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THE TOPOLOGY OF FREE PRODUCTS OF TOPOLOGICAL GROUPS

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1. Introduction

In [3], Graev introduced the free product of Hausdorff topological groups G and H (denoted in this paper by $G \coprod H$) and showed it is algebraically the free product $G * H$ and is Hausdorff. While it has been studied subsequently, for example [4, 6, 7, 8, 11, 12], many questions about its topology remain unsolved. In particular, partial negative results about local compactness were obtained in [7, 11, 12]. In this paper we obtain a complete solution by showing that $G \coprod H$ is locally compact if and only if G, H and $G \coprod H$ are discrete. A similar line of reasoning allows us to show that $G \coprod H$ has no small subgroups if and only if G and H have no small subgroups.

We are able to obtain much stronger results when G and H are k_ω-spaces, a class of spaces which includes, for example, all compact spaces and all connected locally compact groups. In this case we are able to show that the cartesian subgroup, $\text{gp}[G, H] = \text{gp}\{g^{-1}h^{-1}gh : g \in G, h \in H\}$, of $G \coprod H$ is a free topological group, show that certain subgroups of $G \coprod H$ are themselves free products, and show that the topology of $G \coprod H$ depends only on the topologies and not on the algebraic structure of G and H .

2. Definitions and preliminaries

If X is a completely regular Hausdorff space with distinguished point e , the (Graev) *free topological group on* X, $FG(X)$, is algebraically the free group on $X \setminus \{e\}$, with e as identity element and the finest topology making it into a topological group and inducing the given topology on X ; by [2], $FG(X)$ is Hausdorff.

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If G and H are topological groups, their *free product* $G \coprod H$ is a topological group whose underlying abstract group is the algebraic free product $G * H$ and whose topology is the finest topology making it into a topological group and inducing the given topologies on G and H ; by [3], if G and H are Hausdorff then $G \coprod H$ is Hausdorff.

For the remainder of the paper all topological groups and spaces will be presumed Hausdorff.

A topological group is said to be NSS (*or to have no small subgroups*) if there is a neighbourhood of the identity e which contains no subgroup other than $\{e\}$. This property is most important for locally compact groups in that Hilbert's fifth problem yields that a locally compact group is a Lie group if and only if it is NSS.

We require the following algebraic preliminaries: The identity map $G \to G$ and the trivial map $H \to \{e\} \subset G$ extend simultaneously to a homomorphism $\pi_1 : G * H \to G$; by [3], this is also a continuous map from $G \coprod H$ to G . Similarly $\pi_2 : G * H \to H$ is a homomorphism and a continuous map on $G \coprod H$. The map $\pi_1 \times \pi_2 : G * H \to G \times H$ has kernel $\mathrm{gp}[G, H]$, where $[G, H] = \{g^{-1}h^{-1}gh : g \in G, h \in H\}$. Indeed $\mathrm{gp}[G, H]$ is a free group with free basis $[G, H]\backslash\{e\}$. We find it convenient below to introduce a map $c : G \times H \to [G, H]$ given by $c(g, h) = [g, h] = g^{-1}h^{-1}gh$. If w is any element of $G * H$ it has a unique representation $w = ghk$, where $g \in G$, $h \in H$ and $k \in \mathrm{gp}[G, H]$. We define a map $\pi_c : G * H \to \mathrm{gp}[G, H]$ by $\pi_c(w) = k = \pi_2(w)^{-1}\pi_1(w)^{-1}w$: notice that it is *not* a homomorphism. Finally we note that there is a bijection (not a homomorphism) $p : G \times H \times \mathrm{gp}[G, H] \to G * H$ given by $p(g, h, k) = ghk$. The inverse map is $p^{-1}(w) = (\pi_1(w), \pi_2(w), \pi_c(w))$.

