

## On the Pro-Lie Group Theorem and the Closed Subgroup Theorem

Karl H. Hofmann and Sidney A. Morris

Communicated by K.-H. Neeb

**Abstract.** Let  $H$  and  $M$  be closed normal subgroups of a pro-Lie group  $G$  and assume that  $H$  is connected and that  $G/M$  is a Lie group. Then there is a closed normal subgroup  $N$  of  $G$  such that  $N \subseteq M$ , that  $G/N$  is a Lie group, and that  $HN$  is closed in  $G$ . As a consequence,  $H/(H \cap N) \rightarrow HN/N$  is an isomorphism of Lie groups.

*Mathematics Subject Classification 2000:* 22A05.

*Key Words and Phrases:* Pro-Lie Groups, Closed Subgroup Theorem

### 1. Theorem and Proof

In a pro-Lie group  $G$  the set  $\mathcal{N}(G)$  of normal subgroups  $N$  such that  $G/N$  is a Lie group is a filterbasis, and  $G$  is naturally isomorphic to the projective limit  $\lim_{N \in \mathcal{N}(G)} G/N$  (see [4], p. 149ff.). The *Pro-Lie Group Theorem* states that *every limit of a projective system of Lie groups is a pro-Lie group* ([4], Theorem 3.34, p. 157, [5], [1]). The *Closed Subgroup Theorem for Pro-Lie Groups* states that *a closed subgroup of a pro-Lie group is a pro-Lie group* ([4], Theorem 3.5, p. 158); its proof is based on the Pro-Lie Group Theorem and the following facts ([4], Theorem 1.34(i,ii), p. 96f.):

**Theorem 1.1.** *Assume that  $\mathcal{N}(G)$  is a filterbasis of closed normal subgroups of the complete topological group  $G$  and assume that  $\lim \mathcal{N} = 1$ . Let  $H$  be a closed subgroup of  $G$ . Then the following conclusions hold*

(i) *The isomorphism  $\gamma_G: G \rightarrow \lim_{N \in \mathcal{N}} G/N$  maps  $H$  isomorphically onto  $\lim_{N \in \mathcal{N}} \overline{HN}/N$ .*

(ii) *Under the present hypotheses,*

$$H \cong \lim_{N \in \mathcal{N}} H/(H \cap N) \cong \lim_{N \in \mathcal{N}} HN/N \cong \lim_{N \in \mathcal{N}} \overline{HN}/N.$$

The additional assumption, made in [4], that all factor groups  $G/N$  are complete is superfluous, as the referee points out. Recall that a topological group

$G$  is called *almost connected* if its component factor group  $G/G_0$  is compact. We shall prove the following

**Theorem 1.2.** (a) *Let  $H$  be an almost connected closed subgroup of a pro-Lie group  $G$  and let  $M \in \mathcal{N}(G)$ . Then there is a closed normal subgroup  $N$  of  $\overline{HM}$  such that  $N \subseteq M$  and the standard bijection*

$$f_N: H/(H \cap N) \rightarrow HN/N, \quad f_N(h(H \cap N)) = hN,$$

*is an isomorphism of Lie groups.*

(b) *If  $H$  is normal in  $G$ , then  $N$  is constructed to be normal in  $G$ , that is,  $N \in \mathcal{N}(G)$ .*

We shall prove this theorem in several steps through a sequence of lemmas. We are given the pro-Lie group  $G$  with its filter base  $\mathcal{N}(G)$  of closed normal subgroups  $N$  such that  $G/N$  is a Lie group. Then  $G$  may be identified with the projective limit of the system

$$\{p_{MN}: G/N \rightarrow G/M : N \subseteq M, M, N \in \mathcal{N}(G)\}.$$

By Theorem 1.1, a closed subgroup  $H$  of  $G$  gives rise to three projective systems of topological groups:

$$\begin{aligned} &\{q_{MN}: \overline{HN}/N \rightarrow \overline{HM}/M : N \subseteq M, M, N \in \mathcal{N}(G)\}, \\ &\{r_{MN}: HN/N \rightarrow HM/M : N \subseteq M, M, N \in \mathcal{N}(G)\}, \\ &\{s_{MN}: H/(H \cap N) \rightarrow H/(H \cap M) : N \subseteq M, M, N \in \mathcal{N}(G)\}, \end{aligned}$$

