Topology Meets Number Theory

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Abstract. Liouville proved the existence of a set $\mathcal{L}$ of transcendental real numbers now known as Liouville numbers. Erdős proved that while $\mathcal{L}$ is a small set in that its Lebesgue measure is zero, and even its s-dimensional Hausdorff measure, for each $s > 0$, equals zero, it has the Erdős property, that is, every real number is the sum of two numbers in $\mathcal{L}$. He proved $\mathcal{L}$ is a dense G_δ-subset of $\mathbb{R}$ and every dense G_δ-subset of $\mathbb{R}$ has the Erdős property. While being a dense G_δ-subset of $\mathbb{R}$ is a purely topological property, all such sets contain $\mathfrak{c}$ Liouville numbers. Each dense G_δ-subset of $\mathbb{R}$, including $\mathcal{L}$, is homeomorphic to the product $\mathbb{N}^{\aleph_0}$ of copies of the discrete space $\mathbb{N}$ of all natural numbers. Also this product space is homeomorphic to the space $\mathbb{P}$ of all irrational real numbers and the space $\mathcal{T}$ of all transcendental real numbers. Hence every dense G_δ-subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$. Indeed any dense G_δ-subset of $\mathbb{R}$ has a chain $X_m, m \in (0,\infty)$, of homeomorphic dense G_δ-subsets such that $X_m \subset X_n$, for $n < m$, and $X_n \setminus X_m$ has cardinality $\mathfrak{c}$. Finally every real number $r \neq 1$ is equal to a^b, for some $a, b \in \mathcal{L}$.

1. INTRODUCTION. The book *Abstract Algebra and Famous Impossibilities*, [**41**, p. vii], begins "The famous problems of squaring the circle, doubling the cube, and trisecting an angle captured the imagination of both professional and amateur mathematicians for over two thousand years. Despite their enormous efforts and ingenious attempts ... the problems would not yield to purely geometrical methods." It was only the development of abstract algebra which made it possible for Pierre Wantzel (1814–1848) in 1837 to solve the second and third of these problems and reduce the other problem to one in number theory. The solutions came in what may be described as **Geometry Meets Abstract Algebra**. The first problem of squaring the circle remained unanswered until 1882 when Ferdinand von Lindemann (1852–1939) proved that π is a transcendental number. This proved that squaring the circle is impossible in what may be described as **Geometry Meets Number Theory**. Chapter 12 of [**41**] describes how **Calculus Meets Abstract Algebra** resulted in indefinite integration being able to be done using computer algebra packages. Some of these packages implement at least part of the decision procedure of Robert Henry Risch (born 1939). The book [**41**] presents this fascinating material in a manner accessible to college students, perhaps with the occasional assistance of their teachers. More advanced material, suitable for graduate students, is not included in the book. In our article we present material which results when **Topology Meets Number Theory**, more precisely when Point-set Topology Meets Transcendental Number Theory. Our aim too is that the majority of it is accessible to college students.

Paul Erdős (1913–1996) and Underwood Dudley (born 1937) say in [**21**] that as far as they know Leonhard Euler (1707–1783) was the first person to define transcendental numbers as we now know them. The basis for this claim is investigated and explained in [**45**].

In 1768 Johann Heinrich Lambert (1728–1777) proved that the number π is irrational and conjectured that e and π are transcendental numbers. A first year college level proof that π is irrational is outlined in [**41**, Exercises 2.1 #8]. Proofs that e and π are transcendental can be found in many books, including [**41**, Chapter 10].

In his book *Joseph Liouville 1809–1882, Master of Pure and Applied Mathematics* [**37**] Jersper Lützen claims Liouville was the most important French mathematician in the generation between Évariste Galois (1811–1832) and Charles Hermite (1822–1901). In 1844 Liouville was the first to prove that transcendental numbers exist [**35**]. Indeed, he introduced an uncountable set $\mathcal{L}$ of transcendental real numbers now known as Liouville numbers. The first example was the *Liouville constant* $\ell = \sum_{n=1}^{\infty} 10^{-n!}$; that is the real number with the digit in the nth decimal place equal to 0, unless $n = k!$, $k = 1, 2, \dots$, in which case it equals 1. So $\ell = 0.110001000000000000000001 0\dots$. It was not until 1895 that Georg F.L.P. Cantor (1845–1918) introduced the notion of uncountability. Cantor also presented what has become known as the Continuum Hypothesis that in very simple terms says that every infinite subset of the set of real numbers either has a one-to-one correspondence with the set of all natural numbers or with the set of all real numbers. David Hilbert (1862–1943) included the Continuum Hypothesis (CH) as the first of his 23 problems presented to the International Congress of Mathematicians in Paris in 1900. In 1963, Paul J. Cohen (1934–2007), using the 1940 work of Kurt Gödel (1906–1978), proved that CH can be neither proved nor disproved if one assumes only the Zermelo-Fraenkel axioms for set theory with the Axiom of Choice (ZFC); that is, it is independent of ZFC. So when we meet subsets of the set of real numbers it is not sufficient to say that they are countable or uncountable, but rather to state what the cardinality of the set is, if one knows it.

Gentle introductions to Topology, Transcendental Number Theory, and Measure Theory can be found in [**42**], [**4**], and [**44**], respectively. The concepts of Hausdorff dimension and Hausdorff measure are introduced in [**42**]. For more advanced texts on Transcendental Number Theory and Descriptive Set Theory, see [**13**] and [**28**].

Many undergraduate students, especially in the early stages of their studies, struggle to grasp fully, and distinguish, the various notions of size in set theory, topology, and measure theory. The concrete examples in this article show in a practical and instructive manner how these notions differ and can be creatively brought together.

For example, we shall meet the following subsets of $\mathbb{R}$: the set $\mathbb{Q}$ of all rational numbers, the middle-third Cantor set $\mathbb{G}$, the set $\mathbb{P}$ of all irrational real numbers, and the set $\mathcal{L}$ of all Liouville numbers.

- $\mathcal{L}$ is dense in $\mathbb{R}$, is uncountable, has Lebesgue measure zero, has s-dimensional Hausdorff measure equal to zero for every $s > 0$ [**44**, Theorem 2.4], and is totally disconnected.
- $\mathbb{G}$ is not dense in $\mathbb{R}$, is uncountable, has Lebesgue measure equal to zero, has Hausdorff dimension $s = \frac{\log 2}{\log 3}$, has s-dimensional Hausdorff measure equal to 1 [**22**, Theorem 1.14], and is totally disconnected.
- $\mathbb{Q}$ is dense in $\mathbb{R}$, is not uncountable, has Lebesgue measure zero, has Hausdorff dimension equal to zero, and is totally disconnected.
- $\mathbb{P}$ is dense in $\mathbb{R}$, is uncountable, has full Lebesgue measure, has Hausdorff dimension equal to 1, and is totally disconnected.

We shall also see that the topological space $\mathbb{P}$ has a subspace homeomorphic to $\mathbb{G}$, and, perhaps surprisingly, $\mathbb{G}$ has a subspace homeomorphic to $\mathbb{P}$. And to round off the surprises, $\mathbb{P}$ and $\mathcal{L}$ are homeomorphic topological spaces.

2. PRELIMINARIES.

We record some notation:

(i) $\mathbb{R}$ is the topological space of all real numbers with the Euclidean topology;
(ii) $\mathbb{N}$ is the discrete space of all natural numbers $\{1, 2, \dots\}$;
(iii) $\mathbb{Z}$ is the discrete space of all integers;

(iv) $\mathbb{Q}$ is the set of all rational numbers with the topology it inherits as a subspace of $\mathbb{R}$;

(v) $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ is the set of all irrational real numbers with the topology it inherits as a subspace of $\mathbb{R}$;

(vi) $\mathbb{A}$ is the set of all algebraic real numbers with the topology it inherits as a subspace of $\mathbb{R}$;

(vii) $\mathcal{T} = \mathbb{R} \setminus \mathbb{A}$ is the set of all transcendental real numbers with the topology it inherits as a subspace of $\mathbb{R}$;

(viii) $\mathbb{Q}^{>0}$ is the set of all positive rational numbers with the topology it inherits as a subspace of $\mathbb{R}$;

(ix) $\mathfrak{c}$ is the cardinality of the continuum, that is the cardinality of $\mathbb{R}$;

(x) $\aleph_0$ is the cardinality of the set $\mathbb{N}$ of natural numbers;

(xi) $\mathbb{G}$ is the middle-third Cantor set (named after Georg Cantor), [**42**], with the compact topology it inherits as a subspace of $\mathbb{R}$; it consists of those real numbers in $[0,1]$ with ternary (base 3) expansion not using the digit 1, and so $\mathbb{G}$ is an uncountable set. Indeed its cardinality is $\mathfrak{c}$.

