Math. Z. 143, 119–121 (1975)

The Homotopy Groups of an n-Fold Wedge

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The famous Hilton-Milnor Theorem [10, 15, 14, 3] provides a convenient method for computing homotopy groups of wedges $X_1 \vee X_2 \vee \cdots \vee X_n$, where each X_i is the suspension of a path connected CW-complex. While Porter has given in Theorem 1 of [12] a result which implies a method for computing homotopy groups of wedges of non-suspensions, his result appears to have been overlooked. In this paper we prove a semi-simplicial result which seems of interest in itself and which yields Porter's formula on the homotopy type of a wedge. We also give some examples which show the utility of this formula. For example, if $X_1, \ldots, X_n$ are 1-connected spaces of the homotopy type of a CW-complex, then

$$\pi_3(X_1 \vee X_2 \vee \cdots \vee X_n) = \pi_3(X_1) \times \pi_3(X_2) \times \cdots \times \pi_3(X_n) \times \prod_{1 \leqq j < i \leqq n} \pi_3(\Omega X_i \wedge \Omega X_j \wedge S^1).$$

By a *space* we shall mean a pointed k-Hausdorff k-space of the homotopy type of a CW-complex. (A topological space X is a *k-space* if a subset U of X is closed in X whenever $f^{-1}(U)$ is closed in C for all compact Hausdorff C and each continuous map $f\colon C \to X$; further X is k-Hausdorff if $f(C)$ is closed in X for each such C and f.)

Let $X \wedge Y$ denote the smashed product and ΣX the reduced suspension of spaces X and Y. If $Z_1, \ldots, Z_n$ are spaces then we write

$$C(Z_1, \ldots, Z_n) = \bigvee_{i_1, \ldots, i_r} (Z_{i_1} \wedge \cdots \wedge Z_{i_r})$$

where the wedge is over all r-tuples $(i_1, \ldots, i_r)$ with $i_1, \ldots, i_r$ distinct, $i_1 > i_2$, $1 \leqq i_2 < i_3 < \cdots < i_r \leqq n$, and $2 \leqq r \leqq n$. The same notation is used in the case when the Z_i are pointed sets, groups (regarded as pointed sets with identity as base point), pointed simplicial sets, or simplicial groups.

Theorem. *If $X_1, \ldots, X_n$ are path connected spaces then there is a homotopy equivalence*

$$\Omega(X_1 \vee \cdots \vee X_n) \simeq \Omega X_1 \times \cdots \times \Omega X_n \times \Omega\Sigma\big(C(\Omega X_1, \ldots, \Omega X_n)\big).$$

Proof. Without loss of generality we may assume the X_i are realisations $|K_i|$ of pointed connected simplicial sets K_i. It is known [9] that Kan's functor G from pointed connected simplicial sets to simplicial groups satisfies $|G(K)| \simeq \Omega|K|$, so that in particular

$$\Omega X_i \simeq |G(K_i)|, \quad \Omega(X_i \vee \cdots \vee X_n) \simeq |G(K_1 \vee \cdots \vee K_n)|,$$

and

$$C(\Omega X_1, \ldots, \Omega X_n) \simeq \big|C\big(G(K_1), \ldots, G(K_n)\big)\big|.$$

Now Theorem 1 of [10] gives that

$$\Omega\Sigma\big|C\big(G(K_1), \ldots, G(K_n)\big)\big| \simeq \big|F\big(C\big(G(K_1), \ldots, G(K_n)\big)\big)\big|,$$

where F is Milnor's free group functor from pointed simplicial sets to simplicial groups. So the Theorem follows from the formula

$$G(K_1 \vee \cdots \vee K_n) = G(K_1) * \cdots * G(K_n)$$

(where $*$ denotes the free product of simplicial groups) and the following

Proposition (*c.f.* [11]). *Let* $A_1, \ldots, A_n$ *be simplicial groups. Then there is an isomorphism of simplicial sets.*

$$A_1 * \cdots * A_n \cong A_1 \times \cdots \times A_n \times F(C(A_1, \ldots, A_n)).$$

