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VARIETIES OF ABELIAN TOPOLOGICAL GROUPS AND SCATTERED SPACES

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Abstract

The variety of topological groups generated by the class of all abelian k_ω-groups has been shown to equal the variety of topological groups generated by the free abelian topological group on [0, 1]. In this paper it is proved that the free abelian topological group on a compact Hausdorff space X generates the same variety if and only if X is not scattered.

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1. Introduction and main theorem

In [7], it was shown that for every compact Hausdorff topological space X, the free abelian topological group on X, $FA(X)$, is contained in the variety of topological groups generated by $FA[0, 1]$, the free abelian topological group on [0, 1]. Observe that, as a direct consequence of this, the variety of topological groups generated by $FA(X)$ is contained in the variety generated by $FA[0, 1]$. Here we characterize compact Hausdorff spaces X such that the variety of topological groups generated by $FA(X)$ equals the variety of topological groups generated by $FA[0, 1]$.

In this paper we prove the following theorem.

MAIN THEOREM. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

(i) *X is not a scattered space.*
(ii) $\mathfrak{V}(FA(X)) = \mathfrak{V}(FA[0, 1])$.
(iii) *$\mathfrak{V}(FA(X))$ contains $FA[0, 1]$.*
(iv) *$\mathfrak{V}(FA(X))$ contains a Hausdorff group which is not totally path-disconnected.*
(v) *$\mathfrak{V}(FA(X))$ contains $\mathbb{R}$, the additive topological group of all real numbers with the euclidean topology.*
(vi) *$\mathfrak{V}(FA(X))$ contains $\mathbb{T}$, the compact group consisting of the multiplicative group of complex numbers of modulus 1 with its usual euclidean topology.*

We note that, trivially, (ii) implies (iii), (iii) implies (iv), (v) implies (vi) and (vi) implies (iv).

We shall shortly show that (iii) implies (v). The remainder of the paper establishes that the first four statements are equivalent by showing that (i) implies (ii) and (iv) implies (i); this will complete the proof of the main theorem.

2. Preliminaries

For a topological group G, let $|G|$ denote the group obtained from G by 'dropping' the topology. We call $|G|$ the *group underlying* G.

For topological groups G_1 and G_2, we say that G_1 *is topologically isomorphic to* G_2 if there exists a map $f : G_1 \to G_2$ such that f is both an isomorphism of groups and a homeomorphism.

DEFINITION 2.1 [5, Ch. I, Section 9, Part VI]. A topological space X is said to be *scattered* if every nonempty subspace of X has an isolated point.

PROPOSITION 2.2 [14]. *The product of two scattered topological spaces is also a scattered topological space.*

PROOF. Let X and Y be scattered spaces and A a nonempty subspace of $X \times Y$. Let $p_X : X \times Y \to X$ be the projection mapping onto X, and consider $p_X(A)$, a subspace of X. Since X is scattered, $p_X(A)$ has an isolated point, a, say. Let $A_a = \{y \in Y \mid (a, y) \in A\} \subseteq Y$. Clearly, A_a is nonempty and, as it is a subspace of the scattered space Y, it has an isolated point, b. Now let $O \subseteq X$ be a neighbourhood of a such that $O \cap p_X(A) = \{a\}$, and let $U \subseteq Y$ be a neighbourhood of b such that $U \cap A_a = \{b\}$. Then $(a, b) \subseteq O \times U$ and $(O \times U) \cap A = \{(a, b)\}$. Thus A has an isolated point and so $X \times Y$ is scattered. □

A nonempty class $\mathfrak{V}$ of (not necessarily Hausdorff) topological groups is said to be a *variety of topological groups* [8, 11] if it is closed under the operations of forming subgroups, quotient topological groups and arbitrary products (with the Tychonoff product topology). If Ω is a class of topological groups, then the smallest variety containing Ω is said to be the *variety generated by* Ω and is denoted by $\mathfrak{V}(\Omega)$ (see [8, 1]).

We recall the concepts of k_ω-space and k_ω-group.

