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IS A FUNCTION THAT SATISFIES THE CAUCHY-RIEMANN EQUATIONS NECESSARILY ANALYTIC?

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1. THE LOOMAN-MENCHOFF THEOREM — AN EXTENSION OF GOURSAT'S THEOREM

It is well-known ([1]) that a complex-valued function $f = u + iv$, defined and analytic on a domain D in the complex plane satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D. The standard textbooks ([2]) avoid answering the question as to whether or not the converse holds. Most instead offer the following partial converse.

THEOREM 1. (Goursat): *If $f = u + iv$, defined on a domain D, is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in D,*

(ii) u, v satisfy the Cauchy-Riemann equations everywhere in D,

and if further

(iii) f is continuous in D,

(iv) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous in D,*

then f is analytic in D.

The remaining standard texts offer the stronger result:

THEOREM 2. *If $f = u + iv$, defined on a domain D, is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in D,*

(ii) u, v satisfy the Cauchy-Riemann equations everywhere in D,

and if further

(iii) u, v, as functions of two real variables, are differentiable everywhere in D ([3]),

then f is analytic in D.

Recently the authors began a search to discover precisely what is known regarding the converse. The only modern book we were able to find that addresses itself to this problem is Derrick [6]. He points out that far weaker conditions than those of Theorem 2 are known to imply analyticity but that the Cauchy-Riemann equations themselves do not imply analyticity.

Titchmarsh [19, p.70] presents the counterexample

$$f(z) = \begin{cases} e^{-1/z^4} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$$

He shows that f satisfies the Cauchy-Riemann equations everywhere but fails to be analytic at 0. (4,5)

Derrick goes on to suggest that the best converse result appears to be

THEOREM 3. (Looman-Menchoff) *If $f = u + iv$, defined on a domain D is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in D,*

(ii) u, v satisfy the Cauchy-Riemann equations everywhere in D,

and if further

(iii) f is continuous in D,

then f is analytic in D. (6)

Menchoff's proof (see Saks [15, p.199] or Menchoff [12, p.9]) is "..... based on the Lebesgue theory of integration and the Baire theory of categories of sets. It is undoubtedly one of the most elegant and unexpected applications of the modern theory of real functions to the elementary problems of an entirely classical aspect." (7)

Theorem 3 is clearly a significant improvement on Goursat's theorem — the standard result. But hidden in the literature are even better results!

2. EXTENSIONS OF THE THEOREMS OF GREEN, MORERA AND GOURSAT

The earliest contribution to the problem appears to be that of Paul Montel who, in a 1913 note in the Comptes Rendus, asserted the

THEOREM 4. *If $f = u + iv$, defined on a domain D is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in D,*

(ii) u, v satisfy the Cauchy-Riemann equations everywhere in D,

and if further

(iii) f is bounded in D,

then f is analytic in D. (8)

Recall that a function f on D is said to be locally bounded if it is bounded in some neighbourhood of each point of D. As analyticity in D is equivalent to analyticity in some neighbourhood of each point of D, we see that condition (iii) can be replaced by

(iii)' f *is locally bounded in* D.

As every continuous function is locally bounded we see that Theorem 4 with (iii) replaced by (iii)', implies Theorem 3.

Montel did not prove this result in his note [13], nor did he publish a proof elsewhere. (9) Nonetheless he did indicate how the proof is an "immediate application" of a strengthened version of the following classical result on exact differentials.

THEOREM 5 *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ *exist everywhere in K,*

(ii) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *everywhere in K,*

and if further

(iii) P, Q are continuous in K,

(iv) $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ *are continuous in K,*

then

$$\int_C P\,dx + Q\,dy = 0.$$

Of course this is a special case of the classical Green's theorem —
THEOREM 6. (10) *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ *exist everywhere in K,*

(ii) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is integrable in K,* (11)

and if further

(iii) P, Q are continuous in K,

(iv) $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ *are continuous in K,*

then

$$\int_C P\,dx + Q\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$$

In 1923 Looman [9], by weakening the hypotheses involved in Theorem 5, proved Theorem 3. Unfortunately the proof was found to contain a serious gap (see note 21). It was D.E. Menchoff who filled the gap, correct proofs appearing in Saks [15, p.199] and Menchoff [12, p.9].