In §4 we use some additional machinery, that of k_ω-spaces; we rely heavily on [4]. A topological space X is said to be a k_ω-space with decomposition $X = \bigcup X_n$, if $X_1, X_2, \ldots$ are compact subsets of X, $X_n \subset X_{n+1}$ for all n , $X = \bigcup_{n=1}^{\infty} X_n$ and the X_n determine the topology on X in the sense that a subset A of X is closed if and only if $A \cap X_n$ is compact for all n . The decomposition $X = \bigcup X_n$ is essential, in that X may be a union of some other ascending chain of compact subsets which fail to determine the topology. If $X = \bigcup X_n$ and $Y = \bigcup Y_n$ where X_n and Y_n are ascending chains of compact sets, the two ascending chains determine the same topology on X provided each X_n is contained in some Y_k and each Y_n is contained in some X_m .

If G is a topological group and a k_ω-space the decomposition $G = \cup G_n$ may be chosen so that the G_n satisfy two additional conditions: if $g \in G_n$ then $g^{-1} \in G_n$, and if $g \in G_n$, $h \in G_k$ then $gh \in G_{n+k}$.

If X is any subset of a group G , we let $\mathrm{gp}_n(X)$ denote the set of elements of G which are products of at most n elements of X . Hence $\mathrm{gp}_n(G_n) \subset G_{n^2}$.

The class of topological groups which are k_ω-spaces is large enough to include many of the standard examples; in particular, every connected locally compact group is a k_ω-space [12].

We rely heavily on the following result of [4]:

PROPOSITION. *Let G be a topological group and X a subset which generates G algebraically. Let $X = \cup X_n$ be a k_ω-space. Then G has the finest group topology consistent with the original topology on X if and only if G is a k_ω-space with decomposition $G = \cup \mathrm{gp}_n(X_n)$.*

It follows that if $X = \cup X_n$ is a k_ω-space then $FG(X)$ is a k_ω-space with decomposition $FG(X) = \cup \mathrm{gp}_n(X_n)$. If $G = \cup G_n$ and $H = \cup H_n$ are k_ω-spaces then $G \coprod H$ is a k_ω-space with decomposition $G \coprod H = \cup \mathrm{gp}_n(G_n \cup H_n)$.

Finally note that when we say that a continuous map $f : X \to Y$ of topological spaces is *quotient map* we mean that Y has the finest topology for which f is continuous; this is equivalent to requiring that $A \subset Y$ is closed whenever $f^{-1}(A)$ is closed in X .

3. Results for general topological groups

We begin with a few words about Graev's proofs of the existence of free topological groups and free products of topological groups.

Let X be a completely regular space and e a distinguished point of X . Let $G(X)$ be the free group on the set $X \backslash \{e\}$, with e as the identity element of the group. Let $X' = X \cup X^{-1}$. Being completely regular, the topology of X is defined by a family of pseudometrics. Let ρ be a continuous pseudometric on X . Graev extended ρ to a two-sided invariant pseudometric on $G(X)$ as follows: Extend ρ to X' by setting $\rho\left(x^{-1}, y^{-1}\right) = \rho(x, y)$ and

$$\rho\left(x^{-1}, y\right) = \rho\left(x, y^{-1}\right) = \rho(x, e) + \rho(y, e)$$

for x and y in X. For u and v in $G(X)$ we have an infinity of representations $u = x_1 \ldots x_n$, $v = y_1 \ldots y_n$, where x_i and y_i X. Extend ρ to $G(X)$ by setting $\rho(u, v) = \inf\left(\sum_{i=1}^{n} \rho\left(x_i, y_i\right)\right)$, where the infimum is taken over all representations $u = x_1 \ldots x_n$ and $v = y_1 \ldots y_n$. The family of all such two-sided invariant pseudometrics on $G(X)$ yield a topological group $F_s(X)$. (It is shown elsewhere that $F_s(X)$ is the free topological SIN group on X.) Now $F_s(X)$ is Hausdorff; $FG(X)$ is the group $G(X)$ with the finest Hausdorff topology inducing the original topology on X. This topology $FG(X)$ is in general [9] a finer topology than $F_s(X)$.