DEFINITION 2.3 [13]. A topological space X is said to be a *k_ω-space* with *k_ω-decomposition* $X = \bigcup_{n=1}^{\infty} X_n$ if X is a Hausdorff space with compact subspaces X_n, $n = 1, 2, \ldots$, such that:

(i) $X = \bigcup_{n=1}^{\infty} X_n$;
(ii) $X_n \subseteq X_{n+1}$ for all n; and
(iii) a subset A of X is closed in X if and only if $A \cap X_n$ is compact (or closed) for all n.

Further, a topological group that is a k_ω-space is said to be a *k_ω-group*.

Of course, every compact Hausdorff space X is a k_ω-space, with k_ω-decomposition $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n = X$ for all $n = 1, 2 \ldots$. Every connected locally compact Hausdorff group G is a k_ω-group [6, Section 2], with k_ω-decomposition $G = \bigcup_{n=1}^{\infty} K^n$ where K is any compact symmetric neighbourhood of the identity in G [10, Corollaries 1 and 2 to Proposition 8]. For example, $\mathbb{R}$ is a k_ω-space with k_ω-decomposition $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$.

We now recall the definition of a free abelian topological group [3].

DEFINITION 2.4 [3]. Let X be a completely regular Hausdorff space and e a distinguished point in X. The abelian topological group $FA(X)$ is said to be a *free abelian topological group on the space* X if it has the following properties:

(i) X is a subspace of $FA(X)$;
(ii) X generates $FA(X)$ algebraically; and
(iii) for any continuous mapping ϕ of X into any abelian topological group G which maps the point e onto the identity element of G, there exists a continuous homomorphism Φ of $FA(X)$ into G such that $\Phi(x) = \phi(x)$ on X.

The topological group $FA(X)$ has the finest topological group topology on the free abelian group on the set $X \backslash \{e\}$ that satisfies property (i) in Definition 2.4. In [3], Graev showed the existence of the free abelian topological group, $FA(X)$, on each completely regular Hausdorff space X. Graev further showed that any two free abelian topological groups on a given space X are topologically isomorphic; that is, $FA(X)$ is unique up to topological isomorphism.

We note that $|FA(X)|$, the group underlying $FA(X)$, is the free abelian group on the set $X \backslash \{e\}$, and e is the identity element [12, Proposition, p. 376].

Every element w of $FA(X)$ can be represented as a product of members of $X \cup X^{-1}$ in an infinite number of ways. One of these representations is the *reduced representation*[1] with no occurrences of e (unless $w = e$) and where, if $x \in X$ appears in the word, then x^{-1} does not appear in the word.

REMARK 2.5. Let X be a completely regular Hausdorff space, let Y be a subspace of X, and let $n \in \mathbb{N}$. We shall denote by $FA_n(Y)$ the set of all words in $FA(X)$ whose reduced representation has length less than or equal to n with respect to Y.

If X is a k_ω-space with k_ω-decomposition $X = \bigcup_{n=1}^{\infty} X_n$, then $FA(X)$ is a k_ω-space with k_ω-decomposition $FA(X) = \bigcup_{n=1}^{\infty} FA_n(X_n)$ [6, Corollary 1 to Theorem 1]. (Note that in every Hausdorff group topology on $|FA(X)|$ inducing the given topology on X, the set $FA_n(X_n)$ inherits the same compact topology.)

REMARK 2.6. In [7, Theorem A], it was shown that the variety of topological groups generated by $FA[0, 1]$ is exactly the variety generated by the class of all abelian k_ω-groups.

STEP 1. We prove that (iii) implies (v) of the main theorem.

[1] We use the term *reduced representation* where Hall [4] uses the term *reduced word*.

PROOF. As $FA[0, 1] \in \mathfrak{V}(FA(X))$, we know that $\mathfrak{V}(FA[0, 1]) \subseteq \mathfrak{V}(FA(X))$. Therefore every k_ω-group, indeed $\mathbb{R}$, is in $\mathfrak{V}(FA(X))$ (see Remark 2.6). □

We make use of the following result concerning free abelian topological groups, whose proof is straightforward.