and all of them have  $H$  as limit by Theorem 1.1., as is illustrated in the following diagram:

$$\begin{array}{ccccccc} \frac{H}{H \cap M} & \xleftarrow{s_{MN}} & \frac{H}{H \cap N} & \xleftarrow{s_N} & H \cong \lim_{P \in \mathcal{N}(G)} \frac{H}{H \cap P} \\ f_M \downarrow & & f_N \downarrow & & \downarrow \lim_{P \in \mathcal{N}(N)} f_P \\ \frac{HM}{M} & \xleftarrow{r_{MN}} & \frac{HN}{N} & \xleftarrow{r_N} & H = \lim_{P \in \mathcal{N}(G)} \frac{HP}{P} \\ \text{inc} \downarrow & & \text{inc} \downarrow & & \downarrow \text{id}_H \\ \frac{\overline{HM}}{M} & \xleftarrow{q_{MN}} & \frac{\overline{HN}}{N} & \xleftarrow{q_N} & H = \lim_{P \in \mathcal{N}(G)} \frac{\overline{HP}}{P} \\ \text{inc} \downarrow & & \text{inc} \downarrow & & \downarrow \text{inc} \\ \frac{G}{M} & \xleftarrow{p_{MN}} & \frac{G}{N} & \xleftarrow{p_N} & G = \lim_{P \in \mathcal{N}(G)} \frac{G}{P}. \end{array}$$

We note, in particular, that  $q_N(H) = \frac{HN}{N}$ . For the Lie algebra  $\mathfrak{L}(G)$  we write  $\mathfrak{g}$ , and so on.

**Lemma 1.3.** *For each  $N \in \mathcal{N}(G)$ , the quotient  $H/(H \cap N)$  is a Lie group with Lie algebra  $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n})$ .*

**Proof.** For  $N \in \mathcal{N}(G)$  the morphism  $f: H \rightarrow G/N$ ,  $f(h) = hN$  has kernel  $H \cap N$ . The Lie group  $G/N$  has an identity neighborhood  $V$  in which  $\{1\}$  is

the only subgroup. Then  $U \stackrel{\text{def}}{=} f^{-1}(V)$  is an identity neighborhood of  $H$  in which every subgroup is contained in  $H \cap N$ . By the Closed Subgroup Theorem of Pro-Lie Groups [4], 3.35, p. 158,  $H$  is a pro-Lie group. [If  $H$  is assumed to be connected, then this follows already from the arguments yielding [4], 3.29(i) on p. 152.] Since  $\lim \mathcal{N}(H) = 1$ , there is a  $P \in \mathcal{N}(H)$  such that  $P \subseteq U$ , and thus  $P \subseteq H \cap N$ . Then  $H/(H \cap N) \cong (H/P)/((H \cap N)/P)$  is a Lie group as a quotient of a Lie group.

We have  $\mathfrak{L}(H \cap N) = \mathfrak{h} \cap \mathfrak{n}$  since  $\mathfrak{L}$  preserves limits, hence intersections (cf. [4], Theorem 2.25(ii)). By [4], Corollary 4.21(i)  $\mathfrak{L}(H/(H \cap N)) = \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n})$ . [As Glöckner has pointed out in [1], Corollary 4.21(i) of [4] is already available after [4], Lemma 3.24 on p.147f.] ■

**Remark.** We note that for *connected* closed subgroups  $H$  of a pro-Lie group  $G$ , Lemma 1.3 is available in [4] after Lemma 3.29 (i) and (ii) on p. 152. Thus we see that 1.3 is valid for connected  $H$  without invoking the independent proofs of the Pro-Lie Group Theorem in [5] and [1].

For the next step we accept the following facts: An arcwise connected subgroup of a Lie group is an analytic subgroup by the Theorem of Yamabe-Gotô ([2], [6]). If  $A$  is an analytic subgroup of a Lie group  $G$ , then there is a unique connected Lie group topology on  $A$ , possibly finer than the topology induced from  $G$ , making it into a Lie group  $A_{\text{Lie}}$  such that the inclusion map  $A \rightarrow G$  induces an injective morphism of topological groups  $f: A_{\text{Lie}} \rightarrow G$ , where  $\mathfrak{L}(f): \mathfrak{L}(A_{\text{Lie}}) \rightarrow \mathfrak{L}(G)$  is an injection of Lie algebras and  $\text{im } \mathfrak{L}(f) = \mathfrak{L}(A)$ . See for instance [3], Theorem 5.52. The topology of  $A_{\text{Lie}}$  is the *arc component topology* of  $A$  (see [3], A4.1ff.).