Next we let G and H be topological groups. Graev defined a topology τ (not the free product topology, in general) on $G * H$ using the map $p : G \times H \times \text{gp}[G, H] \to G * H$. The method requires us to topologize $\text{gp}[G, H]$ in some way and then topologize $G * H$ to make the map p a homeomorphism. Since p is not a homomorphism it must be checked that this topology τ on $G * H$ is a group topology. (This is in fact quite difficult but our brief comments suppress this difficulty.) Let ρ_G and ρ_H be continuous right invariant pseudometrics on G and H respectively. Define a pseudometric ρ_{gh} on $[G, H]$ by

$$\begin{aligned}\rho_{GH}\left(g_1^{-1}h_1^{-1}g_1h_1, g_2^{-1}h_2^{-1}g_2h_2\right) = \min\left[\min\left(\rho_G\left(g_1, e\right), \rho_H\left(h_1, e\right)\right)\right.\\ \left. + \min\left(\rho_G\left(g_2, e\right), \rho_H\left(h_2, e\right)\right); \rho_G\left(g_1, g_2\right)+\rho_H\left(h_1, h_2\right)\right].\end{aligned}$$

The family of all such ρ_{GH} gives rise to a completely regular topology on $[G, H]$. Next, noting that $\text{gp}[G, H]$ is a free group on $[G, H]\backslash\{e\}$, we topologize $\text{gp}[G, H]$ by putting $\left(\text{gp}[G, H], \tau_1\right) = F_s[G, H]$. Finally we define the topology τ on $G * H$ by making

$$p : G \times H \times \left(\text{gp}[G, H], \tau_1\right) \to (G * H, \tau) \text{ a homeomorphism.}$$

Thompson [13] showed that $F_s(X)$ is NSS if and only if X admits a continuous metric. (Thompson's result is stronger than that of Morris and Thompson [10] which showed that $FG(X)$ is NSS if and only if X admits a continuous metric.)

Now if G is NSS, then G admits a continuous metric [10]; so if G and H are NSS, then $G \times H$ admits a continuous metric. Thus $[G, H]$ with the pseudometric topology described above admits a continuous metric. Hence $F_s[G, H]$ is NSS if G and H are NSS. We are now able to prove the following theorem:

THEOREM 1. *$G \amalg H$ is* NSS *if and only if G and H are* NSS.

PROOF. If $G \amalg H$ is NSS then any subgroup must be NSS. In particular, G and H must be NSS.

If G and H are NSS, then the above discussion yields that $F_s[G, H]$ is NSS. We shall prove that $(G * H, \tau)$ is NSS, as then $G \amalg H$ which has the same algebraic structure but a finer topology will also be NSS. Suppose that $(G * H, \tau)$, which is homeomorphic to $G \times H \times F_s[G, H]$, fails to be NSS. Let N and M be neighbourhoods of e in G and H, respectively, which contain no non-trivial subgroups. Then $\pi_1^{-1}(N) \cap \pi_2^{-1}(M)$ is a neighbourhood of e in $(G * H, \tau)$. Let A be a subgroup contained in $\pi_1^{-1}(N) \cap \pi_2^{-1}(M)$. Since π_1 is a homomorphism and $\pi_1(A) \subset N$ we must have $\pi_1(A) = e$. Similarly $\pi_2(A) = e$. Thus $A \subset F_s[F, G] \subset (G * H, \tau)$. Since $F_s[G, H]$ is NSS, $A = \{e\}$, as desired.

REMARKS. (1) This theorem generalizes the main result of [8] which says that if G and H are connected locally compact groups then $G \amalg H$ is NSS when and only when G and H are Lie groups.

(2) Note that the proof of Theorem 1 actually yields: $(G * H, \tau)$ is NSS if and only if G and H are NSS.

The fact that $(G * H, \tau)$ is homeomorphic to $G \times H \times \mathrm{gp}[G, H]$ leads us to ask if a similar result is true for $G \amalg H$. It is!