LEMMA 2.7. *Let X and Y be completely regular Hausdorff spaces such that there exists a quotient mapping $\phi: X \to Y$. Then ϕ extends to a quotient homomorphism $\Phi: FA(X) \to FA(Y)$, where $FA(X)$ and $FA(Y)$ are the free abelian topological groups on X and Y, respectively.*

3. Proof of the main result

We note the following result proved in [7, Corollary 5.2].

THEOREM 3.1. *Let X be any compact Hausdorff topological space. Then $FA(X)$ is contained in $\mathfrak{V}(FA[0, 1])$.*

We now consider the Cantor space $\{0, 1\}^{\aleph_0}$, which is a compact metric space.

REMARK 3.2 [2, Exercises 6.2.A(c)]. The Cantor space can be characterized as follows. Every nonempty compact totally disconnected perfect (that is, having no isolated points) metrizable space is homeomorphic to the Cantor space.

PROPOSITION 3.3. *Let $G = \{0, 1\}^{\aleph_0}$ be the Cantor space. Then $\mathfrak{V}(FA(G)) = \mathfrak{V}(FA[0, 1])$.*

PROOF. As G is a compact metric space, $FA(G) \in \mathfrak{V}(FA[0, 1])$ by Theorem 3.1, and hence $\mathfrak{V}(FA(G)) \subseteq \mathfrak{V}(FA[0, 1])$.

Now, there exists a continuous mapping ϕ of G onto $[0, 1]$ and, as both G and $[0, 1]$ are compact, ϕ is a quotient mapping. Therefore, by Lemma 2.7, there exists a quotient homomorphism from $FA(G)$ onto $FA[0, 1]$, and the result follows. □

REMARK 3.4. In the proof of the following proposition, we need to find two open nonempty sets in a topological space that have disjoint closures. For a topological space X that has at least two elements and is regular and Hausdorff, we point out that you can indeed find two open nonempty sets that have disjoint closures. To see this, take $a, b \in X$; then there exist open sets U_1 and U_2 such that $a \in U_1$, $b \in U_2$ and $U_1 \cap U_2 = \emptyset$. Note also that $U_1 \subseteq X \backslash U_2$ and so $\overline{U_1} \subseteq X \backslash U_2$, giving $b \notin \overline{U_1}$. Given the regularity of X, there exist open sets O_1, O_2 such that $\overline{U_1} \subseteq O_1$ and $b \in O_2$ with $O_1 \cap O_2 = \emptyset$. Now, we have $O_2 \subseteq X \backslash O_1$ and so $\overline{O_2} \subseteq X \backslash O_1$. Therefore, we have two open sets U_1 and O_2 such that $\overline{U_1} \cap \overline{O_2} = \emptyset$.

As each subset A of X is also regular and Hausdorff, the procedure above applies to A as well.

The technique used in Proposition 3.3 to show that $FA[0, 1] \in \mathfrak{V}(FA(G))$, where G is the Cantor space, suggests that if we can find a quotient mapping from a space

X onto $[0, 1]$, then $FA[0, 1] \in \mathfrak{V}(FA(X))$ by Lemma 2.7. Therefore, the following proposition is the key to proving that (i) implies (ii) of the main theorem.

The following proposition is folklore; however, we include the proof here for completeness. We thank Vladimir Uspenskiĭ for providing an outline of the proof.

PROPOSITION 3.5 [14]. *A compact Hausdorff space admits a continuous mapping onto* $[0, 1]$ *if and only if it is not scattered.*

PROOF. Let X be a compact Hausdorff space that admits a continuous (closed) mapping $f : X \to [0, 1]$. Let Y be a subspace of X such that $f|_Y : Y \to [0, 1]$ is one-to-one and onto. Then $\overline{Y}$ is a closed subspace of X such that $f(\overline{Y}) = [0, 1]$ and no proper closed subset of $\overline{Y}$ is mapped onto $[0, 1]$. Suppose $p \in \overline{Y}$ is an isolated point. Then $\overline{Y}\backslash\{p\}$ is a proper closed (in X) subset of $\overline{Y}$, and hence $f(\overline{Y}\backslash\{p\}) \neq [0, 1]$. Therefore, $f(\overline{Y}\backslash\{p\}) = [0, 1]\backslash\{f(p)\}$ is a closed subspace of $[0, 1]$, and so $[0, 1]$ has an isolated point, which is impossible. Thus, $\overline{Y}$ has no isolated points and so X is not scattered.

