CLOSED SUBGROUPS OF PRODUCTS OF REALS

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1. *Introduction*

It is well known [3; Theorem 9.11] that any closed subgroup of a finite product R^n of reals is topologically isomorphic to $R^a \times Z^b$, where Z denotes the discrete group of integers. For some time the second author has conjectured that an analogous result holds for infinite products of reals. To be precise, the conjecture is that any closed subgroup of an infinite product of reals is topologically isomorphic to a product of copies of R and Z. This conjecture is supported primarily by two known results:

(I) Any closed real vector subspace of an infinite product of reals is topologically isomorphic to a product of reals ([4] and [9]).

(II) Any locally compact subgroup of an infinite product of reals is topologically isomorphic to $R^a \times Z^b$, for some integers a and b [5].

We prove here that any closed *connected* subgroup of a product of reals is topologically isomorphic to a product of reals. In fact, we prove

THEOREM A. *Let G be a closed subgroup of a product $\prod_{\alpha \in I} R_i$ of reals. Then the following conditions are equivalent*:

(i) *G is connected*

(ii) *G is divisible*

(iii) *G is a real vector subspace of $\prod_{\alpha \in I} R_\alpha$*

(iv) *G is topologically isomorphic to a product of reals.*

We also prove:

THEOREM B. *Let G be a closed subgroup of a product $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$, where T_α denotes the compact circle group. Then G is connected if and only if it is divisible. Further G has a maximum compact subgroup K, such that if G is connected then K is connected* (*and divisible*) *and G/K is topologically isomorphic to a product of reals.*

2. *Proofs of theorems*

Proof of Theorem A. (i) $\Rightarrow$ (ii). Let $g \in G$ and n be any positive integer. We have to show that the element $(1/n)\, g \in \prod_{\alpha \in I} R_\alpha$ is, in fact, an element of G. Since G is closed in $\prod_{\alpha \in I} R_\alpha$, it suffices to show that every neighbourhood U of $(1/n)\, g$ inter-

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sects G. Being a neighbourhood in $\prod_{\alpha \in I} R_\alpha$, U contains a set

$$O_{\alpha_1} \times O_{\alpha_2} \times \ldots \times O_{\alpha_m} \times \prod_{\alpha \in J} R_\alpha,$$

where $J = I \diagdown \{\alpha_1, \ldots, \alpha_m\}$ and each O_{α_i} is open in R_{α_i}. Let p be the projection map of $\prod_{\alpha \in I} R_\alpha$ onto $R^m = R_{\alpha_1} \times R_{\alpha_2} \times \ldots \times R_{\alpha_m}$. Then the closure $\overline{p(G)}$ of $p(G)$ in R^m is a closed connected subgroup of R^m. Thus by [3; Theorem 9.11] $\overline{p(G)}$ is topologically isomorphic to R^k, for some k. In particular, $\overline{p(G)}$ is a divisible subgroup of R^m. As $p(g) \in \overline{p(G)}$ we must also have

$$p\left(\frac{1}{n} g\right) = \frac{1}{n} p(g) \in \overline{p(G)}.$$

(We are using the uniqueness of n-th roots in R^m.) Noting that $O_{\alpha_1} \times O_{\alpha_2} \times \ldots \times O_{\alpha_m}$ is an open neighbourhood of $p((1/n) g)$, we see that

$$O_{\alpha_1} \times O_{\alpha_2} \times \ldots \times O_{\alpha_m} \cap p(G) \neq \varnothing.$$

Hence,

$$\left(O_{\alpha_1} \times O_{\alpha_2} \times \ldots \times O_{\alpha_m} \times \prod_{\alpha \in J} R_\alpha\right) \cap G \neq \varnothing$$

which in turn implies that $U \cap G \neq \varnothing$. This is all we need to see that G is divisible.

(ii) $\Rightarrow$ (iii). Let $r \in R$ and $g \in G$. Let q_n be a sequence of rational numbers converging to r. Since G is a divisible subgroup of $\prod_{\alpha \in I} R_\alpha$, we see that $q_n g \in G$, for each n. However, $q_n g$ converges to rg which, since G is closed, implies $rg \in G$. Thus G is a real vector subspace of $\prod_{\alpha \in I} R_\alpha$.

(iii) $\Rightarrow$ (iv). To see this we simply note three facts:

(a) [9; Theorem 1.4]. A real locally convex topological vector space has the weak topology if and only if it is topologically isomorphic to a vector subspace of a product of reals.

(b) [4; Theorem 3]. A real complete locally convex topological vector space has the weak topology if and only if it is topologically isomorphic to a product of reals.

(c) A closed subspace of a complete topological vector space is complete.

(iv) $\Rightarrow$ (i). Trivial.

Theorem A provides a complete answer to the question: what complete connected topological groups can be obtained from R using the operations of forming cartesian products and subgroups? We now investigate what happens if we also allow the operation of forming quotient groups. In the terminology of [**1, 2, 5, 6, 7**]: What complete connected topological groups are in the variety of topological groups generated by the reals? In view of [**1**; Theorem 2], this is equivalent to asking: What are the closed connected subgroups of $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$? Theorem B, while not satisfactorily answering this, provides some interesting information.

LEMMA. *Let G be a closed subgroup of $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$. Then G has a maximum compact subgroup K and G/K is topologically isomorphic to a closed subgroup of $\prod_{\alpha \in I} R_\alpha$.*

Proof. Let K consist of all elements a of G with the property that $p_\alpha(a) = 0$, for all $\alpha \in I$, where p_α denotes the projection of $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$ onto the α-th co-ordinate. Clearly K is closed and is a group. Since $\prod_{\alpha \in J} T_\alpha$ is compact so too is K. Now suppose C is any compact subgroup of $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$. Then $p_\alpha(C)$ is a compact subgroup of R_α, which implies it must be the trivial subgroup $\{0\}$, for each $\alpha \in I$. Hence $C \subseteq K$. Thus K is indeed the maximum compact subgroup of G.