THEOREM 2. *If $\mathrm{gp}[G, H]$ is topologized as a subset of $G \amalg H$, then $G \amalg H$ is homeomorphic to $G \times H \times \mathrm{gp}[G, H]$ (the homeomorphism is given by the map p).*

PROOF. Since $G \amalg H$ is a topological group, the product map $(G \amalg H) \times (G \amalg H) \times (G \amalg H) \to G \amalg H$, given by $(g, h, k) \to ghk$ is continuous, and so is its restriction $p : G \times H \times \mathrm{gp}[G, H] \to G \amalg H$. We must show that the inverse map is continuous. The maps $\pi_1 : G \amalg H \to G$ and $\pi_2 : G \amalg H \to H$ are continuous, so $\pi_c(w) = \pi_2(w)^{-1}\pi_1(w)^{-1}w$ is a product of continuous maps and thus continuous. Hence the map $w \to \left(\pi_1(w), \pi_2(w), \pi_c(w)\right) = (g, h, k)$ is continuous, completing the proof.

THEOREM 3. *Suppose $G \neq \{e\}$ and $H \neq \{e\}$ are topological groups. Then $G \amalg H$ is not a locally compact space or a complete metric space unless G and H are both discrete.* (Of course if G and H are discrete, $G \amalg H$ is also discrete, and consequently locally compact and complete metric.)

PROOF. Suppose $G \amalg H$ is a locally compact space of a complete metric space; then so is the closed subgroup $\mathrm{gp}[G, H]$. But as $\mathrm{gp}[G, H]$ is algebraically a free group it follows from Dudley [1] that $\mathrm{gp}[G, H]$ is discrete. Now G is also discrete: for if $\{g_\delta\}$ is a non-constant net converging to $g \in G$ and $h \in H\backslash\{e\}$, then $\{[g_\delta, h]\}$ is a non-constant net converging to $[g, h]$ in $\mathrm{gp}[G, H]$, which is impossible. Similarly H is discrete. Finally we see $G \coprod H$, which is homeomorphic to $G \times H \times \mathrm{gp}[G, H]$, is also discrete.

REMARK. Theorems 2 and 3 hold (with the same proofs) for any group topology μ on $G * H$ for which the projections $\pi_1 : (G * H, \mu) \to G$ and $\pi_2 : (G * H, \mu) \to H$ are continuous and which induce the given topologies on G and H. Thus it would be of interest to answer:

QUESTION 1.[1] *Is there any group topology* μ *on* $G * H$ *such that either projection* $\pi_1 : (G * H, \mu) \to G$ *or* $\pi_2 : (G * H, \mu) \to H$ *is discontinuous?*

If continuity of π_1 and π_2 could be shown even under the hypothesis that G, H and $(G * H, \mu)$ are locally compact, we could conclude that no group topology on an algebraic free product is locally compact (except trivially).

What is the topology that $\mathrm{gp}[G, H]$ receives as a subset of $G \coprod H$? It is natural to hope that it has a free topological group topology, on an appropriate topology for $[G, H]$.

QUESTION 2. *(a) Does the topology induced on* $\mathrm{gp}[G, H]$ *as a subgroup of* $G \coprod H$ *make it the free topological group* $FG[G, H]$ *?*

(b) Is the topology induced on $[G, H]$ *as a subset of* $G \coprod H$*, the same as the quotient topology under the map* $G \times H \to [G, H]$ *given by* $(g, h) \to [g, h]$ *?*

We have already noted that Graev's Topology $F_s[G, H]$ is not, in general, $FG[G, H]$. Example 1 in §5 shows that 2 *(b)* is also false for Graev's topology; that is, Graev does not give $[G, H]$ the quotient topology. On the other hand we will answer both 2 *(a)* and 2 *(b)* affirmatively when G and H are k_ω-groups.

4. Results for groups which are k_ω-spaces

We begin by answering Question 2 *(b)* for this case.

THEOREM 4. *Let* G *and* H *be topological groups which are* k_ω*-spaces. Then* $\sigma : G \times H \to [G, H] \subset G \coprod H$ *is a quotient map.*

PROOF. Let the k_ω-space decompositions of G and H be $G = \bigcup G_n$ and $H = \bigcup H_n$. In view of the Proposition stated in §2, $G \coprod H$ is a k_ω-space with decomposition $G \coprod H = \bigcup \mathrm{gp}_n(G_n \cup H_n)$. (Thus a set A is closed in $G \coprod H$ if and only if $A \cap \mathrm{gp}_n(G_n \cup H_n)$ is compact for all n, where $\mathrm{gp}_n(G_n \cup H_n)$ is the set of

[1] This question has since been answered in the affirmative.

elements of $G \amalg H$ which are products of at most n elements of $G_n \cup H_n$; it is compact in $G \amalg H$.)