Proof. Let m be a natural number and let B_i be the set of m-simplices of A_i, so that B_i is a group. The group of m-simplices of $A_1 * \cdots * A_n$ is $B_1 * \cdots * B_n$. Now Theorem 5.1 of Gruenberg [4] gives an exact sequence

$$1 \to F(C(B_1, \ldots, B_n)) \xrightarrow{f} B_1 * \cdots * B_n \xrightarrow{p} B_1 \times \cdots \times B_n \to 1 \tag{!}$$

where p is the canonical map and f is given on the basis elements by $(b_{i_1}, \ldots, b_{i_r}) \mapsto [b_{i_1}, \ldots, b_{i_r}]$, where the latter expression is the r-fold commutator. (Our convention for free groups differs from that of [4] in that we take $F(S)$ to be defined for a set S with base point which becomes the identity in $F(S)$ – this convention is essential to give the expression in terms of smashed products.) The sequence (!) is natural with respect to morphisms of groups and implies a bijection

$$\psi\colon B_1 * \cdots * B_n \to B_1 \times \cdots \times B_n \times F(C(B_1, \ldots, B_n)),$$

also natural with respect to morphisms of groups. So ψ determines an isomorphism of simplicial sets, as required by the Proposition. This completes the proof of the Theorem.

In the particular case $n=2$, the Theorem becomes $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$, which has been noted in [7, p. 250], [5, p. 587] and [8]. The case $n=3$ gives the homotopy type of $\Omega(X \vee Y \vee Z)$ as

$$\begin{aligned}\Omega X \times \Omega Y \times \Omega Z \times \Omega\Sigma\,[&(\Omega Y \wedge \Omega X) \vee (\Omega Z \wedge \Omega X) \vee (\Omega Z \wedge \Omega Y)\\ &\vee (\Omega Y \wedge \Omega X \wedge \Omega Z) \vee (\Omega Z \wedge \Omega X \wedge \Omega Y)].\end{aligned}$$

(Note that this is a simpler expression for the case $n=3$ than the one obtained by using the formula for the case $n=2$ twice.) The final term of this product is the loops on a wedge of suspensions; if at least two of X, Y and Z are 1-connected then this final term can be evaluated either by repeated use of our Theorem or by the Hilton-Milnor Theorem.

As another example, let $n=4$ and $X_1 = X_2 = X_3 = X_4 = \mathbb{C}P^\infty$. Then we obtain the homotopy type of $\Omega(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$ as $S^1 \times S^1 \times S^1 \times S^1 \times \Omega(6S^3 \vee 8S^4 \vee 3S^5)$, where the last factor can be evaluated by the Hilton-Milnor Theorem.

For general n one can obtain closed forms for some low dimensional homotopy groups. For example, a connectivity analysis of the terms obtained either by repeating our formula or by using our formula and Hilton-Milnor, gives when each X_i is 1-connected a formula for $\pi_r(X_1 \vee \cdots \vee X_n)$, $2 \leqq r \leqq 6$ as

$$\pi_r(X_1) \times \cdots \times \pi_r(X_n) \times \pi_r(T_{n,2}) \times \cdots \times \pi_r(T_{n,r}),$$

where $T_{n,2}=0$ and for $3 \leqq r \leqq 6$

$$T_{n,r} = \Pi(\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_r} \wedge S^1),$$

the product being taken over all r-tuples $(i_1, \ldots, i_r)$ of distinct numbers with $1 \leqq i_2 < i_1$ and $i_2 < i_3 < \cdots < i_r \leqq n$. To consider higher π_r we would have to examine cross-effect terms in the Hilton-Milnor expansion.

Remark. Sections 1 and 2 of [12] give a formula for $\Omega T_i(X_1, \ldots, X_n)$ where $T_i(X_1, \ldots, X_n)$ is the subset of $X_1 \times \cdots \times X_n$ of points with at least i coordinates at the base point. It would be interesting to have a semi-simplicial version of this result – related work in group theory is given in [1, 2, 6], and in fact our Proposition seems to be related to Theorem 4.1 of [6].

Acknowledgment. The authors wish to acknowledge the financial assistance of the United Kingdom Science Research Council.

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(Received October 19, 1974)