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Varieties of topological groups II

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This paper is a sequel to one entitled "Varieties of topological groups". The variety $\underline{\underline{V}}$ of topological groups is said to be full if it contains every group which is algebraically isomorphic to a group in $\underline{\underline{V}}$. For any Tychonoff space X , the free group F of $\underline{\underline{V}}$ on X exists, is Hausdorff and disconnected, and has X as a closed subset. Any subgroup of F which is algebraically fully invariant is a closed subset of F . If X is a compact Hausdorff space, then F is normal. Let $\underline{\underline{V}}$ be a full Schreier variety and X a Tychonoff space, then all finitely generated subgroups of F are free in $\underline{\underline{V}}$.

A β-variety $\underline{\underline{V}}$ is one for which the free group of $\underline{\underline{V}}$ on each compact Hausdorff space exists and is Hausdorff. For any β-variety $\underline{\underline{V}}$ and Tychonoff space X , the free group of $\underline{\underline{V}}$ exists, is Hausdorff and has X as a closed subset. A necessary and sufficient condition for $\underline{\underline{V}}$ to be a β-variety is given.

The concept of a projective (topological) group of a variety $\underline{\underline{V}}$ is introduced. The projective groups of $\underline{\underline{V}}$ are shown to be precisely the summands of the free groups of $\underline{\underline{V}}$. A finitely generated Hausdorff projective group of a Schreier variety $\underline{\underline{V}}$ is free in $\underline{\underline{V}}$.

We will use the notation and terminology of [8]. Further by abelian (Schreier) variety we will mean a variety of topological groups for which the underlying variety of abstract groups is abelian (Schreier).

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1. Full varieties

DEFINITION. The group F is said to be *moderately free* on the space X, and will be denoted by $F_M(X)$, if,

(i) $\overline{F}$ is a relatively free abstract group with $\overline{X}$ as a free generating set,

(ii) the topology of F is the finest group topology (on $\overline{F}$) which will induce the same topology on $\overline{X}$.

The first theorem is a generalization of Theorems 4 and 22 of [7] and Theorem 4.5 of [8].

THEOREM 1.1. *Let $F_M(X)$ be moderately free on the Hausdorff space X, then $F_M(X)$ is Hausdorff and has X as a closed subset.*

Proof. By p. 32 of [1], X is a completely regular space. The main theorem of [10] then implies that there exists a Hausdorff group topology on $\overline{F_M(X)}$ which induces the given topology on $\overline{X}$ and has X as a closed subset. It then follows immediately from the definition of moderately free that $F_M(X)$ is Hausdorff and has X as a closed subset.

DEFINITION. The variety $\underline{\underline{V}}$ is said to be a full variety if whenever G is any group such that $\overline{G} \in \overline{\underline{\underline{V}}}$, then $G \in \underline{\underline{V}}$.

Clearly the varieties of Example 2.4 (a), (b) and (c) of [8] are full varieties. Note that if F is a free group of a full variety on any space X, then F is moderately free on X.

THEOREM 1.2. *If $\underline{\underline{V}}$ is a full variety, then for any Tychonoff space X, $F(X, \underline{\underline{V}})$ exists.*

Proof. This follows from the main theorem of [10] together with Theorem 2.6 of [8].

We omit the proof of the next theorem because it is similar to that of Theorem 4.4 of [8].

THEOREM 1.3. *Let $F_M(X)$ be moderately free on the space X. If A is a proper subgroup of $F_M(X)$ such that $\overline{A}$ is algebraically fully invariant, then $F_M(X)/A$ is moderately free.*

The following lemma is a corollary to the proof of Theorem 3.3 of [8].

LEMMA 1.4. *Let* $F_M(X)$ *be moderately free on the space* X *and* $\underline{\underline{V}}$ *the intersection of all full varieties containing* $F_M(X)$. *Then* $\underline{\underline{V}}$ *is a full variety and* $F_M(X)$ *is* $F(X, \underline{\underline{V}})$.

Markov [7] showed that if F is a Hausdorff free group then its derived group is a closed subset of F . In a conversation with the author, I.D. Macdonald conjectured that subgroups of F which are algebraically fully invariant are closed subsets of F . Theorem 1.5 shows that this is indeed the case even when F is moderately free.