Conversely, let X be a compact Hausdorff space that is not scattered. Let Y be a subspace of X that has no isolated points. Furthermore, $\overline{Y}$ is compact Hausdorff and has no isolated points. If we can show there exists an onto mapping $f : \overline{Y} \to [0, 1]$, then by the Tietze–Urysohn extension theorem, f can be extended over X. Therefore, without loss of generality, we shall assume that X has no isolated points.

Let V_0 and V_1 be any two open nonempty subsets of X with disjoint closures (see Remark 3.4). Find V_{00} and V_{01}, two open nonempty subsets of V_0 with disjoint closures in V_0, as well as V_{10} and V_{11}, two open nonempty subsets of V_1 with disjoint closures in V_1. Continuing in this manner, we construct open nonempty sets V_s for each finite sequence s of $\{0, 1\}$. Note that the construction is possible as X has no isolated points and so no open set is a singleton set.

Construct the set $F \subseteq X$ consisting of all points $x \in X$ such that $x \in \bigcap_{n=1}^{\infty} \overline{V_{g_1 g_2 \cdots g_n}}$ for some infinite sequence $g_1 g_2 \ldots g_n \ldots$, where $g_n \in \{0, 1\}$ for each $n \in \mathbb{N}$. We shall show that F is closed. Let $f \in \overline{F}$; then there exists an infinite sequence $f_1, f_2, \ldots, f_\alpha, \ldots$ that converges to f, where $f_\alpha \in F$ for each $\alpha = 1, 2, \ldots$. Now, $\overline{F} \subseteq \overline{V_0} \cup \overline{V_1}$, and so f belongs to exactly one of $\overline{V_0}$ and $\overline{V_1}$, which we shall denote by $\overline{V_{g_1}}$. Note that for each $n \in \mathbb{N}$, there exists $\alpha \in \mathbb{N}$, $\alpha > n$, such that $f_\alpha \in \overline{V_{g_1}}$. Thus, there exists an infinite sequence contained in $\overline{V_{g_1}} \cap F$ that converges to f, giving $f \in \overline{\overline{V_{g_1}} \cap F} \subseteq \overline{V_{g_1 0}} \cup \overline{V_{g_1 1}}$. As before, we have f belonging to exactly one of $\overline{V_{g_1 0}}$ and $\overline{V_{g_1 1}}$, which we shall denote by $\overline{V_{g_1 g_2}}$. Continuing in this way, it is clear that $f \in \bigcap_{n=1}^{\infty} \overline{V_{g_1 g_2 \cdots g_n}}$ for some infinite sequence $g_1 g_2 \ldots g_n \ldots$, where $g_n \in \{0, 1\}$ for each $n \in \mathbb{N}$. Therefore $f \in F$ and so F is closed.

