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CONTRIBUTIONS TO THE STRUCTURE THEORY OF CONNECTED PRO-LIE GROUPS

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Abstract. We present some recent results in the structure theory of pro-Lie groups and locally compact groups, improvements of known results, and open problems.

1. Background

A topological group G is a *pro-Lie group* if it is complete and the set $\mathcal{N}(G)$ of all closed normal subgroups N such that G/N is a Lie group is a filter basis converging to 1. A topological group G is *almost connected* if G/G_0 is compact, where G_0 is the connected component of the identity. Any connected group and any compact group is almost connected. Every locally compact almost connected group is a pro-Lie group. Our book [10] establishes a structure theory of connected pro-Lie groups (and to some extent for almost connected pro-Lie groups as well). It is the result of a project that we pursued for a number of years; we reported on this project in this journal at various stages of progress (see [7], [8], [11]). One of our original motivations was to provide a systematic approach to a structure theory of (almost) connected locally compact groups. All results on (almost) connected pro-Lie groups as presented in [10] apply directly to locally compact groups.

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In this context it is of interest to have various alternative conditions for a connected pro-Lie group to be locally compact. We devote Section 2 to a discussion of various topological conditions that one might wish to impose on a pro-Lie group, such as being σ-compact, metrizable, or separable. This involves the consideration of cardinality invariants such as the *weight* and the *density* for the topological space underlying a connected pro-Lie group.

In the structure theory of locally compact groups it was customary since the middle of the last century to focus research on special classes of topological groups such as MAP (=maximally almost periodic) groups, SIN groups (=groups having small invariant neighborhoods) and so on. So far no study of pro-Lie groups in these classes has been made, and we start this discussion in Section 3 with a characterization of connected pro-Lie MAP groups. It clearly reduces to a well-known classical one in the special case of locally compact groups.

There is recent interest in (topological) groups all of whose quotient groups (modulo closed normal subgroups) have a certain property $\mathcal{P}$ but which itself fails to have this property; such groups are called *just non-$\mathcal{P}$ groups.* Typical for this concept are the just non-finite groups. The first approach in the area of topological groups is the article [15] on compact just non-Lie groups. The essential result is that a compact just-non-Lie group is profinite, that is, is a projective limit of discrete groups (cf. [10]). In the absence of compactness this fails to be true as we shall show by an example in Section 3. We shall prove however, that an *abelian* just non-Lie pro-Lie group is isomorphic either to the group $\mathbb{Z}_p$ of p-adic integers or to the group $\mathbb{Q}_p$ of p-adic rationals for some prime.

In Section 4 we offer a list of open problems on the structure of pro-Lie groups—presumably of varying degrees of difficulty.

We shall make frequent reference to results from the books [9] (1998^1, 2006^2) and [10] (2007).

2. Topological properties of connected pro-Lie groups

We first recall some basic facts.

Proposition 2.1. *A pro-Lie group G is locally compact iff $\mathcal{N}(G)$ contains a compact member.*

Proof. If G has a compact identity neighborhood U, then the relation $\lim \mathcal{N}(G) = 1$ implies the existence of an $N \in \mathcal{N}(G)$ with $N \subseteq U$; accordingly N is compact. Conversely, if $N \in \mathcal{N}(G)$ is compact then G is an extension of N by a locally compact Lie group and is therefore locally compact. $\square$

By contraposition this says that whenever a pro-Lie group G fails to be locally compact, none of the $N \in \mathcal{N}(G)$ is compact (or even locally compact). One of the Fundamental Structure Theorems is the following ([10], 12.81–12.86.):

Theorem 2.2. *Let G be a connected pro-Lie group. Then there is a closed subset $M \subseteq G$ and a maximal compact subgroup $C \subseteq G$ such that*

(i) *there is a homeomorphism $\phi\colon \mathbb{R}^J \to M$ for a set J.*
(ii) *Every compact subgroup has a conjugate contained in C.*
(iii) $(v, c) \mapsto \phi(v)c\colon \mathbb{R}^J \times C \to G$ *is a homeomorphism.*