Tolstoff [20] was the first to prove Montel's theorem (Theorem 4). Implicit in his work is the observation that whenever one has a Green-type theorem (see Theorem 6) and a Morera-type theorem one obtains a Goursat-type theorem (Theorem 1). For example, let us see how the classical Goursat theorem follows from the classical Green theorem (more accurately, from its corollary — Theorem 5 — on exact differentials) and the classical Morera theorem.

THEOREM 7. (Morera) (12) *If f, defined on a domain D, is such that*

(i) f is continuous in D,

(ii) $\int_{\partial R} f(z)dz = 0$ *for each rectangle R* (13) *in D,*

then f is analytic in D.

The proof of Goursat's theorem (Theorem 1) goes as follows. For any rectangle R (with sides parallel to the co-ordinate axes)

$$\int_{\partial R} f(z)dz = \int_{\partial R} (u + iv)dz = \int_{\partial R} u\,dx - v\,dy + i\int_{\partial R} v\,dx + u\,dy$$

$= 0$ by Theorem 5 and the Cauchy-Riemann conditions. Hence by Morera's theorem, f is analytic in D.

The moral of this proof is clear. If one can reduce the conditions involved in Morera's theorem and those involved in Green's theorem one can obtain a strengthened version of Goursat's theorem. For instance, one could deduce the Looman-Menchoff theorem (Theorem 3) from the classical Morera theorem and the following extension of Green's theorem due to Paul J. Cohen(14),(15).

THEOREM 8. *Let R be a closed rectangle. If P, Q are real-valued functions of two variables on R such that*

(i) $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ *exist everywhere in R,*

(ii) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is integrable in R,*

and if further

(iii) P, Q are continuous in R,

then

$$\int_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

As a final illustration of the equation Morera + Green → Goursat let us (*d'apres* Tolstoff [20]) weaken both Morera's and Green's theorems in order to prove Theorem 4. First a strong Morera.

THEOREM 9. *If $f = u + iv$, defined on a domain D is such that*

(i) f is integrable in D,

(ii) $\int_{\partial R} f(z)dz = 0$ for each rectangle R in D,

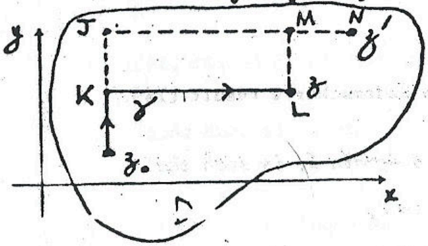
and if further

(iii) f is locally bounded in D,

(iv) f is separately continuous in D, [16]

then f is analytic in D.

PROOF. Fix a point $z_0 = x_0 + iy_0$ in D and for each $z \in D$ put



$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta,$$

the integral being taken along γ as in the diagram.

For $z' \in D$, by condition (ii) applied to the rectangle $JKLM$,

$$F(z') - F(z) = \int_M^N f(\zeta)d\zeta + \int_L^M f(\zeta)d\zeta.$$

By (iii), for some constant C and for all ζ in some neighbourhood of z, $|f(\zeta)| \le C$. Whence, if z' belongs to this neighbourhood

$$|F(z') - F(z)| \le C \quad (|N - M| + |M - L|)$$

and F is continuous at z. Denoting the real and imaginary parts of F by U, V respectively we have

$$U(x,y) = \int_{x_0}^{x} u(t,y)dt - \int_{y_0}^{y} v(x_0,t)dt$$

$$= \int_{x_0}^{x} u(t,y_0)dt - \int_{y_0}^{y} v(x,t)dt.$$

Hence, by the separate continuity of u

$$\frac{\partial U}{\partial x} = u(x,y), \qquad \frac{\partial U}{\partial y} = -v(x,y).$$

Similarly

$$\frac{\partial V}{\partial x} = v(x,y), \qquad \frac{\partial V}{\partial y} = u(x,y),$$

and U, V satisfy the Cauchy-Riemann equations throughout D. The Looman-Menchoff theorem now reveals the analyticity of F in D. Being analytic its derivative $F' = f$ is also analytic. Q.E.D.

Using a lemma of Menchoff [17], Tolstoff proves the following extension of Theorem 5.