We begin our discussion with the fact that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$; that is, each real number ξ can be approximated as closely as we like by a rational number. In other words, given any $\varepsilon > 0$ there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $|\xi - \frac{p}{q}| < \varepsilon$. However we can ask a finer question.
Given a real number ξ, does there exist a rational number $\frac{p}{q}$ such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q}\,?$$

It is easily checked that if $\xi \in \mathbb{N}$, then the answer is in the negative. On the other hand, for any irrational number ξ there exist infinitely-many $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \leq \frac{1}{q}.$$

This result was first proved by Johann P.G.L. Dirichlet (1805–1859) and is known as Dirichlet's approximation theorem. For a proof see, for example, [**43**, Theorem 4.1].

Definition 1. A real number ξ is called a *Liouville number* if for every positive integer n, there exists a pair of integers (p,q) with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}.$$

We see immediately that the Liouville constant ℓ is a Liouville number.

We now show that any sum of a rational number and a Liouville number is a Liouville number.

Proposition 2. *If $r \in \mathbb{Z}$, $s \in \mathbb{N}$, and l is any Liouville number, then $\frac{r}{s} + l$ is a Liouville number.*

Proof. As l is a Liouville number, for every positive integer m, there exists a pair of integers (p,q) with $q > 1$, such that

$$0 < \left| l - \frac{p}{q} \right| < \frac{1}{q^m}. \tag{1}$$

Let $n \in \mathbb{N}$. We choose $m \in \mathbb{N}$ sufficiently large that $q^{m-n} > s^n$, which means $\frac{1}{q^m} < \frac{1}{s^n q^n}$. (We shall see why we make this choice of m soon.)

$$\begin{aligned}
\left| l - \frac{p}{q} \right| &= \left| \left(l + \frac{r}{s} \right) - \left(\frac{p}{q} + \frac{r}{s} \right) \right| \\
&= \left| \left(l + \frac{r}{s} \right) - \frac{ps + rq}{qs} \right| \\
&= \left| \left(l + \frac{r}{s} \right) - \frac{p'}{qs} \right|, \text{ where } p' = ps + rq \text{ ,} \\
&< \frac{1}{q^m} \text{ , by (1)} \\
&< \frac{1}{(qs)^n} \text{ , by the choice of } m \\
&= \frac{1}{(q')^n} \text{ , where } qs = q'; \quad \text{that is, } \left| \left(l + \frac{r}{s} \right) - \frac{p'}{q'} \right| < \frac{1}{(q')^n}.
\end{aligned}$$

So $l + \frac{r}{s}$ is a Liouville number. ■

The above proof is easily modified to prove that if $r \in \mathbb{Z}$, $s \in \mathbb{N}$, and l is any Liouville number, then $\frac{r}{s} \cdot l$ is a Liouville number.

We now use Proposition 2 to prove the more important Proposition 3.

Proposition 3. *The set $\mathcal{L}$ of Liouville numbers is dense in $\mathbb{R}$.*

Proof. It suffices to show that for $a, b \in \mathbb{R}$ with $a < b$ there exists a Liouville number in the open interval (a, b). Let l be any Liouville number and choose any rational number r in the open interval $(a - l, b - l)$. Then $a < l + r$ and $l + r < b$. By Proposition 2, $l + r$ is a Liouville number and we see that it lies in the interval (a, b). ■

3. G_δ-SETS.

Definition 4. A subset of a topological space X is said to be a *G_δ-subset of X* if it is a countable intersection of open subsets of X. A subset of X is said to be a *F_σ-subset of X* if its complement in X is a G_δ-subset of X, that is, it is a countable union of closed subsets of X. A subset of a topological space X is said to be a *Borel set*, (named after Émil Borel (1871–1956)) if it can be obtained from the open sets using only the operations of countable union, countable intersection, and relative complement. (The relative complement of a set A in a set B is $B \setminus A$.) Any continuous image of the space $\mathbb{P}$ is said to be an *analytic* set.

Our focus will be on when $X = \mathbb{R}$ or $X = \mathbb{P}$.

Each of the statements in the following proposition is straightforward to verify.

Proposition 5. (i) *Every open subset of $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$;*

(ii) *every open set of $\mathbb{R}$ is an F_σ-subset of $\mathbb{R}$ since it is a countable union of open intervals in $\mathbb{R}$, and each open interval in $\mathbb{R}$ is a countable union of closed intervals in $\mathbb{R}$;*

(iii) *every closed subset of $\mathbb{R}$ is both a G_δ-subset of $\mathbb{R}$ and an F_σ-subset of $\mathbb{R}$;*

(iv) *every singleton subset $\{x\}$ of $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$;*

(v) *the middle-third Cantor set $\mathbb{G}$ is compact, and so is a closed subset of $\mathbb{R}$, and thus is both a G_δ-subset of $\mathbb{R}$ and an F_σ-subset of $\mathbb{R}$;*

(vi) *if X is a countable subset of $\mathbb{R}$, then $\mathbb{R} \setminus X$ is a G_δ-subset of $\mathbb{R}$;*

(vii) *$\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ is a G_δ-subset of $\mathbb{R}$;*

(viii) *$\mathcal{T} = \mathbb{R} \setminus \mathbb{A}$ is a G_δ-subset of $\mathbb{R}$;*

(ix) *if Y is a G_δ-subset of $\mathbb{R}$ and X is a countable subset of Y, then $Y \setminus X$ is a G_δ-subset of $\mathbb{R}$, since if $X = \{x_i : i \in I\}$, where I is a countable set, then*

$$Y \setminus X = \bigcap_{i \in I} (Y \cap (\mathbb{R} \setminus \{x_i\}));$$

(x) *if X is a G_δ-subset of $\mathbb{R}$ and Y is a G_δ-subset of the topological space X, then Y is a G_δ-subset of $\mathbb{R}$; in particular this is true when $X = \mathbb{P}$ or $X = \mathbb{G}$;*

(xi) *the complement $[0,1] \setminus \mathbb{G}$, of the middle-third Cantor set in the closed unit interval $[0,1]$, is a G_δ-subset of $\mathbb{R}$, as it is an open subset of $[0,1]$ which is a G_δ-subset of $\mathbb{R}$;*

(xii) *if Z_1 and Z_2 are topological spaces, $f : Z_1 \to Z_2$ a continuous mapping, X a G_δ-subset of Z_2, then $Y = f^{-1}(X)$ is a G_δ-subset of Z_1. In particular, this is true when $Z_1 = Z_2 = \mathbb{R}$.*

(xiii) *Every G_δ-set and every F_σ-set is a Borel set, and every Borel subset of $\mathbb{R}$ is an analytic set. Every continuous image of a Borel subset of $\mathbb{R}$ is an analytic set.*

(xiv) *The sets $\mathbb{P}$, $\mathcal{T}$, and $\mathbb{G}$ are Borel sets and analytic sets.*

(xv) *The complement in $\mathbb{R}$ of a Borel set is a Borel set. However, the complement of an analytic set is analytic if and only if it is Borel.*

(xvi) *Every analytic set is Lebesgue measurable.*

(xvii) *Every set of positive Hausdorff dimension is uncountable [**42**, Proposition A4.1.17].*

One of the most important theorems in topology is the Baire Category Theorem, [**42**]. Versions of it are known, for example, for complete metrisable spaces and locally compact Hausdorff spaces. It was proved for $\mathbb{R}$ by William Fogg Osgood (1864–1943) in 1897 and independently in 1899 by René-Louis Baire (1874–1932) for n-dimensional Euclidean space $\mathbb{R}^n$, for any $n \in \mathbb{N}$.

Theorem 6 (Baire Category Theorem). *If X_n, $n \in \mathbb{N}$, are dense open subsets of a nonempty locally compact subspace Y of $\mathbb{R}$ (such as $Y = \mathbb{R}$ or $Y = I$, for any nontrivial open or closed interval I in $\mathbb{R}$, or $Y = \mathbb{G}$), then $\bigcap_{n \in \mathbb{N}} X_i$ is a dense subset of Y. In particular, $\bigcap_{n \in \mathbb{N}} X_i$ is nonempty.*

We note that if A and B are dense subsets of $\mathbb{R}$, then $A \cap B$ need not be a dense subset of $\mathbb{R}$. Indeed if $A = \mathbb{A}$, the set of all real algebraic numbers, and $\mathcal{T}$ is the set of all real transcendental numbers, then A and B are dense subsets of $\mathbb{R}$ but $A \cap B = \emptyset$. It is also true that if X and Y are G_δ-subsets of $\mathbb{R}$, then $X \cap Y$ can equal $\emptyset$. For example this is the case if $X = [0,1]$ and $Y = [2,3]$. However, the situation is quite different for dense G_δ-subsets.