THEOREM 1.5. *Let* $F_M(X)$ *be moderately free on the Hausdorff space* X . *If* A *is a proper subgroup of* $F_M(X)$ *such that* $\bar{A}$ *is a fully invariant abstract subgroup, then* A *is a closed subset of* $F_M(X)$. *Further* $F_M(X)/A$ *is moderately free on* X .

Proof. Let f be the natural mapping of $F_M(X)$ onto $F_M(X)/A$. Then f maps X one to one onto $f(X) = X'$. (See Theorem 4.4 of [8].) Define a topology τ on $\overline{X'}$ as follows: O is open in τ if and only if $f^{-1}(O) \cap X$ is open in X . By the main result of [10], there exists a Hausdorff group topology τ_1 on $\overline{F_M(X)/A}$ which induces the topology τ on $\overline{X'}$.

Clearly $\overline{F_M(X)/A}$ with the topology τ_1 belongs to every full variety containing $F_M(X)$. Thus by Lemma 1.4 the map f , from $F_M(X)$ onto $\overline{F_M(X)/A}$ with the topology τ_1 , is continuous. Therefore the quotient topology of $F_M(X)/A$ is Hausdorff and induces the topology τ on $\overline{X'}$. Consequently A is a closed subset of $F_M(X)$. Noting that $\overline{X'}$ with the topology τ is homeomorphic to X , it follows from Theorem 1.3 that $F_M(X)$ is moderately free on X .

DEFINITION. Let F be an abstract group with generating set X . Then $a \in F$ is said to be of length n with respect to X if n is the least integer N such that $a = x_1^{\varepsilon_1} \ldots x_N^{\varepsilon_N}$, where $\varepsilon_i = \pm 1$ and $x_i \in X$ for $i = 1, \ldots, N$. The set of all elements in F of length not greater than m will be denoted by F_m.

Clearly $F_1 = X \cup X^{-1}$ and F_m, $m > 1$, is the product in F of m copies of $\{X \cup X^{-1} \cup e\}$, where e is the identity of F.

The following lemma can be proved in a similar manner to Theorems 4 and 5 of [2].

LEMMA 1.6. *Let F be a Hausdorff group with a compact subspace X which generates F algebraically. Further, let the topology of F be the finest group topology (on $\overline{F}$) which induces the same topology on $\overline{X}$. Then the set V is open in F if and only if $V \cap F_n$ is open in the induced topology of F_n for each $n = 1, 2, \ldots$. Further, F is a normal space.*

THEOREM 1.7. *Let F be moderately free on the compact Hausdorff space X. Then F is a normal space and V is an open set in F if and only if $V \cap F_n$ is open in the induced topology of F_n for each $n = 1, 2, \ldots$.*

Proof. This follows immediately from Lemma 1.6 and Theorem 1.1.

THEOREM 1.8. *Let F be a Hausdorff group with a compact subspace X which generates F algebraically. If the topology of F has the property that V is open in F if and only if $V \cap F_n$ is open in F_n for each $n = 1, 2, \ldots$, then the topology of F is the finest group topology (on $\overline{F}$) which will induce the same topology on $\overline{X}$. In particular, if $\overline{F}$ is algebraically relatively free on $\overline{X}$, then F is moderately free on X.*

Proof. It is readily seen that every Hausdorff group topology on $\overline{F}$ which induces the given topology on $\overline{X}$ induces the same topology on $\overline{F}_n$, $n = 1, 2, \ldots$. The theorem then follows from Lemma 1.6.

DEFINITION. Let the set X be a free basis for the relatively free abstract group F . The subset Y of F is said to be regularly situated with respect to X if in the subgroup F_Y , generated algebraically by Y , it is impossible to find a sequence of elements with the following properties: the lengths of the elements with respect to Y exceed all bounds and their lengths with respect to X have a uniform bound.

Theorem 1.9 generalizes Theorem 10 of [2] and it can be proved similarly.

THEOREM 1.9. *Let* $\underline{\underline{V}}$ *be a full variety,* X *a compact Hausdorff space and* Y *a compact subset of* $F(X, \underline{\underline{V}})$. *Then the subgroup* F_Y *generated algebraically by* Y *is a closed subset of* $F(X, \underline{\underline{V}})$ *and the topology of* F_Y *is the finest group topology (on* $\overline{F}_Y$*) which induces the given topology on* Y *if and only if* Y *is regularly situated with respect to* X .

