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## VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY SOLVABLE AND NILPOTENT GROUPS

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1. This paper is a sequel to [6]. In [2] it was shown that any connected Lie group in a variety of topological groups generated by solvable (nilpotent) connected Lie groups is solvable (nilpotent). This result was extended in [6] from solvable connected Lie groups to solvable connected locally compact groups. We prove a similar result here for the nilpotent case and also attack the connectedness condition.

Our notation and terminology will be that introduced in [6], and we will use the following basic result [1] on generating varieties:

THEOREM. If  $\Omega$  is any class of topological groups and G is a Hausdorff group in  $V(\Omega)$ , then  $G \in SC\overline{QSD}(\Omega)$ .

2. Our first theorem generalizes Theorem 2 of [6].

THEOREM 1. If  $\Omega$  is a class of locally compact groups each of which has the component of the identity solvable, then any connected locally compact group G in  $V(\Omega)$  is solvable.

Proof. According to the theorem in Section 1,  $G \in SC\overline{QSD}(\Omega)$ ; that is, G is a subgroup of a product  $\prod_{i \in I} A_i$ , where each  $A_i \in \overline{QSD}(\Omega)$ .

We claim that each  $A_i$  has the property that the component of the identity is solvable. Firstly, we observe that each member of  $\overline{SD}(\Omega)$  has this property. Let  $B_i \in \overline{SD}(\Omega)$  and let f be an open continuous homomorphism of  $B_i$  onto  $A_i$ . If  $K_i$  and  $K_i$  are the components of the identity in  $A_i$  and  $B_i$ , respectively, then, by Theorem 7.12 of [3],  $f(K_i)$  is dense in  $K_i$ . Since  $K_i$  is solvable,  $f(K_i)$  is solvable and, consequently,  $K_i$  is solvable; that is, our claim has been proved.

Let  $p_i(G)$  be the projection of G in  $A_i$ , and let  $G_i$  be the closure in  $A_i$  of  $p_i(G)$ . Since G is connected,  $G_i \subset K_i$ . Thus, each  $G_i$  is a solvable connected locally compact group. Finally, noting that  $G \in SC\{G_i : i \in I\}$ , we infer, by Theorem 2 of [6], that G is solvable.

COROLLARY. If  $\Omega$  is a class of solvable locally compact groups and G is a connected locally compact group in  $V(\Omega)$ , then G is solvable.

Remark. We note that if the connectedness restriction on G is removed, then the resulting proposition is false. For example, let  $A_m$ ,  $m=1,2,\ldots$ , be discrete finite solvable groups such that each  $A_m$  is of solvability length not less than m. Then the product group  $\prod_{m=1}^{\infty} A_m$  is a compact non-solvable group in  $\mathbb{V}\{A_m: m=1,2,\ldots\}$ . However, we shall see that we can drop the connectedness restriction on G if we insist instead that G be a Lie group.

LEMMA 2. If  $\Omega$  is any class of topological groups and G is a discrete in  $\mathbf{V}(\Omega)$ , then  $G \in \overline{\mathbf{QSD}}(\Omega)$ .

Proof. We will show that  $G \in \overline{SDQSD}(\Omega)$ . This suffices as clearly [1]

$$\overline{\mathbf{S}}\mathbf{D}\overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}(\varOmega) \subset \overline{\mathbf{S}}\overline{\mathbf{Q}}\mathbf{D}\overline{\mathbf{S}}\mathbf{D}(\varOmega) \subset \overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}\overline{\mathbf{S}}\mathbf{D}(\varOmega) = \overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}(\varOmega).$$

If we put  $\Gamma = \overline{\mathbf{QSD}}(\Omega)$ , then, using the theorem in Section 1, we infer that  $G \in \mathbf{SC}(\Gamma)$ ; that is, G is a subgroup of a product  $\prod_{i \in I} F_i$ , where each  $F_i$  is in  $\Gamma$ . Since G is discrete, there exist  $a_1, \ldots, a_n$  in I such that

$$e = (\prod_{i \in I} O_i) \cap G,$$

where e is the identity element of G, each  $O_i$  is an open set in  $F_i$  and, for  $i \neq a_j$  and some  $j \in \{1, ..., n\}$ ,  $O_i = F_i$ . Let  $F = \prod_{j=1}^n F_{a_j}$  and let p be the natural projection mapping of  $\prod_{i \in I} F_i$  onto F. It is readily seen that p(G) is isomorphic to G. Thus  $G \in \mathbf{SD}(\Gamma)$ . Finally, we note that since G is complete,  $G \in \mathbf{SD}(\Gamma)$ .

THEOREM 3. If  $\Omega$  is a class of solvable locally compact groups, then any Lie group G in  $V(\Omega)$  is solvable.

Proof. Let C be the component of the identity in G. Since C is a connected Lie group in  $V(\Omega)$ , by Theorem 1 we infer that C is solvable. Now, G/C is a totally disconnected Lie group, and hence it is discrete. Therefore, by Lemma 2,  $G/C \in \overline{\mathbb{QSD}}(\Omega)$  and, consequently, must be solvable. Finally, since C and G/C are solvable, so G is too.

THEOREM 4. If  $\Omega$  is a class of locally compact groups each of which has the component of the identity nilpotent, then any connected locally compact group in  $V(\Omega)$  is nilpotent.

Proof. According to Section 4.6 of [5], G has a compact normal subgroup N such that G/N is a Lie group. Further, by Section 4.13 of [5], N is contained in a compact connected subgroup M of G. Since

 $M \in V(\Omega)$ , by Theorem 1 of [6] we infer that M is abelian. Thus N is a compact normal abelian subgroup of G, and, consequently, [4] it is central. We now prove that G/N is nilpotent, and hence G is nilpotent.

Now suppose that each member of  $\Omega$  is connected. Then, as in Theorem 2 of [6],  $V(\Omega) = V(\Gamma)$  for some class  $\Gamma$  of connected nilpotent Lie groups. Since G/N is a connected Lie group in  $V(\Gamma)$ , by Corollary 3.4 of [2] we infer that G/N is nilpotent. Finally, note that the connected restriction on  $\Omega$  can be removed as in Theorem 1.

## REFERENCES

- [1] M. S. Brooks, Sidney A. Morris and Stephen A. Saxon, Generating varieties of topological groups, Proceedings of the Edinburgh Mathematical Society (to appear).
- [2] Su-shing Chen and Sidney A. Morris, Varieties of topological groups generated by Lie groups, ibidem 18 (1972), p. 49-53.
- [3] E. Hewitt and K. A. Ross, Abstract harmonic analysis I, New York 1963.
- [4] Kenkichi Iwasawa, Some types of topological groups, Annals of Mathematics 50 (1949), p. 507-558.
- [5] D. Montgomery and L. Zippin, Topological transformation groups, Interscience tracts in pure and applied mathematics I, New York 1955.
- [6] Sidney A. Morris, On varieties of topological groups generated by solvable groups, Colloquium Mathematicum 25 (1972), p. 225-226.

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