REMARKS ON VARIETIES OF TOPOLOGICAL GROUPS

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§ 1. Introduction

Recently several papers on varieties of topological groups and varieties of locally convex spaces have appeared. (We include a somewhat complete bibliography.) This paper continues the investigation carried on in [10], [11] and [12] and cleans up some points raised there.

§ 2 begins with the definition of a variety of topological groups and a discussion of why we have been so unorthodox as to consider non-Hausdorff groups. The point is that our varieties of topological groups are more closely related to varieties of groups [25] if we do so. Next we glance at the question of how our varieties of topological groups are related to Higman's (rather different varieties of topological groups [9]. In this section we also prove a theorem relating the topological and algebraic structures of free topological groups.

In § 3, the question of how the properties of a subgroup being "topologically fully invariant" and „algebraically fully invariant" are related is investigated. That the latter implies the former is trivial. However we show that for a large class of examples the converse is false.

Some open questions are also presented in the paper.

§ 2. Some Basic Facts

A non-empty class $\mathscr{V}$ of (not necessarily Hausdorff) topological groups is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images.

If $\mathscr{V}$ is a variety of topological groups, then the class of groups, $\mathscr{V}$. which with some topology appear in $\mathscr{V}$ is a variety of groups [25]. (That $\mathscr{V}$ is indeed a variety of groups can be seen from 15.51 of [25].) If we restricted our attention to Hausdorff groups the example below (and Theorem 2.2) shows that this would not be the case.

Example 2.1. Let $\mathscr{V}$ be the class of all topological groups with the property: every neighbourhood of the identity contains a subgroup with only a finite number of cosets. (In the language of [12], $\mathscr{V}$ is the class of all $S(\aleph_0)$-groups.) It is readily verified that $\mathscr{V}$ is a variety of topological groups. However the class Σ of all groups which, with some *Hausdorff* topology, appear in $\mathscr{V}$ is *not* a variety of groups. We can see this by noting that Σ contains all finite groups but does not contain the additive group of reals.

If Ω is a class of topological groups then the smallest variety of topological groups containing Ω is said to be the *variety generated* by Ω and is denoted by $\mathscr{V}(\Omega)$ (or $\mathscr{V}(G)$ if $\Omega = \{G\}$).

Open question. Let Ω be a class of topological groups and Σ be the class of all groups which, with some Hausdorff topology, appear in $\mathscr{V}(\Omega)$. Under what conditions on Ω is Σ a variety of groups?

As a partial answer to this we present:

Theorem 2.2. *Let Ω be a class of connected compact groups and let Σ be the class of all groups which, with some Hausdorff topology, appear in $\mathscr{V}(\Omega)$. Then Σ is a variety of groups if and only if each member of Ω is abelian.*

Proof. If each member of Ω is abelian then, by Theorem 2.5(iv) of [3], $\mathscr{V}(\Omega) = \mathscr{V}(T)$, where T is the circle group with its usual compact topology. It is well-known that every abelian group is algebraically isomorphic to a subgroup of a product of copies of T. Thus Σ is the *variety* of all abelian groups.

Now consider the case where some member of Ω is not abelian. Suppose that Σ is a variety of groups. By Theorem 2 of [1], Σ contains a free group of rank $2^{\aleph_0}$ and hence Σ is the variety of *all* groups. Corollary 3 of [2] then implies that every group is isomorphic to a subgroup of a compact group. This is equivalent to the proposition: Every discrete group is maximally almost periodic [8]. This proposition is shown in [8] to be false. Hence Σ is *not* a variety of groups.

If $\mathscr{V}$ is a variety, X is a topological space and F is a member of $\mathscr{V}$, then F is said to be a *free topological group of $\mathscr{V}$ on X*, denoted by $F(X, \mathscr{V})$, if it has the properties:

(a) X is a subspace of F,

b() X generates F algebraically,

(c) for any continuous mapping γ of X into any member H of $\mathscr{V}$, there exists a continuous homomorphism Γ of F into H such that $\Gamma \mid X = \gamma$.

The following results on free topological groups are proved in [10]:

(i) $F(X, \mathscr{V})$ is unique (up to isomorphism) if it exists,

(ii) $F(X, \mathscr{V})$ exists if and only if there is a member of $\mathscr{V}$ which has X as a subspace,

(iii) $F(X, \mathscr{V})$ is the free group on the set X of the underlying variety of groups $\mathscr{V}$ [25].

A topological group F is said to be *topologically relatively free with free gnerating space* X if X is a subspace of F which generates F algebraically and every continuous mapping of X into F can be extended to a continuous endomorphism of F.