Theorem 7. *If A and B are dense G_δ-subsets of any locally compact subspace Y of $\mathbb{R}$, then $A \cap B$ is a dense G_δ-subset of Y. Further, if $A_1, A_2, \ldots, A_n, \ldots$ are dense G_δ-subsets of Y, then $\bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ-subset of Y. In particular this is the case if $Y = \mathbb{R}$ or $Y = I$, for I any infinite open or closed interval in $\mathbb{R}$, or $Y = \mathbb{G}$.*

Proof. Let $A = \bigcap_{i\in\mathbb{N}} A_i$ and $B = \bigcap_{i\in\mathbb{N}} B_i$, where each A_i and B_i is an open subset of Y. As $A \subseteq A_i$ and $B \subseteq B_i$ for each $i \in \mathbb{N}$, A and B dense in Y implies that each A_i and B_i is dense in Y. So, by Theorem 6, each $A_i \cap B_i$ is a dense G_δ-subset of Y. Applying Theorem 6 again gives that $A \cap B = \bigcap_{i\in\mathbb{N}} (A_i \cap B_i)$ is a dense subset of Y. As a countable intersection of G_δ-subsets is a G_δ-subset, $A \cap B$ is a dense G_δ-subset of Y. The proof that $\bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ-subset of Y is analogous. ∎

Corollary 8. *Let Y be an uncountable locally compact subspace of $\mathbb{R}$ with no isolated points. If X is a dense G_δ-subset of Y, then it is uncountable. In particular this is the case if $Y = \mathbb{R}$, $Y = \mathbb{G}$, or $Y = I$, where I is any nontrivial open or closed interval in $\mathbb{R}$.*

Proof. Suppose there exists an open set U in Y such that U intersects X in only one point x. Then the non-empty open set $U \cap (Y \setminus \{x\})$ has empty intersection with X. This contradicts X being dense in Y. So our supposition is false and every open set in Y intersects X in more than one point. So for each $x \in X$, the set $X \setminus \{x\}$ is dense in Y.

Now suppose that the set X is countable. Then $\bigcap_{x\in X} X \setminus \{x\} = \emptyset$. But this is a countable intersection of dense G_δ-subsets of Y which, by Theorem 7, is not an empty set. So we have a contradiction, and thus X is uncountable. ∎

We shall see in Corollary 29 that every dense G_δ-subset in $\mathbb{R}$ is not only uncountable, but in fact has cardinality $\mathfrak{c}$.

The sets $\mathbb{Q}$ and $\mathbb{A}$ are dense in $\mathbb{R}$ and countably infinite. So Corollary 8, implies that they are not G_δ-subsets of $\mathbb{R}$. On the other hand, we saw above that their complements in $\mathbb{R}$, namely $\mathbb{P}$ and $\mathcal{T}$, are dense G_δ-subsets of $\mathbb{R}$.

Theorem 9. *The set $\mathcal{L}$ of all Liouville numbers is a dense G_δ-subset of $\mathbb{R}$.*

Proof. Recall that a real number ξ is a Liouville number if and only if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}.$$

For $n \in \mathbb{N}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ with $q > 1$, define the set $X_{n,p,q}$ by

$$X_{n,p,q} = \left\{\xi \in \mathbb{R} : 0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}\right\}.$$

Clearly $X_{n,p,q}$ is an open subset of $\mathbb{R}$. Define X_n as follows:

$$X_n = \bigcup_{p\in\mathbb{Z}, q\in\mathbb{N}, q>1} X_{n,p,q}.$$

So X_n is also an open set. Observing that

$$\mathcal{L} = \bigcap_{n \in \mathbb{N}} X_n$$

is a countable intersection of open sets, we see that $\mathcal{L}$ is a G_δ-subset of $\mathbb{R}$. By Proposition 3, $\mathcal{L}$ is dense in $\mathbb{R}$, which completes the proof of the theorem. ■

4. DENSE G_δ-SETS AND $\mathcal{L}$.

Theorem 10. *If X is a dense G_δ-subset of $\mathbb{R}$, then $X \cap \mathcal{L}$ is a dense G_δ-subset of $\mathbb{R}$. Indeed every dense G_δ-subset of $\mathbb{R}$ contains an uncountable set of Liouville numbers.*

Proof. This follows immediately from Theorem 9, Theorem 7, and Corollary 8. ■

To extend this result from $\mathbb{R}$ to say an interval I or to $\mathbb{G}$ requires caution. The question is: if X is dense in $\mathbb{R}$, is $X \cap I$ or $X \cap \mathbb{G}$ dense in I or $\mathbb{G}$, respectively? The answer is "yes" for I but not necessarily for $\mathbb{G}$. In fact, the set $\mathbb{R} \setminus \mathbb{G}$ is dense in $\mathbb{R}$ but has empty intersection with $\mathbb{G}$.

Theorem 10 provides a first justification for the name of this paper. It also leads us to ask if dense G_δ-subsets of $\mathbb{R}$ exist in profusion. We shall see that if f is a suitably-behaved function, then $f(\mathcal{L})$ is a dense G_δ-subset of $\mathbb{R}$. For example, this is the case if $f(x) = rx^n + s$, for n any odd positive integer and $r, s \in \mathbb{R}, r \neq 0$. Also $\log_e(\mathcal{L}^+)$ is a dense G_δ-subset of $\mathbb{R}$, where $\mathcal{L}^+$ is the set of positive Liouville numbers. Theorem 11 and Corollary 12 describe what we mean by suitably-behaved. But first let us settle the existence of $\mathfrak{c}$ dense G_δ-subsets of $\mathbb{R}$.

If r is any positive real number, then the open interval $(0, r)$ is a G_δ-subset of $\mathbb{R}$. By Theorem 9, $\mathcal{L}$ is a dense G_δ-subset of $\mathbb{R}$. So each of the $\mathfrak{c}$ sets $\mathcal{L} \cup (0, r)$ is a dense G_δ-subset of $\mathbb{R}$ and they are distinct sets.

Theorem 11. (cf. [**1, 34**]). *Let Z_1 and Z_2 be topological spaces and $f : Z_1 \to Z_2$ a homeomorphism of Z_1 onto Z_2. If X is a dense G_δ-subset of Z_1, then $f(X)$ is a dense G_δ-subset of Z_2.*

Proof. As f is a homeomorphism of Z_1 onto Z_2, it has a continuous inverse mapping $f^{\leftarrow} : Z_2 \to Z_1$ which is surjective. So $Z_2 = (f^{\leftarrow})^{-1}(Z_1)$. As observed in Proposition 5(xii), the inverse image of a G_δ-subset is a G_δ-subset. So it follows that $f(X)$ is a G_δ-subset of Z_2. That $f(X)$ is dense in Z_2 follows from the fact that a continuous image of a dense set in Z_1 is dense in Z_2. ■

Corollary 12. *If in Theorem 11, $Z_1 = Z_2 = \mathbb{R}$, then $f(\mathcal{L})$ is a dense G_δ-subset of $\mathbb{R}$.*

Theorem 11 applies, for example, for $f = r\sin,\ r\cos,\ r\exp,\ r\log_e$, for a positive real number r and for Z_1 and Z_2 suitable intervals in $\mathbb{R}$. We shall have more to say on this later in this paper, but for now we mention that [**16**, Theorem 3.1] says that $\sin(l), \cos(l), \exp(l), \log_e(l)$, for l any positive Liouville number, are transcendental numbers.

Corollary 12 applies, for example, if $f(x) = rx^n + s$, for n any odd positive integer and $r, s \in \mathbb{R}, r \neq 0$. For example if r is any nonzero real algebraic number, then $r\mathcal{L}$ is a dense G_δ-subset of $\mathbb{R}$ and so, by Theorem 10, $r\mathcal{L}$ contains an uncountable subset of $\mathcal{L}$. However, it is shown in [**17**], that for $r = \sqrt[m]{\frac{2}{3}}$, each $r\mathcal{L} \cap (\mathbb{R} \setminus \mathcal{L}) \neq \emptyset$, where m is any integer ≥ 1.

We now prove a beautiful, and apparently new, property of the set of Liouville numbers.

Theorem 13. *If s is any positive real number with $s \neq 1$, then there exist $a, b \in \mathcal{L}$, with $a, b > 0$, such that $s = a^b$.*

Proof. If s is any positive real number such that $s \neq 1$, put $r = \frac{1}{\log_e(s)}$. So $s = \exp\frac{1}{r}$. Then $f(x) = r\log_e(x)$ is a homeomorphism of $(0, \infty)$ onto $(-\infty, \infty)$.