In particular, G has maximal compact subgroups and they are all conjugate. We call M a *manifold factor* and $\operatorname{card} J$ the *dimension* $\dim M$ of the manifold factor. The manifold factor is homeomorphic to the homogeneous space $G/C = \{gC : g \in G\}$ and G is homeomorphic to $(G/C) \times C$. We shall also call $\dim M = \dim G/C$ the *rank* of G and denote it by $\operatorname{rank}(G)$. Clearly,

Corollary 2.3. *A connected pro-Lie group G is locally compact if and only if its rank* $\operatorname{rank}(G)$ *is finite.*

(See [10], 12.87.)

Thus, additional conditions for G to be locally compact can safely concentrate on the local compactness of weakly complete vector spaces $\mathbb{R}^J$. Recall that a topological group G is *compactly generated* iff there is a compact subset $K \subseteq G$ such that $G = \langle K \rangle$. A space is *σ-compact* iff it is a countable union of compact subsets.

Proposition 2.4. *Let G be a connected pro-Lie group and C one of its maximal compact subgroups. Then the following statements are equivalent:*

(i) *G is compactly generated.*
(ii) *G is σ-compact.*
(iii) *G/C is σ-compact.*

(iv) *G/C is locally compact.*
(v) $\mathrm{rank}(G) = \dim G/C < \infty$.
(vi) *G is locally compact.*

Proof. (i)⇒(ii) is true for every topological group.

(ii)⇒(iii): This follows from Theorem 2.2 since a product of two spaces is σ-compact iff each factor is σ-compact,

(iii)⇔(iv)⇔(v): [10], Proposition A2.17 on p. 644, referring to properties of weakly complete vector spaces.

(iv)⇒(vi): This follows from Theorem 2.2.

(vi)⇒(i): Every connected group is generated by every identity neighborhood and so every locally compact connected group is compactly generated. □

This proposition improves [10], Corollary 12.87.

Recall that the *weight* $w(X)$ of a topological space is $\min\{\mathrm{card}\,\mathcal{B} : \mathcal{B}$ is a basis of the topology of $X\}$. The local weight of a topological group G is $\min\{\mathrm{card}\,\mathcal{B} : \mathcal{B}$ is a basis of the filter of identity neighborhoods of $G\}$. If G is a connected topological group, then weight and local weight coincide.

Proposition 2.5. *Let G be a nonsingleton connected pro-Lie group, C one of its maximal compact subgroups, and T one of its maximal compact connected abelian subgroups. Then*

$$w(G) = \max\{\aleph_0, \mathrm{rank}(G), \dim(C)\} = \max\{\aleph_0, \mathrm{rank}(G), \dim(T)\}.$$

Proof. By Theorem 2.2 we have $w(G) = \max\{w(G/C), w(C)\}$ and $w(G/C) = w(\mathbb{R}^J) = \max\{\mathrm{card}\,J, w(\mathbb{R})\} = \max\{\mathrm{rank}(G), \aleph_0\}$ by [9], EA4.3 following A4.8. The equation $w(C) = w(T)$ was shown in [9], Theorem 9.36(vi). Finally, $w(C) = \max\{\aleph_0, \dim C\}$ and $w(T) = \max\{\aleph_0, \dim T\}$ are established in [9], Theorem 12.25. □

More information on the weight of compact groups is to be found in [9], Chapter 12. For the dimension of compact groups see [9], Theorem 9.52ff.

Since a connected group G is metric iff $w(G) = \aleph_0$ iff G is Polish (that is, is complete and second countable) we get

Corollary 2.6. *Let G be a connected pro-Lie group and C one of its maximal compact subgroups C. Then G is metric if and only if C is metric and* $\mathrm{rank}(G) \leq \aleph_0$.

In particular, there are Polish connected pro-Lie groups which are not locally compact, for instance $\mathbb{R}^{\mathbb{N}}$.