Now let $A \subset [G, H]$ be such that $c^{-1}(A)$ is closed in $G \times H$. We must show A is closed in $[G, H]$. It will suffice to show A is closed in $G \amalg H$. We shall prove that $A \cap \mathrm{gp}_n(G_n \cup H_n) = c\left[c^{-1}(A) \cap \left(G_{n^2} \times H_{n^2}\right)\right] \cap \mathrm{gp}_n(G_n \cup H_n)$ as the right hand side is the intersection of a continuous image of a compact set with a compact set it is compact.

If $n < 4$, both sides are trivial, so assume $n \geq 4$. Now if $w \in \mathrm{gp}_n(G_n \cup H_n)$, $w = x_1 \ldots x_n$, with $x_i \in G_n$ or H_n ; in reduced form $w = g^{-1}h^{-1}gh$, so clearly g is a product of at most n terms from G_n ; hence $g \in G_{n^2}$. Similarly $h \in H_{n^2}$. Since $w = c(g, h)$ we have that $w \in c\left[c^{-1}(A) \cap \left(G_{n^2} \times H_{n^2}\right)\right]$. The other inclusions needed are easy. Hence $A \cap \mathrm{gp}_n(G_n \cup H_n)$ is compact for all n , and A is closed, as required.

Note that it follows from the Proof of Theorem 4 that $[G, H]$ is closed in $G \amalg H$. We now turn to Question 2 *(a)*.

THEOREM 5. *Let G and H be topological groups which are k_ω-spaces. Then the topology on* $\mathrm{gp}[G, H]$ *as a subgroup of $G \amalg H$ is the free topological group topology $FG[G, H]$* .

PROOF. Again let $G = \cup G_n$ and $H = \cup H_n$ be k_ω-space decompositions. Then $G \amalg H = \cup \mathrm{gp}_n(G_n \cup H_n)$ and $[G, H] = \cup([G, H] \cap \mathrm{gp}_n(G_n \cup H_n))$ are k_ω-space decompositions.

Now from the Proposition given in §2, $FG[G, H]$ is a k_ω-space with decomposition $FG[G, H] = \cup \mathrm{gp}_n([G, H] \cap \mathrm{gp}_n(G_n \cup H_n))$. On the other hand, $\mathrm{gp}[G, H]$ is a closed subgroup of $G \amalg H$ and hence a k_ω-space with decomposition $\mathrm{gp}[G, H] = \cup(\mathrm{gp}[G, H] \cap \mathrm{gp}_n(G_n \cup H_n))$.

Clearly each $\mathrm{gp}_n([G, H] \cap \mathrm{gp}_n(G_n \cup H_n))$ is contained in $\mathrm{gp}[G, H] \cap \mathrm{gp}_k(G_k \cup H_k)$, for $k = n^2$; we must show for each n there is an m such that $\mathrm{gp}[G, H] \cap \mathrm{gp}_n(G_n \cup H_n) \subset \mathrm{gp}_m([G, H] \cap \mathrm{gp}_m(G_m \cup H_m))$.

Let $w \in \mathrm{gp}[G, H] \cap \mathrm{gp}_n(G_n \cup H_n)$. Without loss of generality suppose $n \geq 4$ and write $w = g_1h_2g_3 \ldots g_{n-1}h_n$, each $g_i \in G_n$ and each $h_i \in H_n$. We shall discuss a way of writing w as a product of commutators.

$$\begin{aligned} w &= g_1h_2g_3h_4 \cdots g_{n-1}h_n \\ &= \left[g_1^{-1}, h_2^{-1}\right]h_2(g_1g_3)h_4 \cdots g_{n-1}h_n \\ &= \left[g_1^{-1}, h_2^{-1}\right]\left[(g_1g_3)^{-1}, h_2^{-1}\right]^{-1}(g_1g_3)(h_2h_4)g_5 \cdots g_{n-1}h_n \\ &= \left[g_1^{-1}, h_2^{-1}\right]\left[(g_1g_3)^{-1}, h_2^{-1}\right]^{-1}\left[(g_1g_3)^{-1}, (h_2h_4)^{-1}\right] \cdots (g_1 \cdots g_{n-1})(h_2 \cdots h_n) . \end{aligned}$$