THEOREM 1.10. *Let* $\underline{\underline{V}}$ *be a full variety,* X *a Tychonoff space and* Y *a compact subset of* $F(X, \underline{\underline{V}})$. *If* Y *is regularly situated with respect to* X *, then the subgroup* F_Y *algebraically generated by* Y *is a closed subset of* $F(X, \underline{\underline{V}})$ *and the topology of* F_Y *is the finest group topology (on* $\overline{F}_Y$*) which induces the given topology on* Y . *In particular if* $\underline{\underline{V}}$ *is a Schreier variety* [9] *and* $\overline{Y}$ *is a free algebraic basis of* F_Y *, then* F_Y *is* $F(Y, \underline{\underline{V}})$.

Proof. Let $\beta(X)$ be the Stone-Čech compactification of X [5]. Then, by Theorems 1.1 and 1.2, $F(\beta(X), \underline{\underline{V}})$ exists and is Hausdorff. Let ϕ be the imbedding map of X in $\beta(X)$. Then ϕ is a continuous map of X into $F(\beta(X), \underline{\underline{V}})$. Therefore there exists a continuous homomorphism Φ of $F(X, \underline{\underline{V}})$ into $F(\beta(X), \underline{\underline{V}})$ such that $\Phi|X = \phi$. Clearly Φ is an algebraic isomorphism of $F(X, \underline{\underline{V}})$ onto $\Phi(F(X, \underline{\underline{V}}))$. Further, $\Phi(Y)$ is regularly situated in $F(\beta(X), \underline{\underline{V}})$ and is homeomorphic to Y . Therefore, by Theorem 1.9, the subgroup F_1 of $F(\beta(X), \underline{\underline{V}})$ generated algebraically by $\Phi(Y)$ is a closed subset of $F(\beta(X), \underline{\underline{V}})$ and the induced topology of F_1 is the finest group topology (on $\overline{F}_1$) which induces the given topology on $\Phi(Y)$. Thus, since Φ is continuous and $\Phi^{-1}(F_1) = F_Y$, F_Y

has the required properties. The remainder of the theorem follows immediately.

THEOREM 1.11. *Let* X *be a Tychonoff space and* $\underline{\underline{V}}$ *a full variety. If* Y *is a finite subset of* $F(X, \underline{\underline{V}})$ *, then the induced topology on the subgroup* H *, algebraically generated by* Y *, is discrete. Also* H *is a closed subset of* $F(X, \underline{\underline{V}})$ *. Further, if* $\underline{\underline{V}}$ *is a Schreier variety and* $\overline{Y}$ *is a free basis of* $\overline{F}_Y$ *, then* H *is* $F(Y, \underline{\underline{V}})$ *.*

Proof. Since Y is finite, $Y \subseteq K$, for some subgroup K of $F(X, \underline{\underline{V}})$ algebraically generated by a finite subset Z of X . Clearly Z is regularly situated with respect to X and thus by Theorems 1.10 and 1.1, K has the discrete topology. Consequently H , which is a subgroup of K , has the discrete topology. Then by Theorem 5.10 of [*4*] and Theorem 1.1, H is a closed subset of $F(X, \underline{\underline{V}})$. The remaining part of the theorem is now an immediate consequence.

We point out that Theorem 1.11 appears to be new even in the case that $\underline{\underline{V}}$ is the variety of all (all abelian) groups.

The next theorem is in the spirit of Theorems 1.9, 1.10 and 1.11 and follows immediately from §7 of [*6*] together with Theorems 1.1 and 1.2 and Lemma 1.4.

THEOREM 1.12. *Let* $F_M(X)$ *be moderately free on the Hausdorff space* X *. If* Y *is a closed subset of* X *, then the subgroup generated algebraically by* Y *is a closed subset of* $F_M(X)$ *.*

THEOREM 1.13. *If* $F_M(X)$ *is a Hausdorff (non-trivial) moderately free group then it is disconnected.*

Proof. Let $\underline{\underline{V}}$ be the intersection of all full varieties containing $F_M(X)$. By Lemma 1.4, $F_M(X)$ is $F(X, \underline{\underline{V}})$. Let H be any (non-trivial) group in $\underline{\underline{V}}$ with the discrete topology. Define the mapping ϕ of X into H by $\phi(x) = a$ for all x in X , where a is any element of H other than the identity. Clearly ϕ is continuous and thus there exists a continuous homomorphism Φ of $F_M(X)$ into H such that $\Phi|X = \phi$. Then $\Phi^{-1}\{a\}$ is an open and closed proper subset of $F_M(X)$, and the proof is complete.