THEOREM 10. *If P, Q are real-valued functions of two variables defined on a square K such that*

(i) $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ *exist everywhere in K,*

(ii) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *everywhere in K,*

and if further

(iii) P, Q are bounded in K, [18]

then

$$\int_{\partial R} P\,dx + Q\,dy = 0$$

for each rectangle R in K.

From Theorems 9 and 10 Tolstoff deduces [19] the result announced by Montel — Theorem 4.

We have seen that the classical theorems of Goursat, Green and Morera can be significantly improved upon. However, one can find still stronger versions in the literature. We thus conclude this section with a strengthening of Morera due to Rademacher [14] and a strengthening of Green due to Cafiero [3].

THEOREM 11. *If f, defined on a domain D is such that*

(i) f is integrable in D,

(ii) $\int_{\partial R} f(z)dz = 0$ *for each rectangle R in D,*

and if further

(iii) f is separately continuous in D,

then f is analytic in D. [20]

THEOREM 12. *If P, Q are real-valued functions of two variables defined on a domain D such that*

(i) $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ *exist everywhere in D,*

(ii) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is integrable in D,*

and if further

(iii) *P, Q are bounded in D,*

then

$$\int_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

for each rectangle R in D. (21)

3. MORE TECHNICAL RESULTS

We saw in §1 that a function which satisfies the Cauchy-Riemann equations everywhere need not be analytic. With this in mind one cannot fail but to be impressed by the distributional result

THEOREM 13. *If f is a distribution which satisfies the Cauchy-Riemann equations then f is analytic.* (22)

PROOF. The Cauchy-Riemann equations form an elliptic system of partial differential equations. It is known (23) that any distributional solution of a homogeneous elliptic system is in fact a C^∞ function. Thus by Goursat's theorem (Theorem 1) any distributional solution of the Cauchy-Riemann equations is analytic.

The essence of the above proof is as follows. A linear differential operator P can be expressed in terms of Fourier transforms by

$$P\,f(x) = \int e^{i\langle x,\xi\rangle}\, p(x,\xi)\,\hat{f}(\xi)\,d\xi,$$

where p is a polynomial in ξ with smooth functions of x as co-efficients. If f is vector-valued p is matrix-valued. P is said to be elliptic if the terms of highest ξ-degree in p form an invertible matrix for all x and all $\xi \neq 0$. This invertibility of p leads to a weak invertibility property for P, namely that in the class of all Fourier integral operators P is invertible modulo smoothing operators, i.e., one can obtain from $p(x,\xi)$ (by an essentially algebraic procedure) a function $q(x,\xi)$ such that the associated Fourier integral operator Q satisfies

$$QP = I + S, \qquad PQ = I + T$$

where I is the identity and S,T are smoothing operators (i.e., Sf and Tf are C^∞

for all distributions f). If f satisfies $Pf = 0$ we get that $0 = f + Sf$ and so $f = -Sf$ is C^∞. (24)

Let us look at the hurdles to be leapt in transforming this distributional result into a result about honest functions.

Let $f = u + iv$ be a given function. Obviously to apply Theorem 13 f must be a distribution, i.e., it must be locally integrable. Unfortunately this is not enough as the Cauchy-Riemann equations in Theorem 13 refer to the *distributional* derivatives. So one requires conditions on the given f in order that its classical and distributional derivatives agree, i.e., in order that

$$\iint_{R^2} \frac{\partial u}{\partial x}\,\phi\,dx\,dy = -\iint_{R^2} u\,\frac{\partial \phi}{\partial x}\,dx\,dy \text{ etc.},$$

for all test functions ϕ. Clearly a necessary condition for this to hold is that $\frac{\partial u}{\partial x}$ etc., be locally integrable; a sufficient condition (25) being that f is separately absolutely continuous. Finally note that the statement in Theorem 13 that f (as a distribution) is analytic means only that f is equal almost everywhere to an analytic function. To ensure that f is analytic one need only assume f to be separately continuous — see note 20.

From this disucssion we obtain Rademacher's result [14].