We recall that a group F is *relatively free with free generating set* X if, given the indiscrete topology, it is topologically relatively free with free generating space X.

Open questions. If G is topologically relatively free is the underlying group $\bar{G}$ necessarily relativle free?

If G is topologically relatively free with free generating space X and $\bar{G}$ is relatively free with free generating set X, is G necessarily $F(X, \mathscr{V}(G))$? (Of course the converse statement is true.)

Graham Higman [9], using an analogue of "topologically relatively free" inspired by Graev [7], defined his concept of a "variety of topological groups". His work prompts the question:

If F is $F(X, \mathscr{V}(F))$ for some space X, and G is a topological group with the property that every continuous mapping of X into G can be extended to a continuous homomorphism of F into G, does G necessarily belong to $\mathscr{V}(F)$? If not, is it true under the additional assumption that $\bar{G} \in \mathscr{V}(\bar{F})$?

Both of these questions are answered in the negative by Example 2.3.

Example 2.3. Let F be any relatively free group with the discrete topology. Then for some subspace X of F, F is $F(X, \mathscr{V}(F))$. Let F have cardinal m and G be a discrete group of cardinal $n > m$ such that $\bar{G} \in \overline{\mathscr{V}}(\bar{F})$. By Theorems 1.2 and 2.1 of [12], $G \notin \mathscr{V}(F)$. However G clearly has the properties described above.

We now clarify and correct the final remark in § 2 of [10].

Theorem 2.4. *Let X be a space and $\mathscr{V}$ a variety such that $F(X, \mathscr{V})$ exists. If X in an open subset of $F(X, \mathscr{V})$ then, providing $\overline{F(X, \mathscr{V})}$ is not the Klein four--group, $F(X, \mathscr{V})$ has the discrete topology.*

Proof. First, consider the case where X has at least three distinct elements x_1, x_2, and x_3. We will show that $x_1^{1} X \cap x_2^{-1} X \subseteq \{e, x_1^{-1} x_2\}$ and $x_3^{-1} X \cap x_1^{1} X \subseteq \{e, x_1^{1} x_3\}$, where e is the identity of $F(X, \mathscr{V})$.

Let $a \in x_1^{1} X \cap x_2^{1} X$, $a \neq e$. Then $a = x_1^{1} y = x_2^{1} z$, where y and z are in X. Clearly $y \neq z$, $y \neq x_1$, and $z \neq x_2$. If $z \notin \{x_1, x_2, y\}$, then since $F(X, \mathscr{V})$ is algebraically relatively free, $x_1^{1} y = x_2^{1}$. Either $y = x_2$ or $y \notin \{x_1, x_2\}$.

The latter implies $x_1 = x_2$ whilst the former implies $x_1 = x_2^2$. Each of these is obviously false. Thus $z = x_1$. Similarly $y = x_2$. So $a = x_1^{1} x_2 = x_2^{-1} x_1$. Hence $x_1^{1} X \cap x_2^{1} X \subseteq \{e, x_1^{1} x_2\}$ and analogously $x_1^{1} X \cap x_3^{1} X \subseteq \{e, x_1^{1} x_3\}$.

Therefore $x_1^{-1} X \cap x_2^{-1} X \cap x_3^{1} X = \{e\}$. This implies that $\{e\}$ is an open set and consequently $F(X, \mathscr{V})$ has the discrete topology.

Clearly if X has only one element the result is trivial. We are then left with the case $X = \{x_1, x_2\}$. As shown already, unless $x_1^{1}\, x_2 = x_2^{1}\, x_1$. $x_1^{1}\ X \cap$ $\cap\, x_2^{1}\ X = \{e\}$ which again implies that $F(X, \mathscr{V})$ is discrete.

If $x_1^{1}\ x_2 = x_2^{1}\ x_1$ then, since $F(X, \mathscr{V})$ is algebraically relatively free, $x_1^{1} = x_1$ and $x_1\, x_2 = x_2 x_1$. Thus $F(X, \mathscr{V})$ is an abelian group of exponent two and therefore is algebraically isomorphic to the Klein four-group. The proof is complete.

Remark 2.5. The Klein four-group is indeed an exception to the above theorem and not just to the proof. For if $F = \{e, x_1, x_2, x_1 x_2\ \ x_1^2 - x_2^2 - e$ and $x_1 x_2 = x_2 x_1\}$ with an open basis at e for its topology consisting of the set $\{e, x_1 x_2\}$ then X is open in F, where $X = \{x_1, x_2\}$. Also F is $F(X, \mathscr{V}(F))$, but F does not have the discrete topology.