The last line has $n - 3$ commutators. Since $\pi_1(w) = \pi_2(w) = e$ we see that $g_1 \cdots g_{n-1} = h_2 \cdots h_n = e$. So w is a product of $n - 3$ commutators $[g, h]^{\pm 1}$, where each g is a product of at most n factors from G_n and hence lies in G_{n^2} . Similarly for h . So for any $m \geq n^2$ we have

$$[g, h] \in [G, H] \cap \mathrm{gp}_m(G_m \cup H_m)$$

and

$$w \in \mathrm{gp}_m([G, H] \cap \mathrm{gp}_m(G_m \cup H_m)) ,$$

as desired. Thus the topologies of $FG[G, H]$ and $\mathrm{gp}[G, H]$ are the same, completing the proof.

REMARK. It follows that if G and H are topological groups and k_ω-spaces, $G \coprod H$ contains a free topological group $FG[G, H]$ on a k_ω-space $[G, H]$. In this case we can draw somewhat stronger conclusions than Theorem 3; for instance, $G \coprod H$ is (except trivially) not metrizable and not SIN. (A topological group is said to be a SIN *group* if every neighbourhood of e contains a neighbourhood of the identity invariant under inner automorphisms of the group.) This leads us to ask

QUESTION 3. *If G and H are topological groups, at least one of which is not a discrete space, can $G \coprod H$ be*

(a) metrizable, or

(b) a SIN *group?*

By methods exactly similar to those used in Theorem 5 we obtain

THEOREM 6. *Let G and H be topological groups which are k_ω-spaces; let A be a closed subgroup of G and B be a closed subgroup of H. Then the subgroup of $G \coprod H$ generated by $A \cup B$ is closed and is (topologically and algebraically) $A \coprod B$.*

For general G and H , A and B closed does imply that the group generated by $A \cup B$ in $G \coprod H$ is closed; this however requires a careful examination of the Graev topology $(G * H, \tau)$ introduced before Theorem 1. It does not provide an answer to:

QUESTION 4. *Let* G *and* H *be topological groups and* A *and* B *closed subgroups of* G *and* H *respectively. Let* $\mathrm{gp}(A \cup B)$ *denote the subgroup of* $G \coprod H$ *generated by* $A \cup B$. *Algebraically it is* $A * B$. *Is* $\mathrm{gp}(A \cup B)$ *the topological free product* $A \coprod B$ *?*

It is natural to ask whether the topology of $G \coprod H$ depends only on the topologies of G and H or also on the group structures. One may be inclined to conjecture that if $f_1 : G_1 \to H_1$ and $f_2 : G_2 \to H_2$ are hoemomorphisms, perhaps a homeomorphism $f_1 * f_2 : G_1 * G_2 \to H_1 * H_2$ can be constructed by letting $f_1 * f_2(r_1s_1 \ldots r_ns_n) = f_1(r_1)f_2(s_1) \ldots f_1(r_n)f_2(s_n)$, where $r_i \in G_1$ and $s_i \in G_2$. This fails in general! For instance, if $\{s_\delta\}$ is a net converging to e in G_2 , $f_2(e) = e$ and r_1 and r_2 are elements of G_1 with $f_1(r_1)f_1(r_2) \neq f_1(r_1r_2)$, then

$$\lim f_1 * f_2(r_1s_\delta r_2) = \lim f_1(r_1)f_2(s_\delta)f_1(r_2) = f_1(r_1)f_1(r_2)$$

while

$$f_1 * f_2(\lim r_1s_\delta r_2) = f_1 * f_2(r_1r_2) = f_1(r_1r_2) \neq f_1(r_1)f_1(r_2) ,$$

so $f_1 * f_2$ is discontinuous.