2. β-Varieties

In analogy with [6] we introduce the following definition.

DEFINITION. The variety $\underline{\underline{V}}$ is said to be a β-variety if for every compact Hausdorff space X , $F(X, \underline{\underline{V}})$ exists and is Hausdorff.

THEOREM 2.1. *Every full variety is a β-variety.*

Proof. This follows immediately from Theorems 1.1 and 1.2.

THEOREM 2.2. *If* $\underline{\underline{V}}$ *is a β-variety, then for any Tychonoff space* X , $F(X, \underline{\underline{V}})$ *exists. Further,* X *is a closed subset of* $F(X, \underline{\underline{V}})$ *and* $F(X, \underline{\underline{V}})$ *is Hausdorff.*

Proof. Let Y be the Stone-Čech compactification of X . Then $F(Y, \underline{\underline{V}})$ exists and is Hausdorff. Thus X is a subspace of $F(Y, \underline{\underline{V}})$ which, by Theorem 2.6 of [8], implies $F(X, \underline{\underline{V}})$ exists.

Let ϕ be the imbedding mapping of X in Y . Then ϕ is a continuous map of X into $F(Y, \underline{\underline{V}})$ and therefore there exists a continuous homomorphism Φ of $F(X, \underline{\underline{V}})$ into $F(Y, \underline{\underline{V}})$ such that $\Phi|X = \phi$. Since Y is a closed subset of $F(Y, \underline{\underline{V}})$, $\Phi^{-1}(Y) = X$ is a closed subset of $F(X, \underline{\underline{V}})$. It follows immediately that $F(X, \underline{\underline{V}})$ is Hausdorff.

We will now give a characterization of β-varieties and in so doing give an alternative proof of Theorem 2.2

THEOREM 2.3. *Let* X *be the set of non-negative reals not greater than one, with the usual topology. Then* $\underline{\underline{V}}$ *is a β-variety if and only if* $F(X, \underline{\underline{V}})$ *exists and is Hausdorff.*

Proof. Clearly if $\underline{\underline{V}}$ is a β-variety then $F(X, \underline{\underline{V}})$ exists and is Hausdorff. Conversely we will show that if $F(X, \underline{\underline{V}})$ exists and is Hausdorff then for every Tychonoff space Y , $F(Y, \underline{\underline{V}})$ exists and has Y as a closed subset.

By Theorem 7, Chapter 4, of [5], Y can be imbedded in a cartesian product of copies of X . Consequently Y can be imbedded in a cartesian product of copies of $F(X, \underline{\underline{V}})$. Thus by Theorem 2.6 of [8], $F(Y, \underline{\underline{V}})$

exists.

Suppose there is a limit point $y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n}$ of Y not in Y, where $y_i \in Y$ and ε_i is a non-zero integer for each i. Let $b_1, \ldots, b_r$ be the distinct y_i. Choose distinct elements $a_1, \ldots, a_r$ of X. By Theorem 3.6 of [4], there exists a continuous map ϕ of Y into X such that $\phi(b_i) = a_i$, $i = 1, \ldots, r$. Then ϕ is a continuous map of Y into $F(X, \underline{\underline{V}})$, which implies that there exists a continuous homomorphism Φ of $F(Y, \underline{\underline{V}})$ into $F(X, \underline{\underline{V}})$ such that $\Phi|Y = \phi$. Thus $\Phi\left(y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n}\right) = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, where $a_1, \ldots, a_r$ are the distinct x_i.

Since Φ is continuous, $\Phi(\overline{Y}) \subseteq \overline{X}$, and X is closed in $F(X, \underline{\underline{V}})$, we must have $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in X$. Thus $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} t^{-1} = e_1$, where e_1 is the identity of $F(X, \underline{\underline{V}})$ and $t \in X$. This implies, using Theorem 2.8 of [8], that for at least one i, $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i^{-1} = e$. Therefore $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i^{-1}$ is an algebraic law in $\overline{F(X, \underline{\underline{V}})}$. Using Theorem 2.8 of [8] again, we see that $y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n} y_i^{-1} = e_2$, where e_2 is the identity of $F(Y, \underline{\underline{V}})$. Thus $y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n} \in Y$, which is a contradiction. Consequently Y is a closed subset of $F(Y, \underline{\underline{V}})$ and therefore $F(Y, \underline{\underline{V}})$ is Hausdorff.