If a cardinal n is infinite we write $\log n = \min\{m : n \leq 2^m\}$. The density $d(X)$ of a space is $\min\{\operatorname{card} D : X = \overline{D}\}$. The following observation is part of the cardinality folklore:

Lemma 2.7. *If* $\operatorname{card} J \geq \aleph_0$*, then* $d(\mathbb{R}^J) = \log \operatorname{card} J$.

Proof. The inequality $d(\mathbb{R}^J) \leq \log \operatorname{card} J$ follows from the famous Hewitt-Marczewski-Pondiczery Theorem (see for instance [6], Theorem 2.3.15, p. 81). The inequality $w(X) \leq 2^{d(X)}$, that is, $\log w(X) \leq d(X)$, holds for every regular space (see e.g. [6], Theorem 1.5.7, p. 39). From 2.5 we have $w(\mathbb{R}^J) = \operatorname{card} J$, whence $\log \operatorname{card} J \leq d(\mathbb{R}^J)$. □

Proposition 2.8. *Let G be a nonsingleton connected pro-Lie group, C one of its maximal compact subgroups, and T one of its maximal compact connected abelian subgroups. Then* $d(G) =$

$$\max\{\aleph_0, \log \operatorname{rank}(G), \log \dim C\} = \max\{\aleph_0, \log \operatorname{rank}(G), \log \dim T\}.$$

Proof. By Theorem 2.2, $d(G) = \max\{d(\mathbb{R}^{\operatorname{rank}(G)}), d(C)\}$. For $J \neq \emptyset$, the density of $\mathbb{R}^J$ is $\max\{\aleph_0, \log \operatorname{card} J\}$ by 2.7. The density of C is $d(C) = \max\{\aleph_0, \log \dim(C)\}$ (see [9], Theorem 12.25). Moreover, $d(C) = d(T)$ by [9], Theorem 12.24. Putting all things together we arrive at the asserted conclusion. □

If $G = \mathbb{R}$ then $\log \operatorname{rank}(G) = \dim C = 0$, $d(G) = \aleph_0$. We recall that a space X is called *separable* if $d(X) \leq \aleph_0$.

Corollary 2.9. *A connected pro-Lie group G is separable if and only if* $w(G) \leq 2^{\aleph_0}$.

Proof. By Proposition 2.8, G is separable iff $\log \operatorname{rank}(G) \leq \aleph_0$ and $\log \dim C \leq \aleph_0$, that is, iff $\operatorname{rank}(G) \leq 2^{\aleph_0}$ and $\dim C \leq 2^{\aleph_0}$ iff $w(G) \leq 2^{\aleph_0}$ by 2.5. □

Proposition 2.10. *Every connected pro-Lie group is a Baire space.*

Proof. It is shown in [14], Theorem 6, that

(a) *a product of a family of spaces each of which is completely metrizable or locally compact is a Baire space.*

Now let G be a connected pro-Lie group. From 2.2 we know that G is homeomorphic to $\mathbb{R}^J \times C$ where J is a set and C is a maximal compact subgroup of G. By (a), $\mathbb{R}^J \times C$ is a Baire space. The assertion then follows. □

A sample application is the following:

Proposition 2.11. *Let G be a σ-compact topological group, H a connected pro-Lie group, and $f\colon G \to H$ a surjective morphism of topological groups. Then f is open, and H is locally compact.*

Proof. By Proposition 2.10 and Proposition 2.6 of [11], f is open. Clearly, H is σ-compact. Then Proposition 2.4 shows that H is locally compact, as asserted. □

3. Special classes

A topological group G is called *maximally almost periodic* or a MAP-*group* iff the almost periodic functions on G separate the points, and that is the case if and only if there is an injective morphism of topological groups $f\colon G \to K$ into some compact group K. We denote the Lie algebra of a pro-Lie group G by $\mathfrak{L}(G)$ or by $\mathfrak{g}$. The center of a group G is denoted by $Z(G)$, and the center of a pro-Lie algebra $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{g})$.