We define a mapping ϕ from F to $\{0, 1\}^{\aleph_0}$, the Cantor space, as follows. Let $x \in F$ and $x \in \bigcap_{n=1}^{\infty} \overline{V_{g_1 g_2 \cdots g_n}}$; then $\phi(x) = \langle g_1, g_2, \ldots, g_n, \ldots \rangle$. Clearly, ϕ is onto. To show that ϕ is continuous, we first note that for a finite sequence s of $\{0, 1\}$, $(\overline{V_{s0}} \cup \overline{V_{s1}}) \cap F = V_s \cap F$, and this is open in F. Now, let a basic open set in the Cantor space be given by $O = \prod_{i=1}^{\infty} O_i$, where $O_i = \{0, 1\}$ for all but a finite number of values for i, and let m be the largest value for which $O_m \neq \{0, 1\}$. Let $K = \prod_{i=1}^{m} O_i$ and let $k = \langle k_1, \ldots, k_m\rangle \in K$. We claim that $\phi^{-1}(O) = (\bigcup_{k\in K} V_{k_1k_2\ldots k_m}) \cap F$, which is open in F. Let $x \in \phi^{-1}(O)$. Then $x \in F$ and $x \in \bigcap_{n=1}^{\infty} \overline{V_{g_1g_2\cdots g_n}}$ for some infinite sequence $\langle g_1, g_2, \ldots, g_n, \ldots\rangle \in O$. Clearly, $\langle g_1, g_2, \ldots, g_m\rangle \in K$ and $x \in \overline{V_{g_1g_2\cdots g_m0}}$ or $\overline{V_{g_1g_2\cdots g_m1}}$, giving $x \in V_{g_1g_2\cdots g_m}$. Conversely, let $x \in V_{k_1k_2\ldots k_m} \cap F$ where $k = \langle k_1, k_2, \ldots, k_m\rangle \in K$. Now, $x \in \bigcap_{n=1}^{\infty} \overline{V_{h_1h_2\ldots h_n}}$ for some infinite sequence $h = \langle h_1, h_2, \ldots, h_n, \ldots\rangle$ in the Cantor space. Suppose $h_1 \neq k_1$; then $x \in \overline{V_0}$ and $x \in \overline{V_1}$, which is not possible. Thus, $h_1 = k_1$. Similarly, for each $i = 1, \ldots, m$, we have $h_i = k_i$, and so $h \in O$; this gives $\phi(x) = h \in O$, that is, $x \in \phi^{-1}(O)$. Therefore ϕ is continuous.

We therefore have a continuous map from a closed subspace F of X onto the Cantor space, which in turn admits a map onto [0, 1]; that is, F admits a continuous map onto [0, 1]. Thus, by the Tietze–Urysohn extension theorem, X admits a continuous map onto [0, 1]. □

STEP 2. We prove that (i) implies (ii) of the main theorem.

PROOF. As X is not scattered, by Proposition 3.5 there exists a continuous mapping f from X onto [0, 1]. Now f is a quotient mapping and, by Lemma 2.7, $FA[0, 1]$ is a quotient group of $FA(X)$. Thus $\mathfrak{V}(FA[0, 1]) \subseteq \mathfrak{V}(FA(X))$, and we have equality from Theorem 3.1. □

The final part of the proof of our main theorem is not as straightforward as one might first expect.

DEFINITION 3.6. A nonsingleton continuous Hausdorff image of [0, 1] is called a *Peano curve*.

REMARK 3.7. A Peano curve P is uncountable. To see this, note that as P is a continuous image of a compact connected space, P is also compact connected. Further, as P is Hausdorff, this implies that P is a completely regular Hausdorff connected topological space. Thus, there exists a continuous mapping $\Phi : P \to [0, 1]$ such that $\Phi(P)$ is connected and hence contains an interval. Therefore P is uncountable.

Our first lemma concerning Peano curves essentially shows that if a product of topological spaces contains a Peano curve, at least one of the factors also contains a Peano curve—though not necessarily the same curve.

LEMMA 3.8. *For each $i \in I$ where I is some index set, let R_i be a Hausdorff topological space. If $\prod_{i\in I} R_i$ contains a Peano curve, then for some $i \in I$, R_i contains a Peano curve.*

PROOF. Firstly, we note that for each $j \in I$, the projection mapping $p_j : \prod_{i\in I} R_i \to R_j$, given by $p_j(\prod_{i\in I} r_i) = r_j$, is continuous onto R_j. Now, let $f : [0, 1] \to \prod_{i\in I} R_i$ be a continuous mapping such that $f[0, 1] \subseteq \prod_{i\in I} R_i$ is a Peano curve.