LEMMA 2.4. *Let $\underline{\underline{V}}$ be an abelian β-variety and X be the closed interval $[a, b]$ of reals with the usual topology. Then the subgroup A of $F(X, \underline{\underline{V}})$, algebraically generated by $\{a\}$, is a closed subset of $F(X, \underline{\underline{V}})$.*

Proof. Suppose there exists a limit point c of A which is not in A. Define the continuous mapping ϕ of X into $F(X, \underline{\underline{V}})$ by $\phi(x) = a^{-1}x$ for each x in X. Then there exists a continuous endomorphism Φ of $F(X, \underline{\underline{V}})$ such that $\Phi|X = \phi$. Clearly $\Phi(A) = e$, the identity of $F(X, \underline{\underline{V}})$, and since $\{e\}$ is a closed set, $\Phi(c) = e$. However if $c = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, $x_i \in X$ and ε_i a non-zero integer for each i , then $\Phi(c) = a^{-m}c$ where $m = \sum_{i=1}^{n} \varepsilon_i$. This implies c is in A , which is a contradiction.

THEOREM 2.5. *Let V be an abelian β-variety and Z a Tychonoff space. If Y is a (non-empty) closed subset of Z , then the subgroup F_Y of $F(Z, \underline{\underline{V}})$ algebraically generated by Y is a closed subset of $F(Z, \underline{\underline{V}})$.* (cf. Theorem 1.12).

Proof. Suppose $c = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is a limit point of F_Y which is not in F_Y , where the x_i are distinct elements of Z and the ε_i are non-zero integers. Without loss of generality we can assume x_1 is not in Y .

Let the symbols X, a, b and A be as in Lemma 2.4. Now $Y \cup \{x_2, \ldots, x_n\}$ is a closed subset of Z . Therefore there exists a continuous mapping ϕ of Z into X such that $\phi\left(Y \cup \{x_2, \ldots, x_n\}\right) = a$ and $\phi(x_1) = b$. Then there exists a continuous homomorphism Φ of $F(Z, \underline{\underline{V}})$ into $F(X, \underline{\underline{V}})$ such that $\Phi | Z = \phi$. Clearly $\phi(F_Y) = A$ and $\Phi(c) = b^{\varepsilon_1}a^m$, where $m = \sum_{i=2}^{n} \varepsilon_i$. By Lemma 2.4, A is a closed subset of $F(X, \underline{\underline{V}})$ and thus $\Phi(c) \in A$. This implies b^{ε_1} is in A which is clearly a contradiction.

3. Projective groups

Hall [3] introduced the notion of a projective group for the category of Hausdorff abelian groups based upon the requirement that the class of projective groups contains the class of (Hausdorff) free abelian groups. We extend this notion to that of projective in a variety.

We omit all proofs in this section since they are similar to the proofs of the corresponding results in [3].

DEFINITION. A sequence of the form $f : A \to B \to 0$ is said to be exact if A and B are groups and f is a continuous homomorphism of A onto B.

DEFINITION. A group G is projective relative to a family ε of exact sequences of the form

$$A \xrightarrow{f} B \to 0 \tag{1}$$

if for each (1) in ε and each continuous homomorphism g from G to B there exists a continuous homomorphism h from G to A such that $fh = g$.

DEFINITION. Let $\underline{\underline{V}}$ be a variety and $\varepsilon(\underline{\underline{V}})$ be the family of all exact sequences of the form (1), with A and B in $\underline{\underline{V}}$, such that all free groups of $\underline{\underline{V}}$ are projective relative to $\varepsilon(\underline{\underline{V}})$. Then a group G in $\underline{\underline{V}}$ is said to be projective in $\underline{\underline{V}}$ if it is projective relative to $\varepsilon(\underline{\underline{V}})$.

Clearly every free group of a variety $\underline{\underline{V}}$ is projective in $\underline{\underline{V}}$.

LEMMA 3.1. *The sequence* (1) *is in* $\varepsilon(\underline{\underline{V}})$ *if and only if* A *and* B *are in* $\underline{\underline{V}}$ *and there is a continuous function* g *from* B *into* A *such that* fg *is the identity function on* B.

Consequently for every $B \in \underline{\underline{V}}$, the sequence

$$F(B, \underline{\underline{V}}) \xrightarrow{\alpha} B \to 0$$

where α is the natural homomorphism, is in $\varepsilon(\underline{\underline{V}})$.