Proposition 3.1. *Let G be a connected pro-Lie group. Then the following statements are equivalent:*

(i) *G is a* MAP-*group.*

(ii) *G is isomorphic to $\mathbb{R}^J \times C$ for some set J and a unique largest compact subgroup C of G.*

Proof. (i) ⇒(ii). We have an injective morphism $f\colon G \to K$ for a compact group K which we may assume to be connected. This yields an injective morphism $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(K)$ (see [10], 4.20(ii)). Now $\mathfrak{L}(K) = Z(\mathfrak{L}(K)) \times \mathfrak{L}(K')$ where $Z(\mathfrak{L}(K)) \cong \mathbb{R}^I$ for some set I and $\mathfrak{L}(K') = \mathfrak{L}(K)' = \prod_{j\in M} \mathfrak{s}_j$ for a family of simple compact Lie algebras $\mathfrak{s}_j$ (see [10], 12.36). Let $R(G)$ denote the radical of G (see [10], p. 431ff.). Then $f(R(G))$ is a prosolvable connected subgroup of K and thus $\overline{f(R(G))}$ is prosolvable connected subgroup of K by [10], 10.11(v) and 10.18. We may and will assume that $f(G)$ is dense in K. Then $\overline{f(R(G))}$ is normal. So by [10], 10.25, $\overline{f(R(G))} \subseteq R(K) \subseteq Z(K)$, and thus $\mathfrak{L}(\overline{f(R(G))}) \subseteq \mathfrak{z}(\mathfrak{L}(K))$

(see [10], 9.17(ii)). Hence $\mathfrak{L}(R(G)) \cong \mathfrak{L}(f(R(G)))$ is abelian and indeed central in $\mathfrak{L}(G)$. Now $\mathfrak{L}(G) \cong R(\mathfrak{L}(G)) \times \prod_{j\in P} \mathfrak{s}_j$ for a family of simple Lie algebras. (See [10], 7.52 and 7.77.) The analytic subgroup S_j generated by $\mathfrak{s}_j$ in G is injected into the compact group K. Hence S_j is compact, and so $\prod_{j\in P} \mathfrak{s}_j$ is a compact pro-Lie algebra. Thus $\mathfrak{L}(G)$ is procompact ([10], 12.10) and so G is potentially compact ([10], 12.46). The assertion then follows from [10], 12.48. $\square$

A topological group is said to be an SIN-*group* if the filter basis of identity neighborhoods which are invariant under all inner automorphisms converges to 1.

Proposition 3.2. *A connected pro-Lie* SIN *group G is isomorphic to $\mathbb{R}^J \times C$ for some set J where C is a unique maximal compact subgroup of G.*

Proof. We know $G = \lim_{N\in\mathcal{N}(G)} G/N$ such that all factor groups G/N are Lie groups; since G is connected so are all G/N. If G is an SIN-group, then all G/N are SIN-groups and connected Lie groups. Then they are MAP-groups (see e.g. [13], Corollary 12.1 on p. 55). Thus the projective limit G is an MAP-group and hence has the asserted form by Proposition 3.1. $\square$

4. Just Non-Lie Groups

There is considerable recent interest in groups that fail to have a property $\mathcal{P}$ but are such that all nontrivial quotient groups have property $\mathcal{P}$. Such groups are called *just-non-$\mathcal{P}$ groups*, for instance just-nonfinite groups. In the realm of topological groups, "quotient group" has to mean "quotient group modulo a closed normal subgroup". The first paper along these lines is Russo's article on compact just-non-Lie groups [15] in which it is shown that a compact just-non-Lie group is profinite. An essential step was proving that a compact *abelian* just-non-Lie group is isomorphic to the group $\mathbb{Z}_p$ of p-adic integers for some prime p. We note that the group $\mathbb{Q}_p$ of p-adic rationals is a locally compact abelian just-non-Lie group. Recall that a topological group is called *prodiscrete* iff it is a totally disconnected pro-Lie group iff it is complete and the identity element has a neighborhood basis of open subgroups (cf. [10], Definition 3.25 and Proposition 4.23).