**Lemma 1.4.** *Assume that  $H$  is a connected closed subgroup of the pro-Lie group  $G$ . Then for each  $N \in \mathcal{N}(G)$ , the group  $\frac{HN}{N}$  is an analytic subgroup of the Lie group  $G/N$ , and its Lie algebra is  $\frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}}$ . The Lie group  $(\frac{HN}{N})_{\text{Lie}}$  is isomorphic to  $\frac{H}{H \cap N}$ ; indeed it is the image of  $\frac{H}{H \cap N}$  under the standard bijective morphism  $f_N: \frac{H}{H \cap N} \rightarrow \frac{HN}{N}$ ,  $f_N(h(H \cap N)) = hN$ , given the topology making  $f_N$  a homeomorphism.*

**Proof.** Let us write  $L = H/(H \cap N)$ , whence we may write  $\mathfrak{l} = \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n})$  by Lemma 1.3. Set  $\mathfrak{a} \stackrel{\text{def}}{=} \mathfrak{L}(f_N)(\mathfrak{l})$  and  $A \stackrel{\text{def}}{=} HN/N$ . By Lemma 1.3,  $L$  is a Lie group, and since  $H$  is connected, it is connected. Then  $L = \langle \exp_L(\mathfrak{l}) \rangle$  and we have

$$A = f_N(\langle \exp_L(\mathfrak{l}) \rangle) = \langle f_N(\exp_L(\mathfrak{l})) \rangle = \langle \exp_G \mathfrak{L}(f_N)(\mathfrak{l}) \rangle = \langle \exp_G \mathfrak{a} \rangle.$$

This means that  $A$  is the unique analytic subgroup of  $G/N$  generated by the (closed) subalgebra  $\mathfrak{a}$  of  $\mathfrak{L}(G/N)$ . By [3], Theorem 5.52(iii),  $\mathfrak{a} = \mathfrak{L}(HN/N)$ . ■

We recall that  $f_N: \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n}) \rightarrow (\mathfrak{h} + \mathfrak{n})/\mathfrak{n}$  is an isomorphism of pro-Lie algebras (more generally, cf. [4], Theorem A2.12(c), p. 640f.).

We abbreviate  $\mathfrak{L}(\overline{HN})$  by  $\mathfrak{h}_N$  and note  $\mathfrak{L}(\overline{HN/N}) = \mathfrak{h}_N/\mathfrak{n}$  (see [4], Corollary 4.21(i), p. 190). Our passing to the Lie Algebras in our big commutative diagram yields

$$\begin{array}{ccccc}
\frac{\mathfrak{h}}{\mathfrak{h} \cap \mathfrak{m}} & \xleftarrow{s_{MN}} & \frac{\mathfrak{h}}{\mathfrak{h} \cap \mathfrak{n}} & \xleftarrow{s_N} & \mathfrak{h} \cong \lim_{P \in \mathcal{N}(G)} \frac{\mathfrak{h}}{\mathfrak{h} \cap \mathfrak{p}} \\
f_M \downarrow & & f_N \downarrow & & \downarrow \lim_{P \in \mathcal{N}(G)} f_P \\
\frac{\mathfrak{h} + \mathfrak{m}}{\mathfrak{m}} & \xleftarrow{\tau_{MN}} & \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} & \xleftarrow{\tau_N} & \mathfrak{h} = \lim_{P \in \mathcal{N}(G)} \frac{\mathfrak{h} + \mathfrak{p}}{\mathfrak{p}} \\
\downarrow & & \downarrow & & \downarrow \text{id}_{\mathfrak{h}} \\
\frac{\mathfrak{h}_M}{\mathfrak{m}} & \xleftarrow{q_{MN}} & \frac{\mathfrak{h}_N}{\mathfrak{n}} & \xleftarrow{q_N} & \mathfrak{h} = \lim_{P \in \mathcal{N}(G)} \frac{\mathfrak{h}_P}{\mathfrak{p}} \\
\text{inc} \downarrow & & \text{inc} \downarrow & & \downarrow \text{inc} \\
\frac{\mathfrak{g}}{\mathfrak{m}} & \xleftarrow{p_{MN}} & \frac{\mathfrak{g}}{\mathfrak{n}} & \xleftarrow{p_N} & \mathfrak{g} = \lim_{P \in \mathcal{N}(G)} \frac{\mathfrak{g}}{\mathfrak{p}}.
\end{array}$$