Clearly, for each $j \in I$, $p_j \circ f = h_j : [0, 1] \to R_j$ is a continuous mapping into R_j. Further, if $h_j[0, 1] \subseteq R_j$ were a singleton set for each $j \in J$, then $f[0, 1] \subseteq \prod_{j \in J} h_j[0, 1]$ would be a singleton set. However, $f[0, 1]$ is a Peano curve, and therefore cannot be a singleton set. Thus, for some $j \in J$, $h_j[0, 1] \subseteq R_j$ is not a singleton set. Finally, we note that as R_j is Hausdorff, $h_j[0, 1]$ is also Hausdorff. Therefore R_j contains a Peano curve. □

LEMMA 3.9. *Every nonempty open subset O of a Peano curve P contains a Peano curve.*

PROOF. By the Hahn–Mazurkiewicz theorem, P is compact, connected, locally connected and metrizable; so O contains a connected neighbourhood of a point $a \in O$. If this neighbourhood were a singleton, then $\{a\}$ would be a closed and open subset of the connected space P, which is a contradiction. Therefore, O is uncountable because it contains a nonsingleton, connected, completely regular (as P is completely regular) Hausdorff space.

Let $f : [0, 1] \to P$ be a continuous map onto P. Consider $f^{-1}(O)$. As O is open, $f^{-1}(O)$ is an open set in $[0, 1]$ and hence is a countable union of open intervals. Suppose the image of each interval were a singleton. Then O would be countable. However, we know O is uncountable and, therefore, the image of one of the intervals in $f^{-1}(O)$ is not a singleton. Let $[a, b] \subseteq f^{-1}(O)$ be such an interval (if the only one happens to be open, take a smaller closed interval). Then $f([a, b]) \subseteq O$ and hence O contains a Peano curve. □

LEMMA 3.10. *Let P be a Peano curve contained in $A \cup B$ where A and B are closed in the Hausdorff space $A \cup B$. Then A or B contains a Peano curve.*

PROOF. We know that $(A \cup B) \backslash B = A \backslash B$ is open in the space $A \cup B$. Either $(A \backslash B) \cap P$ is nonempty or $P \subseteq B$ (in which case we are done). We note that $(A \backslash B) \cap P$ is an open subset of P and so, by Lemma 3.9, $(A \backslash B) \cap P$ contains a Peano curve. Therefore either A contains a Peano curve or B contains a Peano curve. □

LEMMA 3.11. *Let $\phi : X \to Y$ be a quotient mapping from X onto Y, where both X and Y are compact Hausdorff spaces. Further, let $f : [0, 1] \to Y$ be a nontrivial continuous mapping into Y. Then $[0, 1]$ is a quotient space of X.*

PROOF. Clearly, $f([0, 1])$ is a compact connected Hausdorff space contained in Y. Therefore, there exists a continuous mapping g of $f([0, 1])$ onto $[0, 1]$. Thus, by the Tietze–Urysohn extension theorem, g can be extended to a continuous mapping of Y onto $[0, 1]$. Finally, we see that $g \circ \phi : X \to [0, 1]$ is a continuous surjective map and hence is a quotient map. □

NOTATION.

- Let X be a subset of a group G. We denote the subset $\bigcup_{i=1}^{n}(X \cup X^{-1})^i$ of G by $\mathrm{gp}_n(X)$.
- Let X and Y be disjoint topological spaces. We denote by $X \sqcup Y$ the free union of X and Y; in other words, $X \sqcup Y$ is the set $X \cup Y$ with the coarsest topology inducing the given topologies on X and Y and having X and Y as open subsets.
- Let Ω be a class of (not necessarily Hausdorff) topological groups. Then $S(\Omega)$ denotes the class of all topological groups G such that G is isomorphic to a subgroup of a member of Ω. Similarly, the operators $\overline{Q}$, C and P denote, respectively, Hausdorff quotient group, arbitrary cartesian product with the Tychonoff topology and finite product.

STEP 3. We prove (iv) implies (i) of the main theorem.