Proposition 4.1. *An abelian just-non-Lie pro-Lie group is isomorphic to* $\mathbb{Z}_p$ *or* $\mathbb{Q}_p$.

Proof. Let G be an abelian just-non-Lie pro-Lie group. We know that any abelian pro-Lie group G is of the form $G \cong \mathbb{R}^I \times H$ where H_0 is compact (see [10], Theorem 5.20). By [15], Lemma 1.1, a just-non-Lie group does not contain any nonsingleton closed normal Lie subgroup. Hence we have $I = \emptyset$, that is $G = H$. In [10], Definition 5.4, we defined $\mathrm{comp}(G)$ to mean the union of all compact subgroups; we showed that in an abelian pro-Lie group, this union is a closed fully characteristic subgroup ([10], Theorem 5.5.(i)). If $g \notin \mathrm{comp}(G)$ then $\langle g\rangle \cong \mathbb{Z}$ by Weil's Lemma for pro-Lie groups ([10], Theorem 5.3). Again by [15], Lemma 1.1, such a subgroup does not exist. Hence $G = \mathrm{comp}(G)$. If $G_0 \neq \{0\}$, then G_0 is open by definition of just-non-Lie groups. Since G_0 is compact connected abelian, G_0 is divisible, and thus, being open, is a direct summand algebraically and topologically. Hence if $G_0 \neq G$, the group G_0 is a nontrivial quotient of G and thus is a Lie group, but G does not have proper Lie subgroups by Lemma 1.1 in [15]. By [15], Theorem 2.1, we cannot have $G = G_0$. Thus $G_0 = \{0\}$ follows, that is, $G = \mathrm{comp}(G)$ is totally disconnected and so prodiscrete. Since any nonzero proper subgroup N has a Lie group quotient G/N, it must be open. In particular,

(1) *every nonzero proper compact subgroup C of G is open.*

Since $G = \mathrm{comp}(G)$, it follows that G is locally compact (and nondiscrete as a non-Lie group).

If C is any nonzero proper compact subgroup of G, then due to its openness, it is itself a just-non-Lie group. Hence from [15], Theorem 2.1, we know that there is a prime $p = p_C$ such that $C \cong \mathbb{Z}_p$. Now let C_1 and C_2 two nonzero compact subgroups of G. Then C_1 and C_2 are open, whence both $D \overset{\text{def}}{=} C_1 \cap C_2$ and $C \overset{\text{def}}{=} C_1 + C_2$ are compact open in the nondiscrete group G. Hence D is a nonzero compact subgroup of C_1 and C_2. The closed nonzero subgroups of the group $\mathbb{Z}_p$ of p-adic integers are of the form $p^n\mathbb{Z}_p$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. It follows that $p_{C_1} = p_D = p_{C_2}$. Hence there is a unique prime p such that

(2) *all nonzero compact subgroups of G are isomorphic to* $\mathbb{Z}_p$.

It thus follows that also $C \cong \mathbb{Z}_p$. Hence we find nonzero integers n_1 and n_2 such that $C_k = p^{n_k}{\cdot}C$ for $k = 1, 2$. Without loss of generality we may assume that $n_1 \leq n_2$. Then $C_2 = p^{n_2}{\cdot}C \subseteq p^{n_1}{\cdot}C = C_1$.