Recall that if A is a subgroup of B with the property that every endomorphism of B maps A into itself then A is said to be an *algebraically fully invariant subgroup* of B.

If A and B are topological groups and A is a subgroup of B with the property that every continuous endomorphism of B maps A into itself then A ís said to be a *topologically fully invariant subgroup* of B.

The next theorem is in the same spirit as Theorem 2.4.

Theorem 2.6. *Let X be a space and $\mathscr{V}$ a variety such that $F(X, \mathscr{V})$ exists. Let A be an algebraically fully invariant proper subgroup of $F(X, \mathscr{V})$. If A is open (respectively, closed) in $F(X, \mathscr{V})$, then X is discrete (respectively, Hausdorff).*

Proof. Let x be any element in X. Since A is proper and algebraically fully invariant, $xA \cap X = \{x\}$. From this the results immediately follow.

Our next example shows that Theorem 2.6 cannot be extended to say $F(X, \mathscr{V})$ is discrete (respectively, Hausdorff).

Example 2.7. Let Ω be the class of all groups which are either abelian or have the indiscrete topology. (See Example 3.2 of [12].) It is easily seen that if X is a discrete space then $F(X, \mathscr{V}(\Omega))$ is not even Hausdorff but has the commutator subgroup as an open (algebraically fully invariant) subgroup.

Remark 2.8. Clearly in the above theorem $F(X, \mathscr{V})$ can be replaced by any topological group algebraically isomorphic to $F(X, \mathscr{V})$.

A topological group F is said to be *moderately free* on the space X if

(i) $\bar{F}$ is relatively free with free generating set X, and

(ii) the topology of F is the finest group topology (on $\bar{F}$) which induces the same topology on X.

The importance of moderately free groups is established in [10] and [11].

The final result in this section is used in [15].

Theorem 2.9. *Let Ω be a class of connected locally compact groups. Then the following are equivalent*:

(i) There is a member of Ω which is not compact.

(ii) $Z \in \mathscr{V}(\Omega)$, where Z is the discrete group of integers.

(iii) There exists a Tychonoff space X such that $F(X, \mathscr{V}(\Omega))$ is moderately free on X.

Proof. The equivalence of (i) and (ii) follows from Theorem 2.5 (ii) of [3] and Corollary 3 of [2]. It is obvious that (ii) implies (iii).

Suppose that (iii) is true. Let $x \in X$ and G be the subgroup of $F(X, \mathscr{V}(\Omega))$ generated algebraically by x. By Theorem 2.5(iv) of [3], G is algebraically isomorphic to Z, while by Theorem 1.11 of [11], G has the discrete topology. Thus $Z \in \mathscr{V}(\Omega)$; that is, (ii) is true and hence (i) is also true. The contradiction shows that (iii) implies (i), and the proof is complete.

§ 3. Fully invariant subgroups

In § 5 of [12] we introduced the concept of a fully invariant subgroup. It is obvious that any algebraically fully invariant subgroup is topologically fully invariant. We now show the converse is false.

Theorem 3.1. *Let C be the component of the identity in any topological group A. Then C is a topologically fully invariant subgroup of A.*

Proof. Let Γ be any continuous endomorphism of A. Then $\Gamma(C)$ is a connected set contining the identity. Therefore $\Gamma(C) \subset C$.

Theorem 3.2. *Let $\mathscr{V}$ be any non-indiscrete variety and X a space such that $F(X, \mathscr{V})$ exists. If X is not totally disconnected, then the component C of the identity is not an algebraically fully invariant subgroup of $F(X, \mathscr{V})$.*

Proof. Since X is not totally disconnected there is an $x \in X$ such that the component A of x in X contains $y \in X$, $y \neq x$. Consider xC. Clearly this contains A and so $y \in xC$. Thus $xy^{-1} \in C$.

Suppose C is algebraically fully invariant. Then $xy^{-1} \in C$ implies $x \in C$ which in turn implies $C = F(X, \mathscr{V})$ which is a contradiction to Theorem 6.1 of [11]. Hence C is not algebraically fully invariant.

Remark 3.3. Example 3.4 shows that the above theorem is not necessarily true if we allow X to be totally disconnected.

Example 3.4. Let $\mathscr{V}$ be the class of all topological groups having the property that the intersection of all neighbourhoods of the identity in G contains the commutator subgroup of G. Let X be a discrete space and C be the component of the identity in $F(X, \mathscr{V})$. Obviously C is the commutator subgroup of $F(X, \mathscr{V})$, which is algebraically fully invariant.

Remark 3.5. One might have suspected that for any variety $\mathscr{V}$, X totally disconnected implies $F(X, \mathscr{V})$ totally disconnected. Example 3.4 shows this is not true. Theorem 3.7 is relevant to this.