Now consider G/K. It is a subgroup of $(\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha)/K$. Indeed, since K is compact, G/K is a closed subgroup. Noting that $p_\alpha(a) = 0$ for all $\alpha \in I$, we see that $(\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha)/K$ is topologically isomorphic to $\prod_{\alpha \in I} R_\alpha \times D$, where D denotes $\prod_{\alpha \in J} T_\alpha/K$. Let f be the natural projection of $\prod_{\alpha \in I} R_\alpha \times D$ onto $\prod_{\alpha \in I} R_\alpha$. As D is compact f is a closed mapping. So $f(G/K)$ is a closed subgroup of $\prod_{\alpha \in I} R_\alpha$. From the definition of K, it is obvious that the mapping f, when restricted to G/K, is one to one. Since f is a closed mapping, this implies that f, when restricted to G/K, is a topological group isomorphism—which completes the proof.

Notation. In the proof of Theorem B, K will denote the maximum compact subgroup of G.

Proof of Theorem B. Assume G is connected let $g \in G$ and n be any positive integer. Define S to be the set of all elements x in $\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$ such that $nx = g$. We are required to prove that $S \cap G \neq \varnothing$.

Clearly $S = \prod_{\alpha \in I} P_\alpha \times \prod_{\alpha \in J} P_\alpha$, where for each $\alpha \in I$; P_α is a single point, and for each $\alpha \in J$, P_α is a set with precisely n elements. So S is compact.

Consider any finite subset F of $I \cup J$, and p_F the projection map of

$$\prod_{\alpha \in I} R_\alpha \times \prod_{\alpha \in J} T_\alpha$$

onto

$$R_{\alpha_1} \times R_{\alpha_2} \times \dots \times R_{\alpha_i} \times T_{\alpha_{i+1}} \times T_{\alpha_{i+2}} \times \dots \times T_{\alpha_j},$$

where

$$F = \{\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_j\}.$$

Now the closure $\overline{p_F(G)}$ of $p_F(G)$ is a closed connected subgroup of

$$R_{\alpha_1} \times \dots \times R_{\alpha_i} \times T_{\alpha_{i+1}} \times \dots \times T_{\alpha_j}.$$

This implies, by [**3**; Theorem 9.14], that $\overline{p_F(G)}$ is topologically isomorphic to $R^k \times C$, where C is a compact connected group. However, by [**8**], a compact connected group is divisible. Hence $\overline{p_F(G)}$ is divisible. So there exists an element $y \in \overline{p_F(G)}$ such that

$ny = p_F(G)$. Consequently, $p_F{}^{-1}[\overline{p_F(G)} \cap p_F(S)] \neq \varnothing$; that is, $p_F{}^{-1}(\overline{p_F(G)}) \cap S \neq \varnothing$. If we put $R_F = p_F{}^{-1}((\overline{p_F G}) \cap S)$, we see that R_F is a closed non-empty subset of S.

For any two finite subsets F_1 and F_2 of $I \cup J$, we clearly have

$$R_{F_1} \cap R_{F_2} \supseteq R_{F_1 \cup F_2} \neq \varnothing.$$

Thus the collection $\{R_F : F \text{ finite} \subseteq I \cup J\}$ has the finite intersection property. Since S is compact and each R_F is closed, we then have that

$$\bigcap_{\substack{F \text{ finite} \\ F \subseteq I \cup J}} R_F \neq \varnothing.$$

Thus there exists a $w \in S$, such that for each finite subset F of $I \cup J$,

$$p_F(w) \in \overline{p_F(G)} \tag{1}$$

We claim that $w \in S \cap G$. To see this, let U be any open neighbourhood of w. Without loss of generality, we can assume

$$U = O_{\alpha_1} \times \ldots \times O_{\alpha_i} \times O_{\alpha_{i+1}} \times \ldots \times O_{\alpha_i} \times \prod_{a \in J \setminus F} R_\alpha \times \prod_{\alpha \in J \setminus F} T_\alpha,$$

where $F = \{\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_j\}$ and each O is open. Then $O_{\alpha_1} \times \ldots \times O_{\alpha_j}$ is an open neighbourhood of $p_F(w)$, which, by (1), implies

$$O_{\alpha_1} \times \ldots \times O_{\alpha_j} \cap p_F(G) \neq \varnothing.$$

So $U \cap G \neq \varnothing$. Since G is closed, w must be an element of G—as required. So G is divisible.

Conversely, assume G is divisible. Let a be any element of K and n any positive integer. Since $a \in K$, by the lemma, $p_\alpha(a) = 0$ for all $\alpha \in I$. Since G is divisible, there exists an element $b \in G$ such that $nb = a$. Obviously $p_\alpha(b) = 0$, for all $\alpha \in I$. So $b \in K$. Consequently, K is divisible. Note that, by **[8]**, K is divisible if and only if it is connected.

As G is divisible, G/K is divisible. By the lemma and Theorem A, this implies that G/K is topologically isomorphic to a product of reals.

Finally, to see that G is connected simply note that K and G/K are connected. This completes the proof.

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Note added in proof. Since this paper was written, Ronald Brown, Philip J. Higgins and Sidney A. Morris have shown that any closed subgroup of a countably infinite product of reals is topologically isomorphic to a product of copies of R and Z. Details of this and more general results will appear in a paper entitled " Countable products and sums of lines and circles: their closed subgroups, quotients and duality properties ".