Therefore
(3) *The set of compact subgroups of G is totally ordered with respect to containment "$\subseteq$".*
If G is itself compact, then $G \cong \mathbb{Z}_p$ (see also [15]). Let us now assume that G is not compact and fix one compact open subgroup C. Since $C \cong \mathbb{Z}_p$, the set $\{p^n C : n \in \mathbb{N}_0\}$ is a basis for the filter of identity neighborhoods in G and contains all compact subgroups contained in C. Now let K be a compact subgroup of G which is not contained in C. Then we have $K \cong \mathbb{Z}_p$ by (2) and $C \subseteq K$ by (3). Hence there is an $m \in \mathbb{N}_0$ such that $p^m \cdot K = C$, and since $K \cong \mathbb{Z}_p$, the equation $K = p^{-m} \cdot C$ is meaningful. Thus
(4) *The set of nonzero compact subgroups is $\{p^n \cdot C : n \in \mathbb{Z}\}$ and*

$$G = \bigcup_{n \in \mathbb{N}_0} p^{-n} \cdot C = \operatorname{colim}\{C \xrightarrow{\text{incl}} \frac{1}{p} \cdot C \xrightarrow{\text{incl}} \frac{1}{p^2} \cdot C \xrightarrow{\text{incl}} \cdots\}.$$

Moreover, any isomorphism $\phi_0 \colon \mathbb{Z}_p \to C = p^m \cdot K$ of additive compact abelian groups is the restriction of a unique isomorphism $p^{-m}\mathbb{Z}_p \to K$, that is, for each $n \in \mathbb{N}_0$, an isomorphism $\phi_0 \colon \mathbb{Z}_p \to C$ induces an isomorphism
$\phi_n \colon \frac{1}{p^n} \cdot \mathbb{Z}_p \to \frac{1}{p^n} \cdot C$, giving us a morphism of topological groups $\Phi \colon \mathbb{Q}_p \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}_p \arrow[r, "\text{incl}"] \arrow[d, "\phi_0"'] & \frac{1}{p} \cdot \mathbb{Z}_p \arrow[r, "\text{incl}"] \arrow[d, "\phi_1"'] & \frac{1}{p^2} \cdot \mathbb{Z}_p \arrow[r, "\text{incl}"] \arrow[d, "\phi_2"'] & \cdots & \mathbb{Q}_p \arrow[d, "\Phi"] \\
C \arrow[r, "\text{incl}"'] & \frac{1}{p} \cdot C \arrow[r, "\text{incl}"'] & \frac{1}{p^2} \cdot C \arrow[r, "\text{incl}"'] & \cdots & G.
\end{tikzcd}$$

Since all ϕ_n are isomorphisms, Φ is an isomorphism. This completes the proof. $\square$

The following example exposes a 3-dimensional locally compact connected (hence pro-Lie) group which is a just-non-Lie group:

Example 4.2. Let $\widetilde{S}$ be the simply connected covering group of $\mathrm{SL}(2, \mathbb{R})$ and $Z \cong \mathbb{Z}$ its center with z being one of its two generators. Then $G = (\mathbb{Z}_p \times \widetilde{S})/D$, $D = \{(-n, z^n) : n \in \mathbb{Z}\}$ is a 3-dimensional connected locally compact just non-Lie group whose nonsingleton closed normal subgroups are $(p^n\mathbb{Z}_p \times \{1\})D/D \cong p^n\mathbb{Z}_p$.

A few additional comments follow rather immediately: If G is a connected pro-Lie just-non-Lie group and the coreductive radical N is nonsingleton, then G/N is a reductive Lie group. If $N = \{1\}$

then G is reductive, and $\mathfrak{L}(G) \cong \mathbb{R}^K \oplus \prod_{j\in J} \mathfrak{s}_j$ for a set K and a family of simple Lie algebras $\mathfrak{s}_j$. Let S_j be the analytic subgroup generated by $\mathfrak{s}_j$. Now $\overline{S_j}$ is a normal subgroup M such that G/M is a Lie group. Then $\mathfrak{L}(G/M) \cong \mathbb{R}^K \oplus \prod_{i\in J\setminus\{j\}} \mathfrak{s}_i$. We conclude that K and $J \setminus \{j\}$, and thus J, are finite. Hence

if $\mathfrak{g}$ is the Lie algebra of a connected pro-Lie just-non-Lie group and $\mathfrak{n}$ its coreductive radical, then $\mathfrak{g}/\mathfrak{n}$ is finite dimensional.