Again we observe that  $q_N(\mathfrak{h}) = \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}}$ .

**Lemma 1.5.** *For each  $M \in \mathcal{N}(G)$  there is an  $N = N_M \in \mathcal{N}(G)$  contained in  $M$  such that  $\frac{\mathfrak{h}_N}{\mathfrak{n}} = \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N \cap \mathfrak{m}}{\mathfrak{n}}$ .*

**Proof.** Applying [4], Lemma 3.18 = Theorem A2.12 we find that for each  $M \in \mathcal{N}(G)$  there is an  $N = N_M \in \mathcal{N}(G)$  such that  $q_{MN}(\frac{\mathfrak{h}_N}{\mathfrak{n}}) \subseteq q_M(\mathfrak{h}) = \frac{\mathfrak{h} + \mathfrak{m}}{\mathfrak{m}}$ . Let  $X + \mathfrak{n} \in \frac{\mathfrak{h}_N}{\mathfrak{n}}$ . Then we find an element  $Y \in \mathfrak{h}$  such that  $X \in X + \mathfrak{m} = p_{MN}(X + \mathfrak{n}) = q_{MN}(X + \mathfrak{n}) = q_M(Y) = Y + \mathfrak{m}$ . Thus there is a  $U \in \mathfrak{m}$  such that  $X - Y = U$ . Now  $U + \mathfrak{n} \subseteq \mathfrak{h}_N \cap \mathfrak{m}$  since  $\mathfrak{h} + \mathfrak{n} \subseteq \mathfrak{h}_N$ . So  $X + \mathfrak{n} = Y + U + \mathfrak{n} \in \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N \cap \mathfrak{m}}{\mathfrak{n}}$ . This implies the claim. ■

Now we let  $M^*$  be that subgroup of  $\overline{HN} \cap M \subseteq M$  containing  $N$  for which  $\frac{M^*}{N} = \left( \frac{\overline{HN} \cap M}{N} \right)_0$  in the Lie group  $G/N$ . Then  $M^*$  is normal in  $\overline{HN}$  and characteristic in  $\overline{HN} \cap M$  modulo  $N$ .

As a consequence of Lemma 1.5, we get

**Lemma 1.6.** *For each  $M \in \mathcal{N}(G)$  there is an  $N = N_M \in \mathcal{N}(G)$  contained in  $M$  such that*

$$\frac{\overline{HN}}{N} = \frac{HN}{N} \cdot \left( \frac{\overline{HN} \cap M}{N} \right)_0 = \frac{HN}{N} \cdot \frac{M^*}{N} = \frac{HM^*}{N}.$$

**Proof.** We have  $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N / \mathfrak{n} \rangle$ , further  $\frac{HN}{N} = \langle \exp_{G/N} \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} \rangle$  by Lemma 1.6. Finally

$$\frac{\mathfrak{h}_N}{\mathfrak{n}} \cap \frac{\mathfrak{m}}{\mathfrak{n}} = \ker q_{MN} \text{ and } \frac{\overline{HN}}{N} \cap \frac{M}{N} = \ker q_{MN},$$

whence  $\left( \frac{\overline{HN} \cap M}{N} \right)_0 = \langle \exp_{G/N}(\frac{\mathfrak{h}_N \cap \mathfrak{m}}{\mathfrak{n}}) \rangle$ . Therefore,  $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N / \mathfrak{n} \rangle = \left\langle \exp_{G/N} \left( \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N \cap \mathfrak{m}}{\mathfrak{n}} \right) \right\rangle = \langle \exp_{G/N} \frac{\mathfrak{h} + \mathfrak{n}}{\mathfrak{n}} \rangle \langle \exp_{G/N} \frac{\mathfrak{h}_N \cap \mathfrak{m}}{\mathfrak{n}} \rangle = \frac{HN}{N} \cdot \left( \frac{\overline{HN} \cap M}{N} \right)_0 = \frac{HN}{N} \cdot \frac{M^*}{N} = \frac{HM^*}{N}$ . This completes the proof. ■

