

Let G be a closed subgroup of a product $\prod_{i\varepsilon I} L_i$ of locally compact abelian groups, for some index set I. If G is connected or locally connected then G is topologically isomorphic to a product of locally compact abelian groups. Indeed, if G is connected then it is topologically isomorphic to $\prod_{j\varepsilon J} R_j \times K$, where K is a compact connected group and J is some index set. If G is locally connected then it is topologically isomorphic to $\prod_{j\varepsilon J} R_j \times K_1 \times D$, where K_1 is a compact connected locally connected group and D is a discrete group.

PROOF.

Let G be connected. Define P_i, iεI, to be the projection mappings of $\prod_{i\varepsilon I} L_i$ onto L_i. Then $p_i(G)$ is connected. So the closure of $p_i(G)$, $\overline{p_i(G)}$, is a connected locally compact abelian group. Further, G is a closed subgroup of $\prod_{i\varepsilon I} \overline{P_i(G)}$. As each $\overline{p_i(G)}$ is a connected locally compact abelian group, it is topologically isomorphic to a product $R^{n_i} \times K_i$, where n_i is a non-negative integer and K_i is a compact group. (Theorem 9.14 of Hewitt and Ross, 1963). So G is topologically isomorphic to a closed subgroup of a product $\prod_{m \in M} R_m \times K$, where M is some index set and $K = \prod_{i \in I} K_i$. As G is connected, Theorem A implies that G is topologically isomorphic to $\prod_{j \in J} R_j \times K_1$, where K_1 is compact. As G is connected, K_1 must be connected too.

Now let G be locally connected. If A denotes the component of the identity in G, then A is a closed connected subgroup of $\prod_{i \in I} L_i$ and so, by the above paragraph, is topologically isomorphic to $\prod_{j \in J} R_j \times K_1$. As K_1 is compact and connected, it is divisible (Theorem 24.25 of Hewitt and Ross, 1963). So A is an open divisible subgroup of G and 6.22(b) of Hewitt and Ross, 1963 implies that G is topologically isomorphic to $A \times (G/A)$; that is, G is topologically isomorphic to $\prod_{j \in J} R_j \times K_1 \times D$, where D is the discrete group G/A. Finally observe that as K_1 is a continuous open image of the locally connected group G, it too is locally connected.

COROLLARY

If G is a locally connected closed subgroup of a product $\prod_{i \in I} R_i \times K$, where K is a compact abelian group and I is an index set, then G is topologically isomorphic to a product $\prod_{j \in J} R_j \times K_1 \times Z^n \times F$, where K_1 is a compact connected locally connected abelian group, n is a non-negative integer and F is a finite discrete group.

PROOF

This is an immediate consequence of the above theorem and the fact, proved by Morris (1972), that a discrete subgroup of $\prod_{i \in I} R_i \times K$ is isomorphic to $Z^n \times F$.

REMARK

It would be interesting to know if a closed subgroup of a countable product of locally compact abelian groups is necessarily topologically isomorphic to a product of locally compact abelian groups. A positive answer would mean that Noble's result (1970), that every closed subgroup of countable product of locally compact abelian groups satisfies Pontryagin duality would follow from Kaplan's result (1948), that every product of locally compact abelian groups satisfies Pontryagin duality.

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