THEOREM 14. *If $f = u + iv$, defined on a domain D, is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist a.e. in D,*

(ii) u, v satisfy the Cauchy-Riemann equations a.e. in D,

and if further

(iii) f is locally integrable in D,

(iv) f is separately absolutely continuous in D,

(v) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are locally integrable in D,*

then f is analytic in D.

Due to the presence of conditions (iii), (iv) and (v) this result does not appear to be particularly strong. (26) However, the weakening of conditions (i) and (ii) to "a.e." suggest that one might be able to weaken the Looman-Menchoff theorem to, say,

If $f = u + iv$, defined on a domain D is such that

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ **exist a.e. in** D,

(ii) u, v **satisfy the Cauchy-Riemann equations a.e. in** D,

and if further

(iii) f **is continuous in** D,

then f **is analytic in** D.

This conjecture is false — a counterexample is given by Maker [10, p.267]. The difficulty lies in weakening the condition that the partial derivatives exist everywhere. That some such weakening is possible is illustrated by the version of Looman-Menchoff appearing in Saks [15, p.199].

THEOREM 15. *If* $f = u + iv$, *defined on a domain* D *is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in* D *except on a countable set,*

(ii) u, v *satisfy the Cauchy-Riemann equations a.e. in* D,

and if further

(iii) f *is continuous in* D,

then f *is analytic in* D.

If one wishes to weaken still further the conditions on the partial derivatives of f **one must impose more stringent conditions on** f **itself. For instance one may insist that** f **be separately continuous. (Of course, as mentioned in note 8, if the partial derivatives of** u, v **exist everywhere in** D f **is separately continuous.) Results along these lines have been obtained by Cafiero [4] and Fesq [7], the latter deriving the following.**

THEOREM 16. *If* $f = u + iv$, *defined on a domain* D *is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in* D *except on a countable union of sets of finite one-dimensional Hausdorff measure* (27, 28, 29),

(ii) u, v *satisfy the Cauchy-Riemann equations a.e. in* D,

and if further

(iii) f *is locally bounded in* D,

(iv) f *is separately continuous in* D,

then f *is analytic in* D.

The attempts by Shapiro [17], Cafiero [3] et al., to obtain a "best possible" Green's theorem led Fesq [7] to the

THEOREM 17. *Let R be a rectangle. If P, Q are real-valued functions of two variables on R such that*

(i) $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ *exist everywhere in R except on a countable union of closed sets of finite one-dimensional Hausdorff measure,*

(ii) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is Lebesgue integrable on R,*

and if further,

(iii) *P, Q are locally bounded in R,*

(iv) *P, Q are separately continuous in R except on a closed set of one-dimensional Hausdorff measure zero,*

then

$$\int_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

We conclude with two questions which, as far as we know, have not been answered.

QUESTION 1. *Suppose $f = u + iv$, defined on a domain D is such that*

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist everywhere in D,*

(ii) *u, v satisfy the Cauchy-Riemann equations everywhere in D,*

and suppose further that

(iii) *f is integrable in D.*

Does it follow that f is analytic in D?

QUESTION 2. *Suppose $f = u + iv$, defined on a domain D is such that u, v satisfy the Cauchy-Riemann equations everywhere in D. How "large" can the set of points of non-analyticity of f be? In particular, can it be of positive measure?*

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NOTES AND REMARKS

(1) This is a rare instance of a well-known result that is indeed well-known.

(2) For instance those authored by Ahlfors, Ash, Cartan, Churchill, Duncan, Fuchs and Shabat, Greenleaf, Jameson, Knopp, Pennisi, Sansone and Gerretson.

(3) u is said to be differentiable at (x_0,y_0) if there are real numbers α, β such that $|u(x_0 + h, y_0 + k) - u(x_0,y_0) - (\alpha h + \beta k)|/\sqrt{h^2 + k^2} \to 0$ as $(h,k) \to (0,0)$. If u is differentiable at (x_0,y_0) the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ exist there and necessarily $\alpha = \frac{\partial u}{\partial x}(x_0,y_0)$, $\beta = \frac{\partial u}{\partial y}(x_0,y_0)$. However, the mere existence of these partial derivatives at (x_0,y_0) does not imply the differentiability of u there. Such an implication can be made if further the partial derivatives exist near (x_0,y_0) and are continuous at (x_0,y_0). See M. Spivak, "Calculus on manifolds", Benjamin, N.Y. (1965) p.31.