**Main Lemma 1.7.** *Let  $H$  be a closed connected subgroup of the pro-Lie group  $G$ . Then for each  $M \in \mathcal{N}(G)$  there is an  $N_M \in \mathcal{N}(G)$  such that  $N_M \subseteq M$*

and that, for the subgroup  $M^* \leq \overline{HN_M} \cap M \subseteq \overline{HN_M}$  containing  $N_M$  such that  $\frac{M^*}{N_M} = \left( \frac{\overline{HN_M} \cap M}{N_M} \right)_0$  the following statements are true:

- (i)  $N_M \subseteq M^* \subseteq M$  and  $\frac{\overline{HN_M}}{M^*} = \frac{\overline{HM^*}}{M^*}$  is a Lie group.
- (ii) The analytic subgroup  $\frac{HM^*}{M^*}$  of the Lie group  $\frac{\overline{HM^*}}{M^*}$  is closed.
- (iii) The natural morphism  $\frac{H}{H \cap M^*} \rightarrow \frac{HM^*}{M^*}$  is an isomorphism of Lie groups.

**Proof.** We choose  $N_M \in \mathcal{N}(G)$  as in Lemma 1.6.

Proof of (i). We have  $N_M \subseteq M^* \subseteq \overline{HN} \cap M$  by the definition of  $M^*$  and note  $\overline{HM^*} \subseteq \overline{HN_M} \subseteq \overline{HM^*}$ . Since  $N_M \in \mathcal{N}(G)$ , the quotient  $G/N_M$  is a Lie group and thus the quotient  $\frac{\overline{HN_M}}{M^*} \cong \frac{\overline{HN_M}/N_M}{M^*/N_M}$  is a Lie group as a quotient of a Lie group.

Proof of (ii). The quotient group  $\frac{HN_M}{N_M}$  is an analytic subgroup of the Lie group  $G/N_M$  by Lemma 1.4. The group  $\overline{HM^*}/M^*$  is a continuous image of this arcwise connected group under the morphism  $hN_M \mapsto hM^*$  and therefore, as an arcwise connected group, an analytic subgroup of  $\overline{HM^*}/M^*$ . By Lemma 1.6 we have  $\frac{\overline{HN_M}}{N_M} = \frac{HM^*}{N_M}$ . Passing to the quotient modulo  $M^*/N_M$  on both sides yields  $\frac{\overline{HN_M}/N_M}{M^*/N_M} = \frac{HM^*/N_M}{M^*/N_M}$ , which is equivalent to  $\frac{\overline{HN_M}}{M^*} = \frac{HM^*}{M^*}$ ; this implies that  $\frac{HM^*}{M^*}$  is closed and is, therefore, a Lie group.

Proof of (iii). The natural bijective morphism of topological groups

$$h(H \cap M^*) \mapsto hM^* : \frac{H}{H \cap M^*} \rightarrow \frac{HM^*}{M^*}$$

between two connected Lie groups is an isomorphism by the Open Mapping Theorem for Locally Compact Groups. ■

The situation is illustrated by the following diagram:

$$\begin{array}{ccccccc}
 \frac{HM}{M} & \longleftarrow & \frac{HM^*}{M^*} & \longleftarrow & \frac{HN_M}{N_M} & \longleftarrow & H = \lim_{P \in \mathcal{N}(G)} \frac{HP}{P} \\
 \text{inc} \downarrow & & = \downarrow & & \downarrow \text{inc} & & \downarrow \text{id}_H \\
 \frac{\overline{HM}}{M} & \longleftarrow & \frac{\overline{HM^*}}{M^*} & \longleftarrow & \frac{\overline{HN_M}}{N_M} & \xleftarrow{q_N} & H = \lim_{P \in \mathcal{N}(G)} \frac{\overline{HP}}{P} \\
 \text{inc} \downarrow & & & & \downarrow \text{inc} & & \downarrow \text{inc} \\
 \frac{G}{M} & \xleftarrow{p_{MN}} & \frac{G}{N_M} & \xleftarrow{=} & \frac{G}{N_M} & \xleftarrow{p_N} & G = \lim_{P \in \mathcal{N}(G)} \frac{G}{P}.
 \end{array}$$