The set $\mathcal{L}^+$ of positive numbers in $\mathcal{L}$ is $\mathcal{L} \cap (0, \infty)$ and is a dense G_δ-subset of $(0, \infty)$. Then by Theorem 11, $f(\mathcal{L}^+)$ is a dense G_δ-subset of the set of all real numbers, and so by Theorem 10, $f(\mathcal{L}^+)$ contains a dense G_δ subset L_0 of the set of all Liouville numbers. So for any $l_2 \in L_0$, there is an $l_1 \in \mathcal{L}^+$ such that $f(l_1) = r\log_e(l_1) = l_2$.

As l_2 is a Liouville number, so too is $b = \frac{1}{l_2}$. Thus $\frac{1}{r} = \log_e(a^b)$, where $a = l_1$. Hence $\exp(\frac{1}{r}) = a^b = s$, as required. ■

It is clear from the proof of Theorem 13 that for each positive real number $s \neq 1$, there is an uncountable number of different pairs (a, b) which satisfy the theorem, since l_2 can be chosen to be any member of the set L_0, which is a dense G_δ-subset of $\mathcal{L}$ and hence also of $\mathbb{R}$, so by Corollary 8, is uncountable.

We stated Theorem 13 for the set $\mathcal{L}$ of Liouville numbers, rather than a more general dense G_δ-subset of $\mathbb{R}$ or even a dense G_δ-subset of $\mathcal{L}$. We identify just one place in the proof where we used a special property of $\mathcal{L}$, namely that if $l_2 \in \mathcal{L}$, then $\frac{1}{l_2} \in \mathcal{L}$. However it is readily seen that if X is a dense G_δ-subset of $\mathbb{R}$, then it has a dense G_δ-subset Y with the property that $y \in Y \implies \frac{1}{y} \in Y$. The next lemma provides the proof.

Lemma 14. *Every dense G_δ-subset X of $\mathbb{R}$ has a dense G_δ-subset Z such that $y \in Z \setminus \{0\} \implies \frac{1}{y} \in Z$.*

Proof. Let f be the homeomorphism $f(x) = \frac{1}{x}$ of $\mathbb{R} \setminus \{0\}$ onto itself. So the set $f(X \cap (\mathbb{R} \setminus \{0\}))$ is a dense G_δ-subset of $\mathbb{R} \setminus \{0\}$. Thus

$$Y = f(X \cap (\mathbb{R} \setminus \{0\})) \cap (X \cap (\mathbb{R} \setminus \{0\}))$$

is a dense G_δ-subset of $\mathbb{R} \setminus \{0\}$, and in particular is not the empty set.

If $y \in Y$, then $f(y) \in f(X \cap (\mathbb{R} \setminus \{0\}))$ and, by the definition of f,

$$f(y) \in f(f(X \cap (\mathbb{R} \setminus \{0\})) = X \cap (\mathbb{R} \setminus \{0\}).$$

So we see that $f(y) \in Y$. Thus $y \in Y \implies f(y) = \frac{1}{y} \in Y$. We complete the proof by putting $Z = Y \cup \{0\}$. ■

So we obtain the following more general result from Lemma 14 and the discussion following Theorem 13.

Theorem 15. *If X is any dense G_δ-subset of $\mathbb{R}$ and s is any positive real number $\neq 1$, then there exist $a, b \in X$, with $a, b > 0$, such that $s = a^b$.*

Noting Theorem 13, it is reasonable to ask for which values of real numbers x and y is x^y a transcendental number. The seventh problem of David Hilbert's 23 problems stated in 1900, asked if x is an algebraic number with $x \neq 0, 1$, and y is an algebraic number which is not a rational real number, is x^y a transcendental number? David Hilbert (1862–1943) regarded this problem as the hardest of his 23 problems and certainly harder than proving or disproving the Riemann Hypothesis [**12**]. (The Riemann Hypothesis, named after Bernhard Riemann (1826–1866), is not only unproved today but is one of the seven Millennium Prize Problems selected in 2000 by the Clay Mathematics Institute. The Clay Institute has pledged a prize of one million dollars for the first correct solution to each problem.) However Hilbert's seventh problem was solved independently in 1934 by Osipovich Gelfond (1906–1968) and by Theodor Schneider (1911-1988). In the Gelfond-Schneider Theorem they provided a positive answer to this problem. See [**41**, Chapter 10] and [**23**]. [**39**] proved that a^l is a transcendental number for every algebraic number $a \notin \{0,1\}$, and $l \in \mathcal{L}$ (or more generally $l \in U$, where U is the Mahler set U [**13, 5**] of transcendental numbers; Mahler sets are discussed in our §6.). Theorem 13 shows that $l_1^{l_2}$ is not necessarily a transcendental number for $l_1, l_2 \in \mathcal{L}$. Further information on this kind of question is provided in [**50**].

Corollary 16. *For each $r \in \mathbb{R} \setminus \{0\}$ and each $n \in \mathbb{N} \setminus \{1\}$,*

$$r = l_1^{l_2^{l_3^{\cdot^{\cdot^{\cdot^{l_n}}}}}}, \textit{ for some } l_i \in \mathcal{L}.$$

See also Theorem 20 below.

Our next theorem was proved in 1962 by Paul Erdős in [**20**].

Theorem 17. *Let X be any dense G_δ-subset of $\mathbb{R}$, $r, s \in \mathbb{R}$, $s \neq 0$. Then there exist $a, b, c, d \in X$ such that*

(i) $r = a + b$; *(Erdős property);*
(ii) $s = c \cdot d$; *(multiplicative Erdős property).*

In particular this is the case if X is the set of all Liouville numbers.

Proof. Define a map f of $\mathbb{R}$ into $\mathbb{R}$ by $f(x) = r - x$. Then f is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, and so $f(X)$ is a dense G_δ-subset of $\mathbb{R}$. So $f(X) \cap X \neq \emptyset$. Let $a \in f(X) \cap X$ and put $f(a) = b$, so that $a, b \in X$. Then $f(a) = b = r - a$; that is, $r = a + b$, as required in (i).

For (ii), let $g(x) = \frac{x}{s}$, for $s \in X, s \neq 0$. Then g is a homeomorphism of $\mathbb{R}$ onto itself. As in (i), $g(X) \cap X$ is a dense G_δ-subset of $\mathbb{R}$ and in particular is nonempty. Let $d \in X \cap g(X)$. For $d \neq 0$, $g(d) \neq 0$. Put $g(d) = \frac{1}{c}$, so that $g(d) = \frac{1}{c} = \frac{d}{s}$; that is, $s = c \cdot d$, as required in (ii). ∎

Note that for c and d in Theorem 17, we could have chosen any member of an uncountable dense G_δ-set, it being the intersection of two dense G_δ-sets. So in Theorem 17, there is an uncountable number of a, b, c, d which satisfy (i) and (ii) for any given r and s.

We now present a result which may be of no great importance, but it is pretty.

Proposition 18. *Let r be any positive real number, and X a dense G_δ-subset of $\mathbb{R}$. Then there exist $l_1, l_2 \in X$ such that $r = l_1 \sin(l_2)$. In particular this is the case if $X = \mathcal{L}$.*

Proof. By Lemma 14 we can, without loss of generality, assume that if $y \in X \setminus \{0\}$, then $\frac{1}{y} \in X$. The function $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(x) = \frac{1}{r} \cdot \sin(x)$ is a homeomorphism of $(0, \frac{\pi}{2})$ onto $(0, \frac{1}{r})$. So f maps the dense G_δ -subset $X \cap (0, \frac{\pi}{2})$ of $(0, \frac{\pi}{2})$ onto the dense G_δ-subset $f(X \cap (0, \frac{\pi}{2}))$ of $(0, \frac{1}{r})$. So $(X \cap (0, \frac{\pi}{2})) \cap f(X \cap (0, \frac{\pi}{2})) \neq \emptyset$.

Let $a \in (X \cap (0, \frac{\pi}{2})) \cap f(X \cap (0, \frac{\pi}{2})$, so that $a = f(b)$, where $b \in X \cap (0, \frac{\pi}{2})$. Thus $a, b \in X \setminus \{0\}$ and $a = \frac{1}{r} \cdot \sin(b)$. So $r = \frac{1}{a} \cdot \sin(b)$. As $a \in X \setminus \{0\}$, $\frac{1}{a} \in X$. Putting $l_1 = \frac{1}{a}$ and $l_2 = b$, we obtain $r = l_1 \cdot \sin(l_2)$. This completes the proof. ∎

Of course the function $\sin$ in Proposition 18 can be replaced by any suitably well-behaved function.

Before moving from this topic, let us have an example using the power of the second part of Theorem 7. But first, we state a definition.