In the k_ω-space case, another approach succeeds:

THEOREM 7. *Let* G_i *and* H_i *be topological groups which are* k_ω*-spaces, for* $i = 1, 2$. *If* G_i *is homeomorphic to* H_i , $i = 1, 2$ *then* $G_1 \coprod G_2$ *is homeomorphic to* $H_1 \coprod H_2$.

PROOF. As $G_1 \coprod G_2$ is homeomorphic to $G_1 \times G_2 \times FG[G_1, G_2]$ and $H_1 \coprod H_2$ is homeomorphic to $H_1 \times H_2 \times FG[H_1, H_2]$ and as $FG(X)$ and $FG(Y)$ are homeomorphic if X and Y are homeomorphic (independent of the choice of basepoints) it will suffice to show that $[G_1, G_2]$ is homeomorphic to $[H_1, H_2]$. Let $f_i : G_i \to H_i$ be a homeomorphism for $i = 1, 2$; since topological groups are homogeneous, we may assume that the f_i have been chosen so that $f_i(e) = e$ for each i . Hence the diagram

$$\begin{array}{ccc} G_1 \times G_2 & \xrightarrow{f_1 \times f_2} & H_1 \times H_2 \\ \Big\downarrow c & & \Big\downarrow c \\ [G_1,\ G_2] & \xrightarrow[j]{} & [H_1,\ H_2] \end{array}$$

is commutative, where $j([g_1, g_2]) = [f_1(g_1), f_2(g_2)]$, and as each vertical map is a quotient map, j is a homeomorphism. This completes the proof.

In view of this it appears that general solutions to Question 2 *(a)* and 2 *(b)* would allow a general solution of:

QUESTION 5. *Let* G_i *and* H_i *be topological groups for* $i = 1, 2$. *If* G_i *is homeomorphic to* H_i *for* $i = 1, 2$ *is* $G_1 \coprod G_2$ *necessarily homeomorphic to* $H_1 \coprod H_2$ *?*

It was shown in Ordman [12] that if G and H are arcwise connected topological groups, then the fundamental group

$$\pi(G \coprod H) = \pi(G \times H) \times L = \pi(G) \times \pi(H) \times L$$

for some group L . It was conjectured that L is always trivial. We now see that $\pi(G \coprod H) = \pi(G) \times \pi(H) \times \pi(\mathrm{gp}[G, H])$, where $\mathrm{gp}[G, H]$ has the induced topology from $G \coprod H$. Further if G and H are k_ω-spaces, then

$$\pi(G \coprod H) = \pi(G) \times \pi(H) \times \pi(FG[G, H]) .$$

So the group L has now been identified. However we have been unable to prove that $\pi(FG[G, H])$ is trivial in any case other than the one covered in [12]; that is, when G and H are countable CW-complexes with exactly one-zero-cell. It seems reasonable to conjecture that if G and H are simply connected then $\pi(G \coprod H) = \pi(G) \times \pi(H)$. However for this we need to answer

QUESTION 6. *If* X *is simply connected is* $FG(X)$ *necessarily simply connected? Is it true under the additional assumption that* X *is a* k_ω*-space?*

5. Examples

We conclude by giving two elementary examples which bear on the preceding.

EXAMPLE 1. The map $c : G \times H \to [G, H] \subset (G * H, \tau)$ is not a quotient map, in general, where τ is Graev's topology. Let $G = H = R$, the additive group of reals with the usual topology. Consider the sequence $a_n = (n, 1/n)$ in $R \times R$. Now $c(a_n)$ converges to e in $(R * R, \tau)$, for

$\rho(c(a_n), e) = \min(|n|, |1/n|) = 1/n \to e$, where ρ is the metric (described in §3) arising from the usual metric on each copy of R . However $c(a_n)$ fails to converge to e in $R \amalg R$. To see this note that R is a k_ω-space with decomposition $R = \bigcup[-n, n]$. Since $\{c(a_k) : k = 1, 2, \ldots\}$ has finite intersection with each $\text{gp}_n([-n, n] \cup [-n, n])$ (here the first $[-n, n] \subset R = G$, the second $[-n, n] \subset R = H$), it is a closed set in $R \amalg R$ and hence does not converge to e .