In fact, if N happens to be a nonsingleton coreductive radical, then all $G/N^{[[n]]}$, $n = 0, 1, \dots$ are Lie groups and $G \cong \lim_n G/N^{[[n]]}$. Thus

A connected pro-Lie just-non-Lie group with a nondegenerate coreductive radical is Polish.

5. Open problems on connected pro-Lie groups

Proposition 2.4 above suggests the following question

Problem 5.1. Is a compactly generated pro-Lie group locally compact?

Even for abelian pro-Lie groups, it is not known whether a compactly generated prodiscrete group without nontrivial compact subgroups is finitely generated free. (Cf. [10], p. 237.) A contribution to Problem 5.1 even for abelian pro-Lie groups would be welcome.

Let G be a pro-Lie group and N a closed normal subgroup. We know examples in which G is abelian connected and N is prodiscrete and algebraically a free group in countably many generators such that G/N fails to be complete (see [10], Corollary 4.11, p. 179). We also know that G/N is complete if N is almost connected and G/G_0 is complete (see [10], Theorem 4.28(i), p. 202). We do not know whether G/G_0 has to be complete. An answer to the following problem would be therefore significant:

Problem 5.2. Is the component factor group G/G_0 of a pro-Lie group complete?

Despite a full chapter on the structure theory of abelian pro-Lie group presented in [10], such a theory is far from complete. Therefore we formulate:

Problem 5.3. Develop a structure and character theory of prodiscrete abelian groups.

(a) Special case: compact-free prodiscrete groups. As a typical example, the kernel $F = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ of the morphism $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \to \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}/\mathbb{Z})$ is a closed and nondiscrete subgroup of $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{\mathbb{R}}$ and is algebraically isomorphic to $\mathbb{Z}^{(\mathbb{N})}$. The character group and bicharacter group of F have not been investigated. (See [10], pp. 173ff.)
(b) Special case: prodiscrete groups consisting of compact elements. A relevant example, due to Banaszczyk [4], p. 159f., is as follows: Let $\mathbb{Z}(2) = \mathbb{Z}/2\mathbb{Z}$ be the group of 2 elements and Ω the first uncountable ordinal. The group $\mathbb{Z}(2)^{(\Omega)}$ has a nondiscrete group topology making it a prodiscrete (hence pro-Lie) group T such that the filter basis consisting of the subgroups $\mathbb{Z}(2)^{(\{\nu:\alpha<\nu\})}$ is a basis for the filter of identity neighborhoods. Then T is a torsion group, hence T is the union of all compact subgroups. The bidual $\hat{\hat{T}}$ is discrete (see [4], p. 160); the evaluation morphism $T \to \hat{\hat{T}}$ is bijective and open but not continuous.

The following comment may be helpful. It requires that we know the concept of a *nuclear group*, for whose definition we refer to W. Banaszczyk's monograph [4] or to Außenhofer's survey [2]. The class of nuclear groups is closed under the formation of arbitrary products and passage to subgroups, hence under the formation of projective limits. Since $\mathbb{R}$, $\mathbb{T}$, and all discrete abelian groups are nuclear, so are all Lie groups, being isomorphic to $\mathbb{R}^m \times \mathbb{T}^n \times D$ for a discrete group D. Thus we note:

Remark 5.4. All commutative pro-Lie groups are nuclear groups.

The relevance of the remark in the context of Problem 5.3 is that nuclear groups have a comparativley good duality theory that has been investigated by Banaszczyk [4], Aussenhofer [2], [3], and other authors (cf. [2]).

We saw earlier that Theorem 2.2 has formidable consequences not all of which we discussed here. It is reasonable to expect that the answer to the following question is positive:

Problem 5.5. Does Theorem 2.2 remain valid for almost connected pro-Lie groups G?

Similarly, it is not unreasonable that the answer to the following question is yes:

Problem 5.6. Let G be an almost connected pro-Lie group. Does there exists a profinite subgroup P such that $G = G_0P$?

The answer is yes for locally compact G.

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