Definition 19. The real numbers $y_1, y_2, \ldots, y_n$ are said to be *algebraically independent* if $\{y_1, y_2, \ldots, y_n\}$ do not satisfy any nontrivial polynomial equation with coefficients in $\mathbb{A}$.

Theorem 20. *Let Y be any countably infinite subset of the set of all positive real numbers. Then there exists a dense G_δ-subset X of $\mathcal{L}^+$ such that $l^y \in X$ for all $y \in Y$ and $l \in X$. Indeed this is the case when Y is the set of* (i) *all positive rational numbers,* (ii) *all constructible positive real numbers [**41**],* (iii) *all positive real algebraic numbers,* (iv) *all computable positive real numbers [**46**], or* (v) *the set consisting of π and all positive real numbers which are not algebraically independent of π.*

Proof. As Y is a countably infinite set we can write $Y = \{y_1, y_2, \ldots, y_n, \ldots\}$. First we shall find a set X_n for each y_n. Consider $f_{y_n} : (0,\infty) \to (0,\infty)$ given by $f_{y_n}(r) = r^{y_n}$. By Theorem 11, $f_{y_n}(\mathcal{L}^+)$ is a dense G_δ-subset of $(0,\infty)$. So

$$f_{y_n}(\mathcal{L}^+) \cap \mathcal{L}^+ = X_n \text{ is a dense } G_\delta\text{-subset of } (0,\infty).$$

Observe that $f_{y_n}(l) = l^{y_n} \in X_n$, for each $l \in X_n$. As each X_n is a dense G_δ-subset of $(0,\infty)$, $\bigcap\limits_{n\in\mathbb{N}} X_n$ is a dense G_δ-subset of $(0,\infty)$ and so is nonempty. We define X to be this intersection. We now have that X is a dense G_δ-subset of $(0,\infty)$ which is contained in $\mathcal{L}^+$ and satisfies the theorem. ∎

Theorem 20 complements [**40**, Theorem 6.2].

5. TOPOLOGY AND CARDINALITY. In this section we shall use a beautiful characterization of the topological space $\mathbb{P}$, the set of all irrational real numbers, from [**53**].

Definition 21. A topological space X is said to be *topologically complete* (or *completely metrisable*) if the topology of X is the same as the topology induced by a complete metric on X.

Of course any complete metric space has a topologically complete topology. We shall see that, for example, $\mathbb{P}$ is topologically complete.

The next result follows from $(1) \implies (3)$ and $(4) \implies (1)$ in [**53**, Theorem A.6.3].

Theorem 22. *A subspace Y of a separable metric topologically complete space X is a G_δ-subset of X if and only if Y is topologically complete.*

Definition 23. A topological space X is said to be *nowhere locally compact* if no point of X has a neighborhood with compact closure.

Now [**53**, Theorem 1.9.8] gives the following result.

Theorem 24. *The space of all irrational real numbers $\mathbb{P}$ is topologically the unique nonempty, separable, metrisable, topologically complete, nowhere locally compact, and zero-dimensional space.*

Lemma 25. *Let X be a dense subspace of $\mathbb{P}$. Then X is nowhere locally compact.*

Proof. We are required to show that X is nowhere locally compact; that is, no point x of X has an open neighborhood N with compact closure $\overline{N}$ in $\mathbb{P}$. Without loss of generality we can assume that $N = (c,d) \cap X$ for some $c, d \in \mathbb{R}$. As X is dense in $\mathbb{P}$, $N = (c,d) \cap X$ is dense in (c,d). Suppose the closure in $\mathbb{P}$, $\overline{N}$, of N is compact in $\mathbb{P}$, then it is compact in $\mathbb{R}$; that is $\overline{(c,d) \cap X}$ is compact in $\mathbb{R}$. This implies $\overline{N} = \overline{(c,d) \cap X} = [c,d]$. But $\overline{N}$ is a subset of $\mathbb{P}$ which $[c,d]$ is not. This contradiction shows X is nowhere locally compact. ■

Corollary 26. *The set $\mathcal{T}$ of all transcendental real numbers is nowhere locally compact.*

The next result is a consequence of Theorem 22 and Lemma 25 and Theorem 24.

Corollary 27. *Every dense G_δ-subset of the set $\mathbb{P}$ of all irrational real numbers is homeomorphic to $\mathbb{P}$ and to $\mathbb{N}^{\aleph_0}$, a countably infinite product of copies of the discrete space of all natural numbers. In particular, the space $\mathcal{T}$ of all transcendental real numbers and the space $\mathcal{L}$ of all Liouville numbers, with their subspace topologies from $\mathbb{R}$, are both homeomorphic to $\mathbb{P}$ and to $\mathbb{N}^{\aleph_0}$.*

A useful and immediate consequence of Corollary 27 is the following. Until now we knew only that such sets are uncountable.

Theorem 28. *Let X be a dense G_δ-subset of $\mathbb{P}$. Then X has cardinality $\mathfrak{c}$.*

Proof. By Corollary 27, X is homeomorphic to $\mathbb{P}$ and so has cardinality $\mathfrak{c}$. ■

Corollary 29. *Let Y be a dense G_δ-subset of $\mathbb{R}$. Then X has cardinality $\mathfrak{c}$.*

Proof. Put $X = Y \cap \mathbb{P}$. As Y and $\mathbb{P}$ are dense G_δ-subsets of $\mathbb{R}$, their intersection is a dense G_δ-subset of $\mathbb{R}$ and of $\mathbb{P}$. By Theorem 28, therefore, X has cardinality $\mathfrak{c}$. So Y has cardinality $\mathfrak{c}$. ■

6. MAHLER'S PARTITION OF THE SET OF REAL NUMBERS. In his influential book [**5**, p. 85], Fields Medalist Alan Baker (1939–2018) introduces the chapter on Mahler's Classification as follows: "*A classification of the set of transcendental numbers into three distinct aggregates, termed S-, T-, and U-numbers, was introduced by Mahler in 1932, and it has proved to be of considerable value in the general development of the subject.*"

We follow the presentation in [**13**, Chapter 3] and almost all the results in this section are stated and explained in that chapter. The classification of Mahler partitions of the set $\mathbb{R}$ into four sets (the fourth set in fact turns out to be the set of all algebraic numbers), is characterized by the rate with which a nonzero polynomial with integer coefficients approaches zero when evaluated at a particular number.

Given a polynomial $P(X) \in \mathbb{C}[X]$, the height of P, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of P. Given a complex number ξ, a positive integer n, and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| : P(X) \in \mathbb{Z}[X],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}.$$

Furthermore, we set

$$w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(\xi, H)}{\log H}$$

and

$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

With the above notation in mind, Kurt Mahler (1903–1988) partitions the set $\mathbb{R}$ as follows:

Definition 30. Let ξ be a real number. The number ξ is

- an $\mathbb{A}$-number if $w(\xi) = 0$,
- an S-number if $0 < w(\xi) < \infty$,
- a T-number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$,
- a U-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer n_0.

As observed in [**13**, Chapters 3 and 7], the $\mathbb{A}$-numbers are the algebraic numbers and there exist an infinity of $\mathbb{A}$-numbers, S-numbers, U-numbers and T-numbers. It was an open question for 36 years on whether the set of T-numbers is non-empty. It was answered in 1970 in the positive by Wolfgang M. Schmidt (born 1933) who won the Frank Nelson Cole Prize in Number Theory for work on Diophantine Approximation.

The following theorem of Mahler, see [**13**, Theorem 3.2], records a fundamental property of the Mahler classes.

Theorem 31. *If $\xi, \eta \in \mathbb{R}$ are algebraically dependent (that is, not algebraically independent) then they belong to the same Mahler class.*

In 1932 Mahler proved that the Mahler set S (of all S-numbers) has full Lebesgue measure and the Lebesgue measure of each of the Mahler sets $\mathbb{A}$, U, and T is therefore zero, see [**13**, Chapter 3]. In 1950, see [**13**, Chapter 7], it was proved that the Hausdorff dimension of the set of T-numbers and the set U-numbers is zero. Further [**13**, §3.3–3.5] records that $\mathcal{L} \subset U$, the number $e \in S$, and the number $\pi \in S \cup T$. This shows that neither e nor π is a Liouville number.

Whether $\pi + e$ is a transcendental number, or even an irrational number, has been an open problem for at least decades. However if π is in the Mahler set T (rather than the Mahler set S), then by Theorem 31, not only is $\pi + e$ a transcendental number but so is each number $a_i\pi^n + b_i e^m$, for $n, m \in \mathbb{N}$, and $a_i, b_i \in \mathbb{A} \setminus \{0\}$.