Since $c(a_n) \in [R, R]$ for all n and $e \in [R, R]$, it follows that $[R, R]$ is topologized differently in $(R * R, \tau)$ than in $R \amalg R$. Hence answering Question 2 will require more than an appeal to Graev's topology.

Incidentally the above argument also shows that the topology constructed in Ordman [11 (I)] also yields a topology on $R * R$ other than the free product topology.

EXAMPLE 2. While the free product of compact groups is a k_ω-space, it is very large. Although every discrete subgroup of a compact group is finite, the free product $T \amalg T$ of two circle groups contains a discrete subgroup which is not even finitely generated. Consider the subgroup $\{e, a\}$ of order 2 of the first factor and the subgroup $\{e, b, b^2\}$ of order 3 of the second factor. The free product $\{e, a\} \amalg \{e, b, b^2\}$ is discrete and by Theorem 6 it is a subgroup of $T \amalg T$. Hence its subgroup $\text{gp}[\{e, a\}, \{e, b, b^2\}]$, the free group on the two generators $x = [a, b]$ and $y = [a, b^2]$ is discrete. This group in turn contains the free group on the countable set $\{x, yxy^{-1}, y^2xy^{-2}, \ldots\}$.

On the other hand, compact subgroups of $T \amalg T$ are very small. Every compact subset of $T \cup T$ is contained in some group $\text{gp}_n(T \cup T)$; that is, has bounded word length. However the only subgroups of $T * T$ with bounded word length are those which are conjugates of subgroups of one of the two factors. Hence every compact subgroup $T \amalg T$ is either finite, or a conjugate of one of the two factors and hence itself a circle group.

QUESTION 7. *What are the locally compact subgroups of* $T \amalg T$ *?*

References

[1] R.M. Dudley, "Continuity of homomorphisms", *Duke Math. J.* 28 (1961), 587-594. MR25#141.

[2] М.И. Граев [M.I. Graev], "Свободные топологические группы" [Free topological groups], *Izv. Akad. Nauk SSSR Ser. Mat.* 12 (1948), 279-324; *Amer. Math. Soc. Transl.* 35 (1951), pp. 61; reprinted *Amer. Math. Soc. Transl.* (1) 8 (1962), 305-364. MR10,11.

[3] М.И. Граев [M.I. Graev], "О свободных произведениях топологических групп" [On free products of topological groups], *Izv. Akad. Nauk SSSR Ser. Mat.* 14 (1950), 343-354. MR12,158.

[4] J. Mack, Sidney A. Morris and E.T. Ordman, "Free topological groups and the projective dimension of a locally compact abelian group", *Proc. Amer. Math. Soc.* 40 (1973), 303-308.

[5] А.А. Марков [A.A. Markov], "О свободных топологических группах" [On free topological groups], *Izv. Akad. Nauk SSSR Ser. Mat.* 9 (1945), 3-64; *Amer. Math. Soc. Transl.* 30 (1950), 11-88; reprinted *Amer. Math. Soc. Transl.* (1) 8 (1962), 195-272. MR3,36.

[6] Sidney A. Morris, "Free products of topological groups", *Bull. Austral. Math. Soc.* 4 (1971), 17-29. MR43#410.

[7] Sidney A. Morris, "Local compactness and free products of topological groups I and II", submitted.

[8] Sidney A. Morris, "Free products of Lie groups", *Colloq. Math.* (to appear).

[9] Sidney A. Morris and H.B. Thompson, "Invariant metrics on free topological groups", *Bull. Austral. Math. Soc.* 9(1973), 83-88.

[10] Sidney A. Morris and H.B. Thompson, "Free topological groups with no small subgroups", *Proc. Amer. Math. Soc.* (to appear).

[11] E.T. Ordman, "Free products of topological groups with equal uniformities I and II", *Colloq. Math.* (to appear).

[12] E.T. Ordman, "Free products of topological groups which are k_ω-spaces", *Trans. Amer. Math. Soc.* (to appear).

[13] H.B. Thompson, "Remarks on free topological groups with no small subgroups", *J. Austral. Math. Soc.* (to appear).

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