PROOF. Let G be a nontotally path-disconnected Hausdorff topological group contained in $\mathfrak{V}(FA(X))$. Clearly, G contains a Peano curve. Now, by [1, Theorem 2], $G \in SC\overline{Q}P(FA(X))$. Thus G is a subgroup of $H = \prod_{i \in I} H_i$, where each $H_i \in \overline{Q}P(FA(X))$. Note that H contains a Peano curve. By Lemma 3.8, there exists $i \in I$ such that $H_i \in \overline{Q}P(FA(X))$ contains a Peano curve. Now $FA(X) \times FA(X)$ is topologically isomorphic to $FA(X_1 \sqcup X_2)$, where X_1 and X_2 are copies of X [9, Theorem 6]. Therefore, for $K \in P(FA(X))$, K is topologically isomorphic to $FA(X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n)$, where each X_i is a copy of X. Thus, we have $H_i \in \overline{Q}(FA(X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n))$, and H_i contains a Peano curve. Noting that $Y = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ is compact, we see that $FA(Y)$ is a k_ω-group and hence H_i is a k_ω-group. Let $\theta : FA(Y) \to H_i$ be the quotient homomorphism onto H_i. Then a k_ω-decomposition of H_i is given by $H_i = \bigcup_{j=1}^{\infty} \mathrm{gp}_j(\theta(Y))$, as $\mathrm{gp}_j(\theta(Y)) = \theta(FA_j(Y))$. Moreover, every compact subspace of H_i lies in $\mathrm{gp}_m(\theta(Y))$ for some $m \in \mathbb{N}$ (see [6, Section 2]). So we choose m to be the smallest value such that $\mathrm{gp}_m(\theta(Y))$ contains the Peano curve P_i in H_i. Let $g : [0, 1] \to H_i$ be the mapping such that $P_i = g([0, 1])$; then $g : [0, 1] \to \mathrm{gp}_m(\theta(Y))$ is continuous and nontrivial. We now have that $U = P_i \cap (\mathrm{gp}_m(\theta(Y)) \backslash \mathrm{gp}_{m-1}(\theta(Y)))$ is an open nonempty subset of the Peano curve P_i and so, by Lemma 3.9, contains a Peano curve. Consider

$$\begin{aligned}
\mathrm{gp}_m(\theta(Y)) \backslash \mathrm{gp}_{m-1}(\theta(Y)) &= \theta(FA_m(Y)) \backslash \theta(FA_{m-1}(Y)) \\
&\subseteq \theta(FA_m(Y) \backslash FA_{m-1}(Y)) \\
&= A_1 \cup A_2 \cup \cdots \cup A_{2^m} = A,
\end{aligned}$$

where each $A_k = \theta(Y^{\varepsilon_1} Y^{\varepsilon_2} \cdots Y^{\varepsilon_m}) = \theta(Y)^{\varepsilon_1} \cdots \theta(Y)^{\varepsilon_m}$, with $\varepsilon_i = \pm 1$ for $i = 1, \ldots, m$. Now, each A_k is compact and A contains a Peano curve. Thus, by Lemma 3.10, A_k for some k contains a Peano curve, P_k. Now,

$$\begin{aligned}
A_k &= \theta(X_1 \sqcup \cdots \sqcup X_n)^{\varepsilon_1} \ldots \theta(X_1 \sqcup \cdots \sqcup X_n)^{\varepsilon_m} \\
&= [\theta(X_1)^{\varepsilon_1} \sqcup \cdots \sqcup \theta(X_n)^{\varepsilon_1}] \cdots [\theta(X_1)^{\varepsilon_m} \sqcup \cdots \sqcup \theta(X_n)^{\varepsilon_m}],
\end{aligned}$$

so we see that A_k is the union of closed sets of type $\theta(X_{l_1})^{\varepsilon_1} \cdots \theta(X_{l_m})^{\varepsilon_m}$, where each X_{l_i} is a copy of X. It follows from Lemma 3.10 that for some collection $l_1, \ldots, l_m$, $\theta(X_{l_1})^{\varepsilon_1} \cdots \theta(X_{l_m})^{\varepsilon_m}$, which is homeomorphic to $\theta(X^n)$, contains a Peano curve. As X^n is compact, $\theta(X^n)$ is a compact quotient space of X^n. Applying Lemma 3.11, we obtain that [0, 1] is a quotient space of X^n, and so X^n is not scattered. Finally, if X were scattered, then by Proposition 2.2 X^n would also be scattered. Therefore X is not scattered. □

This completes the proof of the main theorem.

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