Using Mahler sets it is easily proved in [**16**] that if $x \in \mathcal{L}$, then the numbers $\sin x$, $\cos x$, $\exp x$, and $\log_e x$ are all transcendental numbers.

Proposition 32. *[31] The Mahler sets S, T, U, and $\mathbb{A}$ are analytic sets.*

7. THE CANTOR-LIOUVILLE SET. Kurt Mahler in [**38**] expressed interest in the intersection of various sets of real numbers with the middle-third Cantor set $\mathbb{G}$. A first step is to examine the *Cantor-Liouville set* which is defined to be $\mathbb{G} \cap \mathcal{L}$.

There are some well-known facts about the middle-third Cantor set $\mathbb{G}$:

(i) the real number $r \in \mathbb{G}$ if and only if it has a ternary expansion (that is, an expansion to base 3) which has only 0s and 2s. (Note, for example, that $1/3 \in \mathbb{G}$ has the two ternary expansions 0.1 and $0.0\overline{2} = 0.022\ldots2\ldots$.)

(ii) $\mathbb{G}$ is a compact subset of $[0, 1]$;

(iii) $\mathbb{G}$ has cardinality $\mathfrak{c}$;

(iv) $\mathbb{G}$ has Lebesgue measure 0, has Hausdorff dimension $s = \frac{\log 2}{\log 3}$, and has s-dimensional Hausdorff measure equal to 1 [**22**, Theorem 1.14];

(v) $\mathbb{G}$ is homeomorphic to $\{0,1\}^{\aleph_0}$, that is the product of a countably infinite number of discrete 2 point spaces;

(vi) $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\aleph_0}$ which clearly has a closed subspace homeomorphic to $\{0,1\}^{\aleph_0}$, which is homeomorphic to $\mathbb{G}$;

(vii) [**28**, p. 17] $\mathbb{G}$ has a dense subspace X homeomorphic to $\mathbb{P}$. (Let 0^n denote $00\ldots0$ (n times). Noting that $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\aleph_0}$ and $\mathbb{G}$ is homeomorphic to $\{0,1\}^{\aleph_0}$, we define a map $f : \mathbb{N}^{\aleph_0} \to \{0,1\}^{\aleph_0}$. The map $f(x) = 0^{x_1-1}10^{x_2-1}10^{x_3-1}1\ldots0^{x_n-1}1\ldots$, where $x = (x_1, x_2, \ldots, x_n, \ldots)$, is a homeomorphism of $\mathbb{N}^{\aleph_0}$ onto a subset S of $\{0,1\}^{\aleph_0}$. Clearly S consists of those elements of $\{0,1\}^{\aleph_0}$ with an infinite number of 1s. So the complement of S in $\{0,1\}^{\aleph_0}$ consists of those elements with only a finite number of 1s and so is countably infinite. Indeed this countably infinite set is homeomorphic to $\mathbb{Q}$. It follows from this analysis and (iv) above that the set X, which has been shown to be homeomorphic to $\mathbb{P}$, has somewhat surprisingly Hausdorff dimension equal to $s = \frac{\log 2}{\log 3}$.)

(viii) [**3**, Theorem 4.1] Every closed uncountable subset of $\mathbb{R}$ has a subspace homeomorphic to $\mathbb{G}$. (This is an easy consequence of the Cantor-Bendixson Theorem [**28**, Theorem 6.4] and [**28**, Theorem 13.6].)

(ix) It follows from (viii) that every uncountable F_σ-set in $\mathbb{R}$ has a subspace homeomorphic to $\mathbb{G}$ and so has cardinality $\mathfrak{c}$.

Proposition 33. *Let X be a subset of $\mathbb{R}$ with positive Lebesgue measure or finite positive Hausdorff measure. Then X has a subspace homeomorphic to $\mathbb{G}$ and so has cardinality $\mathfrak{c}$.*

Proof. If X has positive Lebesgue measure, then by the regularity of Lebesgue measure, the set X has an F_σ-subset Y with the same Lebesgue measure as X. So Y is uncountable. The result then follows from (ix) above.

The proof for the Hausdorff measure case is analogous since it is proved in [**10**] that every set of finite positive Hausdorff measure has an F_σ-subset with the same Hausdorff measure. ∎

Using the main theorem of [**19**] and [**53**, Corollary 1.5.13], we obtain the following powerful theorem.

Theorem 34. *If X is an uncountable analytic subset of $\mathbb{R}$, then it has a subspace homeomorphic to $\mathbb{G}$. In particular, X has cardinality $\mathfrak{c}$. If Y is an analytic subset of $\mathbb{R}$ with finite positive Hausdorff dimension, then it has cardinality $\mathfrak{c}$ and contains a maximal algebraically independent subset of $\mathbb{R}$ of cardinality $\mathfrak{c}$.*

Theorem 34 also implies the facts already observed in (viii) and (ix) above.

Using Theorem 34 and Theorem 31, one readily obtains:

Corollary 35. *Let X be an analytic subset of $\mathbb{R}$ having finite positive Hausdorff dimension. Then the intersection of X with each Mahler set S, T, and U is infinite.*

Noting that the middle-third Cantor set has Hausdorff dimension equal to $\frac{\log 2}{\log 3}$, we obtain:

Corollary 36. *The intersection of the middle-third Cantor set $\mathbb{G}$ with each Mahler set S, T, and U is infinite.*

Definition 37. Let ξ be a real number. Then ξ is said to have *irrationality exponent* $m(\xi)$ if $m(\xi)$ is the infimum of the set R of all m such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^m}$$

has at most finitely-many solutions $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. If $R = \emptyset$, then ξ is said to have *infinite irrationality exponent.*

It is a routine exercise to verify that for each $m \in (2, \infty)$, the set of real numbers with irrationality exponent equal to m is an analytic set.

It is not easy to determine the irrationality exponent of specific numbers. However all rational numbers have irrationality exponent 1, the Liouville numbers have infinite irrationality exponent, the number e has irrationality exponent 2, and the Thue-Siegel-Roth Theorem (1955), for which Klaus Roth (1925–2015) won a Fields Medal in 1958, says that every irrational algebraic number has irrationality exponent 2 [**15**, Theorem 1, Chapter VI]. We see that the bigger the irrationality exponent is the better the real number can be approximated, or perhaps we should say "the more quickly" it can be approximated. For further discussion of irrationality exponent, see [**2, 11, 7, 8, 9, 24, 27, 47, 48**]. In particular we mention that the set of real numbers of irrationality exponent equal to 2 has full Lebesgue measure, that the set of real numbers of exponent $m \in (2, \infty)$ has Lebesgue measure 0 and Hausdorff dimension equal to $\frac{2}{m}$,[**27**].

Proposition 38. *For each $m \in [2, \infty)$, let E_m be the set of real numbers of irrationality exponent equal to m. Then $\mathbb{G} \cap E_m$ has cardinality $\mathfrak{c}$ and a subset homeomorphic to $\mathbb{G}$.*

Proof. It is proved in [**14**] that for each $m \in [2, \infty)$, $E_m \cap \mathbb{G}$ is uncountable. As E_m is the intersection of two analytic sets, Theorem 34 implies that $E_m \cap \mathbb{G}$ has cardinality $\mathfrak{c}$ and a subset homeomorphic to $\mathbb{G}$. ■

We mentioned in §6 that it took 36 years to find a proof of the fact that the Mahler set T is nonempty and has cardinality $\mathfrak{c}$. Our next proposition, using the fact that the Mahler set T is nonempty shows not only that it has cardinality $\mathfrak{c}$ but that it contains a number of irrationality exponent m, for each $m \in (2, \infty)$.

Noting that each Mahler set is nonempty and that for each $m \in (2, \infty)$, the set E_m of real numbers of irrationality exponent equal to m has Hausdorff dimension $\frac{2}{m}$, and there are $\mathfrak{c}$ disjoint sets E_m, we deduce the following result from Corollary 35 and Theorem 34.

Theorem 39. *For each $m \in (2, \infty)$, let E_m be the set of real numbers of irrationality exponent equal to m. Then each E_m has infinite intersection with each of the Mahler sets S, T, and U. Thus S, T, and U each have cardinality $\mathfrak{c}$.*

That the mysterious Mahler set T contains an element of each irrationality exponent $m \in (0, \infty)$ appears to be a new result. It complements the result in [**14**] that $\mathbb{G}$ has the same property.

For each $m \in (2, \infty)$ and each $q \in \mathbb{Q}$, $q + E_m = E_m$, $q + S = S$, $q + T = T$, and $q + U = U$. With Theorem 39 this leads to the following corollary.

Corollary 40. *For the Mahler sets S, T, and U, each of the sets $E_m \cap S$, $E_m \cap T$, and $E_m \cap U$ is dense in $\mathbb{R}$.*

Noting that the Mahler set S has full Lebesgue measure, [**13**], Proposition 33 yields the next two corollaries.