Theorem 3.6. *Let $\mathscr{V}$ be any abelian variety which contains a finitely generated Hausdorff free group of $\mathscr{V}$. Let X be any space such that $F(X, \mathscr{V})$ exists. If C is* *any non-trivial connected subgroup of $F(X, \mathscr{V})$, then C is not algebraically fully invariant.*

Proof. Let $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ be any element in C, where $x_i \neq x_j$ for $i \neq j$, and $x_i^{\varepsilon_i} \neq e$, the identity, for any i and j.

Suppose C is algebraically fully invariant. Then $x_1^{\varepsilon_1} \in C$. Let $F(\{a\}, \mathscr{V})$ be a Hausdorff free group of $\mathscr{V}$. Define a mapping γ of X into $F(\{a\}. \mathscr{V})$ by $\gamma(X) = a$. Then since γ is continuous, there exists a continuous homomorphism Γ of $F(X, \mathscr{V})$ into $F(\{a\}, \mathscr{V})$ such that $\Gamma \mid X = \gamma$. Since $F(\{a\}, \mathscr{V})$ is totally disconnected, $\Gamma(C) = e_1$, the identity of $F(\{a\}, \mathscr{V})$. However $\Gamma(x_1^{\varepsilon_1}) = a^{\varepsilon_1} \neq \neq e_1$, which is a contradiction. Hence C is not algebraically fully invariant.

Theorem 3.7. *Let X be a 0-dimensional Hausdorff space and $\mathscr{V}$ a variety such that $F(X, \mathscr{V})$ exists. Further, let $\mathscr{V}$ be such that for each discrete n-element space $Y(n), F(Y(n), \mathscr{V})$ exists and is Hausdorff. Then $F(X, \mathscr{V})$ is totally disconnected.*

Proof. Let C be the component of the identity e in $F(X, \mathscr{V})$. Suppose $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is an element of C other than e. Let $\{a_1, \ldots, a_m\}$ be the distinct x_i. Since X is 0-dimensional and Hausdorff, $X = 0_1 \cup 0_2 \cup \ldots \cup 0_m$, where $a_i \in 0_i$, $i = 1, \ldots, m$ and $0_i \cap 0_j = \emptyset$ for $i \neq j$, and each 0_i is an open subset of X.

Let $Y(m) = \{b_1, \ldots, b_m\}$ be a discrete m-element space. Then $F(Y(m), \mathscr{V})$ is Hausdorff. Define a continuous mapping γ of X into $F(Y(m), \mathscr{V})$ by $\gamma(0_i) = b_i$, $i = 1, \ldots, m$. Then there exists a continuous homomorphism Γ of $F(X, \mathscr{V})$ into $F(Y(m), \mathscr{V})$ such that $\Gamma \mid X = \gamma$. Since $F(Y(m), \mathscr{V})$ is totally disconnected, $\Gamma(C) = e_1$, the identity of $F(Y(m), \mathscr{V})$. However, $\Gamma(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}) \neq \neq e_1$. This is a contradiction and hence $F(X, \mathscr{V})$ is totally disconnected.

A variety $\mathscr{V}$ is said to be a *β-variety* if for each Tychonoff space X, $F(X. \mathscr{V})$ exists and is Hausdorff. (See [11], [12] and [20]).

Corollary 3.8. *Let X be a 0-dimensional Tychonoff space and $\mathscr{V}$ a β-variety. Then $F(X, \mathscr{V})$ is tctally disconnected.*

Theorem 3.9. *Let $\mathscr{V}$ be any variety and X any space such that $F(X, \mathscr{V})$ exists. Let A be any open and closed subset of X. If K is a connected set in $F(X, \mathscr{V})$ and $K \supset A$, then $K \cap X = A$.*

Proof. Clearly the result is true if $A = X$. Therefore assume A is a proper subset of X and let B the complement in X of A.

Since X is not indiscrete, $F(X, \mathscr{V})$ is not indiscrete and so $\mathscr{V}$ is not an indiscrete variety. Therefore $\mathscr{V}$ contains a non-trivial countable Hausdorff group H. Let h be any element of H other than the identity, e. Define a continuous mapping γ of X into H by $\gamma(A) = h$ and $\gamma(B) = e$. Then there exists a continuous homomorphism Γ of $F(X, \mathscr{V})$ into H such that $\Gamma \mid X = \gamma$. Clearly $\Gamma(K) = h$, since H is totally disconnected, whilst $\Gamma(B) = e$. Therefore $K \cap \cap X = A$.

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