THEOREM 3.2. *If the group* G *is a summand* [3] *of a projective group of* $\underline{\underline{V}}$, *then* G *is a projective group of* $\underline{\underline{V}}$.

THEOREM 3.3. *A group is projective in a variety* $\underline{\underline{V}}$ *if and only if it is a summand of a free group of* $\underline{\underline{V}}$.

COROLLARY 3.4. *If* P *is projective in* $\underline{\underline{V}}$ *then* $\overline{P}$ *is a projective abstract group of* $\overline{\underline{\underline{V}}}$. (See [9].) *Further, if* $\underline{\underline{V}}$ *is a Schreier variety then* $\overline{P}$ *is free in* $\overline{\underline{\underline{V}}}$.

The next theorem is a generalization of Theorem 2 of [3]. We point out, however, that Theorem 2 of [3] could be deduced from Theorems 3.3 and 1.11, whilst the theorem below could not.

THEOREM 3.5. *Let* P *be a Hausdorff finitely generated projective group of the Schreier variety* $\underline{\underline{V}}$. *Then* P *is a free group of* $\underline{\underline{V}}$.

THEOREM 3.6. *Let* $P_1, \ldots, P_n$ *be projective groups of the abelian variety* $\underline{\underline{V}}$. *Then the direct product group of the* P_i *is projective in* $\underline{\underline{V}}$.

4. Other results

THEOREM 4.1. *Let* G *be a relatively free group with free generating space* X . *Then* G *is a continuous algebraic isomorphic image of a quotient group* F/A *of the free group* F *on* X , *where* F/A *has generating space* X *and* A *is a fully invariant subgroup of* F .

Proof. Noting Lemma 4.3 of [8] we only have to show A is a fully invariant subgroup of F . Clearly it is sufficient to show that for every continuous endomorphism δ of F there exists a continuous endomorphism ξ of G such that $\Phi\delta = \xi\Phi$, where Φ is the natural homomorphism of F onto G .

Let δ be a continuous endomorphism of F . Then $\delta|X$ is a continuous mapping of X into F and $\Phi\delta|X$ is a continuous mapping of X into G . Since G is relatively free on X , there exists a continuous endomorphism ξ of G such that $\xi|X = \Phi\delta|X$. It is readily verified that $\Phi\delta = \xi\Phi$.

Neumann has various equivalents of "the free abstract group of an algebraic variety", namely 13.11, 13.21, 13.22 and 13.23 of [9]. In §5 of [8] the relationships between topological analogues of these properties were examined. In particular it was shown that these analogues are not equivalent. Here we modify the topological analogues and show that the modified ones indeed are equivalent.

(4.2) G is relatively free with generating space X and the topology of G is the finest group topology (on $\overline{G}$) which induces the same topology on X .

(4.3) G has a generating space X such that every relator of X is a law in G and the topology of G is the finest group topology (on $\overline{G}$) which induces the same topology on X .

(4.4) G has a representation $G \cong F/A$ as the quotient group of the free group F on X by a fully invariant subgroup A of F and the natural homomorphism Φ of F onto G maps X homeomorphically onto $\Phi(X)$.

(4.5) G has a representation $G \cong F/R$, such that every continuous endomorphism of the free group F on X induces the natural endomorphism of F , and the natural homomorphism Φ of F onto G maps X homeomorphically onto $\Phi(X)$.

THEOREM 4.6. *The properties* (4.2), (4.3), (4.4) *and* (4.5) *are equivalent.*

Proof. Clearly (4.4) and (4.5) are equivalent. By Theorems 5.8 and 5.11 of [8], (4.2) and (4.3) are equivalent. Also, by Theorem 4.1, property (4.2) implies (4.4). Using Theorem 5.8 of [8] we see that property (4.4) implies (4.2). The proof of the theorem is complete.

We leave the proof of the final theorem to the reader.

THEOREM 4.7. *Let* $F_1, \ldots, F_n$ *be free groups of the abelian variety* $\underline{\underline{V}}$. *Then the direct product of the* F_i *is a free group of* $\underline{\underline{V}}$.

It is not true, in general, that the direct product of an arbitrary set of free groups of $\underline{\underline{V}}$ is a free group of $\underline{\underline{V}}$. (see Example 2 of [3].)

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