Corollary 41. *The Mahler set S and every subset of S of finite positive Lebesgue measure has cardinality $\mathfrak{c}$ and has a subset homeomorphic to $\mathbb{G}$.*

Corollary 42. *Every subset of $\mathbb{G}$ of finite positive Hausdorff measure has cardinality $\mathfrak{c}$ and has a subset homeomorphic to $\mathbb{G}$.*

Proposition 43. *The Cantor-Liouville set $\mathbb{G} \cap \mathcal{L}$ is infinite. Further, $\mathbb{G} \cap \mathcal{L}$ is a dense subset of $\mathbb{G}$.*

Proof. First observe that the number $s = 2 \sum_{n=1}^{\infty} \frac{1}{3^{n!}}$ has only 0s and 2s in its ternary expansion. So $s \in \mathbb{G}$. It is also clear that, like the Liouville constant ℓ, s is a Liouville number. So $\mathbb{G} \cap \mathcal{L} \neq \emptyset$.

We saw in Proposition 2 that if $l \in \mathcal{L}$ and $q \in \mathbb{Q}$, then $l + q \in \mathcal{L}$. In particular this is the case if q is a rational number with a terminating ternary expansion. Adding a terminating ternary expansion to l is akin to changing a finite number of the digits of the ternary expansion of l. However it is easily seen that changing a finite number of digits in a Liouville number results in a Liouville number. So we can change finitely-many digits of s to either 0 or 2 and it remains a member of $\mathbb{G}$. So we now know that $\mathbb{G} \cap \mathcal{L}$ is an infinite set. Indeed $\mathbb{G} \cap \mathcal{L}$ contains $s + q$, for all rational numbers q with terminating ternary expansion. Thus $\mathbb{G} \cap \mathcal{L}$ is dense in $\mathbb{G}$. ∎

It is easy to modify the proof of Proposition 43 to show that the set $([0, 1] \setminus \mathbb{G}) \cap \mathcal{L}$ is infinite.

Theorem 44. *The Cantor-Liouville set $\mathbb{G} \cap \mathcal{L}$ has cardinality $\mathfrak{c}$. Further, $\mathbb{G} \cap \mathcal{L}$ has a subspace homeomorphic to $\mathbb{G}$.*

Proof. By the remark (vii) preceding Proposition 33 we see that $\mathbb{G}$ has a subspace Y which is homeomorphic to $\mathbb{P}$ such that $\mathbb{G} \setminus Y$ is a countably infinite set of rational numbers.

We have proved in Proposition 43 that $\mathbb{G} \cap \mathcal{L}$ is dense in $\mathbb{G}$ and contains no rational numbers as $\mathcal{L}$ does not. Therefore $Y \cap \mathcal{L} = \mathbb{G} \cap \mathcal{L}$ is dense in $\mathbb{G}$ and in Y. So by Theorem 28, $\mathbb{G} \cap \mathcal{L}$ has cardinality $\mathfrak{c}$.

That $\mathbb{G} \cap \mathcal{L}$ has a subspace homeomorphic to $\mathbb{G}$ now follows immediately from Theorem 34 as $\mathbb{G} \cap \mathcal{L}$ is an uncountable analytic set. ∎

Of course in $\mathbb{R}$ there are $\mathfrak{c}$ subspaces homeomorphic to the Cantor space $\mathbb{G}$. One may wonder whether each of these homeomorphic images of $\mathbb{G}$ has a nontrivial intersection with the set $\mathcal{L}$ of Liouville numbers. The answer is in the negative, since we saw in Corollary 41 that the Mahler set S has a subset homeomorphic to $\mathbb{G}$ but it is known that $S \cap \mathcal{L} = \emptyset$.

Definition 45. A real number x is said to be *very well approximable* if there exists an $\varepsilon > 0$ such that

$$\left| x - \frac{p}{q} \right| < q^{-(2+\varepsilon)}, \quad \text{for infinitely-many } (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

The set of very well approximable numbers is denoted by VWA.

It is clear from the definition that $\mathcal{L} \subseteq VWA$. Mahler was interested to know whether $(VWA \setminus \mathcal{L}) \cap \mathbb{G}$ is nonempty. It was in fact proved in [**36**] that the Hausdorff dimension of $(VWA \setminus \mathcal{L}) \cap \mathbb{G}$ is finite and positive. As every countable set has

Hausdorff dimension equal to zero, we see that $(VWA \setminus \mathcal{L}) \cap \mathbb{G}$ is uncountable. So by Proposition 33,

$$(VWA \setminus \mathcal{L}) \cap \mathbb{G} \text{ has a subset homeomorphic to } \mathbb{G} \text{ and has cardinality } \mathfrak{c}. \tag{1}$$

There is a significant amount of literature, for example [**6, 30, 32, 33**], on the intersection of the middle-third Cantor set with translations of it because, since Jules Henri Poincaré (1854–1912) in the late 1800s, it plays a role in studying nonlinear dynamical systems. (See [**18**].) We shall briefly discuss a related question. What can be said about the intersection of a translate of the middle-third Cantor set $\mathbb{G}$ with the set $\mathcal{L}$ of Liouville numbers?

To set the stage we prove Theorem 46 which is surprising even though it generalizes the well-known result that for some $r \in \mathbb{R}$, there is a translation $r + \mathbb{G}$ of the middle-third Cantor set which contains only irrational numbers. En route we mention the class of computable real numbers, introduced by Alan Turing (1912–1954), [**46**], which is a very broad class of numbers, albeit countably infinite, including all algebraic numbers as well as numbers such as π, e, e^{π}, Apéry's constant $= \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ (named after Roger Apéry (1916–1994) who proved in 1979 that it is an irrational number, see [**52**]), the Liouville constan ℓ, and the Euler-Mascheroni Constant $\gamma = \lim\limits_{n\to\infty} (-\log_e n + \sum\limits_{k=1}^{n} \frac{1}{k})$. (It is named after Leonhard Euler (1707-1783) and Lorenzo Mascheroni (1750–1800). It is not known where it is an irrational number or not.) We shall see that there is an uncountable number of translations of the middle-third Cantor set which contain no computable numbers.

Theorem 46. *Let X be any subset of $\mathbb{R}$ of Lebesgue measure zero and S a countably infinite subgroup of $\mathbb{R}$. Then there exists a real number r such that $r + X$ has empty intersection with S.*
In particular this is the case if S is the group $\mathbb{Q}$ of all rational numbers or the group $\mathbb{A}$ of all algebraic real numbers or the group of all computable real numbers and also if X is the middle-third Cantor set $\mathbb{G}$ or the set $\mathcal{L}$ of all Liouville numbers or the Mahler set U or the Mahler set T or the set of all real numbers of irrationality exponent m, for each $m \in (2, \infty)$.
Further, the set of real numbers r with this property has full Lebesgue measure in $\mathbb{R}$ and so, in particular, has cardinality $\mathfrak{c}$ and is dense in $\mathbb{R}$.

Proof. Put $Y = \bigcup\limits_{s\in S} (s + X)$. Clearly $s + Y = Y$, for all $s \in S$. We shall show that if $r \in \mathbb{R}$ is such that $r + Y$ contains one element of S, then $r + Y$ contains all of S.

For a fixed $r \in \mathbb{R}$, let $s \in S$ be such that $s = r + y$, for some $y \in Y$. Now let $s_1 \in S$. Then $s - s_1 \in S$; that is, $r + y - s_1 = s_2 \in S$. So

$$\begin{aligned} s_1 &= r + (y - s_2) \\ &= r + (s_3 + x - s_2), \text{ for some } x \in X, s_3 \in S \\ &= r + (x + s_4), \text{ for } s_4 \in S \\ &\in r + Y. \end{aligned}$$

Thus we have $\quad S \cap (r + Y) \neq \emptyset \implies S \subseteq r + Y.$

Suppose $0 \in r + Y$ for all $r \in \mathbb{R}$. Then $-r \in Y$, for all $r \in \mathbb{R}$; that is, $Y = \mathbb{R}$. This is a contradiction as Y, being a countable union of sets of Lebesgue measure zero, has

Lebesgue measure zero and so does not equal $\mathbb{R}$. It is now clear that $0 \notin r + Y$, for those r in a set of full Lebesgue measure, that is the complement in $\mathbb{R}$ of this set has Lebesgue measure zero, which completes the proof. ∎

Corollary 47. *Let X be any subset of $\mathbb{R}$ of full Lebesgue measure and Y a countably infinite subgroup of $\mathbb{R}$. Then there exists a real number r such that $r + X$ contains Y. In particular this is the case if X is the set of all normal numbers, [**13**], or the set of all real numbers of irrationality exponent equal to 2. The set of real numbers r with this property has full Lebesgue measure in $\mathbb{R}$, has cardinality $\mathfrak{c}$, and is dense in $\mathbb{R}$.*

In this article we investigate intersections of translations by only rational numbers of the middle-third Cantor set $\mathbb{G}$ with the sets $\mathcal{L}$ and VWA of Liouville numbers and very well approximable numbers, respectively.