(4) Indeed $(f(z) - f(0))/z \to \infty$ as $z \to 0$ with $\arg z = \pi/4$. This example appears in Looman [9, p.107].

(5) Observe that this example has an essential singularity at 0. It could not have a pole because a complex-valued function f for which $\frac{\partial f}{\partial x}$ exists at z_0 cannot have a pole at z_0. (Indeed it suffices to assume that f is continuous in x at z_0.)

(6) Warning. This theorem might suggest that if a function is continuous at a point z_0 and satisfies the Cauchy-Riemann equations at z_0 it is necessarily analytic at z_0. This is not the case! For example, [6, p.15], f, defined by $f(z) = z^5/|z|^4$ if $z \neq 0$, $f(0) = 0$ is not analytic at 0 but is continuous everywhere and does satisfy the Cauchy-Riemann equations at 0. To the best of our knowledge the strongest result in this direction is the standard one: if $f = u + iv$ is such that (i) u, v are differentiable at z_0, (ii) u, v satisfy the Cauchy-Riemann equations at z_0, then f is (complex) differentiable at z_0. See G.J.O. Jameson, "A first course in complex functions." Chapman and Hall, London (1970) p.35.

(7) Quote from Saks' review (Zentralblatt, <u>14</u> (1936), 167) of Menchoff [12].

(8) Although this result appears to be quite strong observe that condition (i) implies the separate continuity of f which in turn implies its measurability. See note 16.

(9) The result was however stated as a theorem in Menchoff's monograph [12] — one in a series edited by Paul Montel.

(10) See T.M. Apostol, "Mathematical analysis." Addison-Wesley (1957) p.289.

(11) We include (ii) as it is clearly necessary; of course (iv) implies (ii). Throughout this paper the term integrable means Lebesgue integrable and all integrals are Lebesgue integrals. However every bounded Riemann integrable function is Lebesgue integrable. Thus, with the exception of Theorem 11 and Question 1 at the end of the paper, Lebesgue can be replaced by Riemann throughout.

(12) See Saks and Zygmund [16, p.120].

(13) Throughout this paper all rectangles are assumed to have their sides parallel to the co-ordinate axes.

(14) See Cohen [5] — the same Cohen of Continuum Hypothesis fame.

(15) Such a proof would reverse the chronological order of things. Indeed Cohen's ideas were inspired by the proof of the Looman-Menchoff theorem. Incidentally, the extension of Green's theorem from rectangles to more general regions has been investigated by amongst others K. Menger, "On Green's formula", Proc. Nat. Acad. Sci. U.S.A., 26 (1940) 660-664; D.H. Potts, "A note on Green's theorem", J. Lond. Math. Soc., 26 (1), (1951) 302-304 — (see also Apostol, op. cit.); J. Ridder, "Über den Greenschen Satz in der Ebene", Nieuw. Arch. Wiskunde (2), 21 (1941) 28-32 and S. Verblunsky, "On Green's formula", J. Lond. Math. Soc., 24 (1949) 146-148.

(16) Note that a function f of two variables x, y which is separately continuous (i.e., continuous in x for each fixed y and continuous in y for each fixed x) need *not* be jointly continuous (i.e., continuous as a function of (x,y)). For example, if $f(x,y) = xy/(x^2 + y^2)$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$, f is jointly discontinuous at the origin but separately continuous there. René Baire has remarked that as late as 1898 analysts were surprised to discover that the two concepts are not equivalent. However, a separately continuous function is at least measurable in (x,y). See J.H. Michael and B.C. Rennie, "Measurability of functions of two variables", J. Aust. Math. Soc., 1 (1) (1959) 21-26. Note also that conditions (iii) and (iv) together imply condition (i).

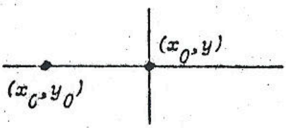
(17) See the second lemma on page 198 of Saks [15].

(18) Tolstoff fails to mention condition (iii) in his statement of the theorem but does use it in his proof. In fact it suffices to assume that for some

integrable functions ϕ, ψ, $|P(x,y)| \leq \phi(x)$ for all y, $|P(x,y)| \leq \psi(y)$ for all x, with similar conditions on Q. See E.W. Hobson "The theory of functions of a real variable. II". Dover, N.Y. (1926) §227, p.326.