Proposition 48. *Let $q \in \mathbb{Q}$ be any rational number. Then the sets $(q + \mathbb{G}) \bigcap \mathcal{L}$ and $(q + \mathbb{G}) \bigcap (VWA \setminus \mathcal{L})$ each have cardinality $\mathfrak{c}$ and a subspace homeomorphic to $\mathbb{G}$.*

Proof. By Proposition 2, $q + \mathcal{L} = \mathcal{L}$. So

$$(q + \mathbb{G}) \bigcap \mathcal{L} = (q + \mathbb{G}) \cap (q + \mathcal{L}) = q + (\mathbb{G} \cap \mathcal{L})$$

which by Theorem 44 has cardinality $\mathfrak{c}$. As $(q + \mathbb{G}) \cap (q + \mathcal{L})$ is an analytic set and $(q + \mathbb{G}) \bigcap \mathcal{L}$ has cardinality $\mathfrak{c}$, it follows from Theorem 34 that it has a subspace homeomorphic to $\mathbb{G}$.

Similarly $(q + \mathbb{G}) \bigcap (VWA \setminus \mathcal{L})$ has a subspace homeomorphic to $\mathbb{G}$. ∎

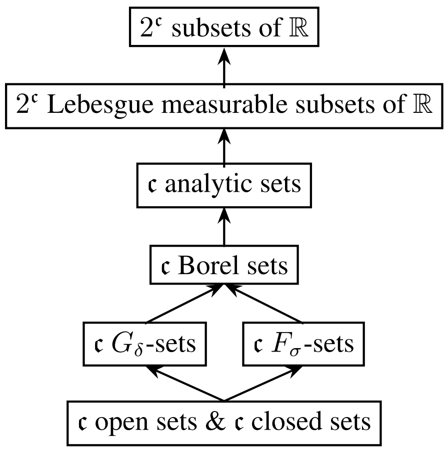
We should record the relationships applying to various approaches of describing the size of a subset of $\mathbb{R}$.

(i) Each set is finite, countably infinite, or uncountable.
(ii) A dense subset of $\mathbb{R}$ is either countably infinite or uncountable. However, even sets of cardinality $\mathfrak{c}$ need not be dense in $\mathbb{R}$.
(iii) The Hausdorff dimension [**42**] of a countably infinite or finite set is zero. Every subset of $\mathbb{R}$ of positive Hausdorff measure has cardinality $\mathfrak{c}$.
(iv) The largest uncountable sets in $\mathbb{R}$ have cardinality $\mathfrak{c}$, as do $\mathbb{R}$, $\mathbb{P}$, and $\mathcal{T}$. There are $\mathfrak{c}$ finite subsets of $\mathbb{R}$, $\mathfrak{c}$ countably infinite subsets of $\mathbb{R}$, and $2^{\mathfrak{c}}$ subsets of $\mathbb{R}$ of cardinality $\mathfrak{c}$. However there are only $\mathfrak{c}$ uncountable closed subsets of $\mathbb{R}$. (See [**44**, Lemma 5.2].)
(v) The Lebesgue measure of any finite or countably infinite set is zero. If a subset of $\mathbb{R}$ has positive Lebesgue measure then it has cardinality $\mathfrak{c}$. Some uncountable sets have zero Lebesgue measure.
(vi) A subset of $\mathbb{R}$ of full Lebesgue measure is dense in $\mathbb{R}$.
(vii) A subset of $\mathbb{R}$ with the Erdős property has cardinality $\mathfrak{c}$ but can have zero Hausdorff dimension and zero Lebesgue measure and need not be homeomorphic to $\mathbb{P}$.
(viii) A dense G_δ-subset of $\mathbb{R}$ is homeomorphic to $\mathbb{P}$ and so has cardinality $\mathfrak{c}$ but can have zero Hausdorff dimension and zero Lebesgue measure. By [**49**, Theorem 3.3] every comeager set, and in particular every dense G_δ-subset of $\mathbb{R}$, has packing dimension [**51**] equal to one.
(ix) An uncountable closed subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$.
(x) An uncountable F_σ-subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$.
(xi) An uncountable analytic subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$.

Relevant to the above statements, we state some examples.

(a) The set $\mathbb{Q}$ is a countably infinite dense subset of $\mathbb{R}$.
(b) The closed interval $[a, b]$, for each $a, b \in \mathbb{R}$ with $a < b$, and the middle-third Cantor set $\mathbb{G}$ have cardinality $\mathfrak{c}$ but are not dense in $\mathbb{R}$. There are $\mathfrak{c}$ such intervals $[a, b]$.
(c) The Mahler set S [**13**] has cardinality $\mathfrak{c}$, full Lebesgue measure, is dense in $\mathbb{R}$, has the Erdős property, but is not homeomorphic to $\mathbb{P}$.
(d) The Mahler sets T and U have cardinality $\mathfrak{c}$, have zero Lebesgue measure, and are dense in $\mathbb{R}$. The Mahler set U contains $\mathcal{L}$ which has the Erdős property, and so U also has the Erdős property.
(e) The set $\mathcal{L}$ of all Liouville numbers has cardinality $\mathfrak{c}$, is a dense G_δ-subset in $\mathbb{R}$, has zero Hausdorff measure, zero Lebesgue measure and is homeomorphic to $\mathbb{P}$.
(f) The set of real numbers of irrationality exponent (previously known as irrationality measure) m, for any $m \in (2, \infty)$ has zero Lebesgue measure, has cardinality $\mathfrak{c}$, is dense in $\mathbb{R}$, and is not a G_δ-subset of $\mathbb{R}$.

Readers may find the following helpful.



In conclusion, in a paper which focused so much on dense G_δ-subsets of $\mathbb{R}$ and of $\mathcal{L}$, we settle how many of them there are. We shall once again see the power of the topological approach.

Theorem 49. *There are $\mathfrak{c}$ distinct dense G_δ-subsets L_m of $\mathcal{L}$ such that $m < n \implies L_m \subset L_n$, where $m, n \in (0, \infty)$.*

Proof. Let us firstly consider the analogous problem for the topological space $\mathbb{P}$ of all irrational real numbers. As every open interval is a G_δ-subset of $\mathbb{R}$ and the union of two G_δ-subsets of $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$, for $m, n \in (0, \infty)$, each of the sets $X_m = \mathcal{L} \cup ((0, m) \cap \mathbb{P})$ is a dense G_δ-subset of $\mathbb{P}$ with the property that $m < n \implies X_m \subset X_n$. Of course there are $\mathfrak{c}$ sets X_m.

Next we note that, by Corollary 27, the spaces $\mathcal{L}$ and $\mathbb{P}$ are homeomorphic, and the properties of being dense and being a G_δ-subset are purely topological properties. If f is a homeomorphism of $\mathbb{P}$ onto $\mathcal{L}$, then each $L_m = f(X_m)$ is a dense G_δ-subset of $\mathcal{L}$ and the L_m have the properties described in the statement of this theorem. ∎

Having found $\mathfrak{c}$ dense G_δ-subsets of $\mathcal{L}$ and noting that there are $2^{\mathfrak{c}}$ subsets of $\mathcal{L}$, it is reasonable to ask how many dense G_δ-subsets of $\mathcal{L}$ are there? By the Laverentieff Theorem, [**53**, Theorem A8.5], named after Mikhail Alekseevich Laverentieff

(1900–1980), there are at most $\mathfrak{c}$ subspaces of $\mathbb{R}$ which are homeomorphic. So there are precisely $\mathfrak{c}$ dense G_δ-subsets of $\mathbb{R}$ since each is homeomorphic to $\mathbb{P}$.

8. FURTHER READING. Readers of our article may like to continue investigating the interplay between topology and number theory in the book [**25**] by the geometric topologist Allen Hatcher (born 1944).

9. POSTSCRIPT. One often hears that one's research should influence one's teaching and that sometimes one's teaching influences one's research. The authors of this article set out to write an entirely expository note, but en route they discovered results which they have not found in the literature or are stronger versions than they found in the literature, including Theorem 10, Theorem 13, Theorem 15, Corollary 16, Proposition 33, Corollary 35, Theorem 39, Corollary 40, Corollary 41, Theorem 46, Proposition 48, and Theorem 49. Of course some of these may be known to experts in the field.

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