(19) By the method outlined immediately after Theorem 7.

(20) Actually Rademacher's theorem is: if $f = u + iv$ is integrable in (x,y) and if for each x, u and v are integrable in y and for each y they are integrable in x, and if $\int_{\partial R} f(z)dz = 0$ for each rectangle R, then f is equal almost everywhere to an analytic function. However, a separately continuous function equal almost everywhere to an analytic function is in fact analytic. This follows from the Proposition: if ϕ, ψ are separately continuous real-valued functions of two variables which are equal almost everywhere then in fact they are equal everywhere. As suppose $\phi(x_0,y_0) \neq \psi(x_0,y_0)$, then ϕ, ψ must disagree at all points on some line segment through (x_0,y_0) parallel to the x-axis. For each point (x_0,y) on this segment, ϕ and ψ must disagree at all points on some line segment through (x_0,y) parallel to the y-axis. The union of all these line segments is not of measure zero.



(21) Interestingly enough (as Fesq [7] shows), even though $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$ do not occur in Green's formula, some assumptions regarding them must be made in order to obtain the formula. Indeed, even if the right-hand side of the formula is zero the statement is false without such assumptions. It was Tolstoff [21] who first realized this. Unfortunately this error appears in the papers of both Montel and Looman. See also Wilkosz [23].

(22) Compare Theorem 13 with Weyl's lemma which states: if u, v are L^2 functions on the unit disc D and if

$$\iint_D \left(u \frac{\partial\phi}{\partial x} - v \frac{\partial\phi}{\partial y}\right) dx\, dy = 0, \qquad \iint_D \left(u \frac{\partial\phi}{\partial y} + v \frac{\partial\phi}{\partial x}\right) dx\, dy = 0$$

for all test functions ϕ (i.e: C^∞ functions with compact support in D), then there is an analytic function $\tilde{u} + i\tilde{v}$ such that $u = \tilde{u}$ almost everywhere and $v = \tilde{v}$ almost everywhere. The conditions on u, v imply that their distributional derivatives satisfy the Cauchy-Riemann equations. The usual proof of Weyl's lemma is a short-circuited version of the proof outlined below for Theorem 13. (It proceeds directly to the smoothing operators S_ρ. See G. Springer, "An introduction to Riemann surfaces", Addison-Wesley (1957) p.199 for details.)

(23) See Lars Hörmander, "Linear partial differential operators". Spinger-Verlag, Berlin (1964). Corollary 4.1.7 on page 102 and Corollary 4.1.2 on page 101.

(24) In fact Q can only be constructed locally on compact subsets, but this is no problem since smoothness is a local property. When P is the Cauchy-Riemann operator on the unit disc the relevant family of smoothing operators S_ρ having the property $S_\rho f(z) = -f(z)$ for $|z| \leq 1-\rho$ is given by

$$S_\rho f(z) = -\frac{3}{\pi\rho^6}\int_0^{2\pi}\int_0^\rho f(z+re^{i\theta})(\rho^2-r^2)^2\, r dr d\theta.$$

(25) Perhaps not the strongest such.

(26) Nonetheless, it is not contained in any of the others.

(27) The one-dimensional Hausdorff measure of a set E in the plane is defined to be

$$\sup_{\varepsilon>0} \inf \{ \textstyle\sum_1^\infty diam\,(E_n) : E = \bigcup_1^\infty E_n,\ diam\,(E_n) < \varepsilon\}.$$

(See P.R. Halmos, "Measure theory". Van Nostrand, N.Y. (1950) p.53.) Any contour of finite length has finite one-dimensional Hausdorff measure, whereas a (non-trivial) rectangle has infinite one-dimensional Hausdorff measure.

(28) Condition (i) dates back to Besicovitch [1] who proved that if a function, defined on a simply-connected domain D, is continuous everywhere and (complex) differentiable everywhere except on a countable union of sets of finite linear measure, then in fact it is (complex) differentiable everywhere in D.

(29) Cafiero [4] has replaced condition (i) by a condition on the Dini derivatives of u, v. See also Menchoff [12].

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