We observe that  $M^*$  need not be a member of  $\mathcal{N}(G)$ . If  $H$  is a normal subgroup, then  $\overline{HN_M} \cap M$  is normal in  $G$ , and since  $\frac{M^*}{N_M} = \left( \frac{\overline{HN_M} \cap M}{N_M} \right)_0$  is characteristic in  $\frac{G}{N_M}$ , the group  $M^*$  is invariant under all automorphisms of  $G$  leaving  $N_M$  invariant, and so is certainly invariant under all inner automorphisms of  $G$ ; and thus  $M^*$  is a member of  $\mathcal{N}(G)$ . Therefore we have the

**Lemma 1.8.** *Let  $H$  be a closed connected normal subgroup of the pro-Lie group  $G$ . Then for each  $M \in \mathcal{N}(G)$  there is an  $M^* \in \mathcal{N}(G)$  such that  $M^* \subseteq M$  and that the following statements are true:*

- (i) *The analytic subgroup  $\frac{HM^*}{M^*}$  of the Lie group  $\frac{G}{M^*}$  is closed.*
- (ii)  *$h(H \cap M^*) \mapsto hM^* : \frac{H}{H \cap M^*} \rightarrow \frac{HM^*}{M^*}$  is an isomorphism of Lie groups. ■*

At this time we have proved Theorem 1.2 for *connected* closed subgroups  $H$ . We now complete the proof of 1.2 by showing that the results of Lemma 1.7 remain intact for an *almost* connected closed subgroup  $H$  of a pro-Lie group  $G$ .

Thus let  $H$  be a closed almost connected subgroup of the pro-Lie group  $G$  and let  $M \in \mathcal{N}(G)$ . We apply Lemma 1.6 to  $H_0$  in place of  $H$  and find that there is a closed normal subgroup  $M^*$  of  $\overline{H_0 M}$  contained in  $M$  such that  $H_0 M^*/M^*$  is a connected Lie group. In particular,  $H_0 M^*$  is a closed subgroup of  $G$ . Since  $M^*$  is characteristic in  $\overline{H_0 N_M} \cap M$  modulo  $N_M$ , and  $\overline{H_0 N_M} \cap M$  is normal in  $\overline{H N_M}$ , the group  $M^*$  is normal in  $\overline{H N_M}$ . So  $M^*$  is normalized by  $H$  and thus by  $HM^*$ . We must show that  $HM^*/M^*$  is a Lie group; the remainder of Theorem 1.2 then follows. Now  $H/H_0$  is assumed to be compact. The continuous morphism

$$hH_0 \mapsto hH_0 M^* : H/H_0 \rightarrow HM^*/H_0 M^*$$

is surjective. Hence  $HM^*/H_0 M^*$  is a compact group. Thus  $HM^*/M^*$  is a locally compact group as an extension of the Lie group  $H_0 M^*/M^*$  by the compact group  $HM^*/H_0 M^*$ . The morphism

$$h(H \cap N_M) \mapsto hM^* : H/(H \cap N_M) \rightarrow HM^*/M^*$$

is a surjective morphism from a  $\sigma$ -compact group onto a locally compact group and is therefore open. The group  $H/(H \cap N_M)$  is a Lie group by Lemma 1.3. Thus  $HM^*/M^*$  is a Lie group. This completes the proof of Theorem 1.2.

## 2. Applications

In the formulation of the Closed Subgroup Theorem for Projective Limits 1.34 in [4], a complete group  $G$  was considered for which there is a filter basis  $\mathcal{N}$  of closed normal subgroups  $N$  such that  $\lim \mathcal{N} = 1$  and that  $G/N$  is complete for each  $N \in \mathcal{N}$ . For a closed subgroup  $H$  of  $G$  it was asserted in 1.34(iv) that

- (\*) *for each  $N \in \mathcal{N}$  the standard morphism  $f_N : H/(H \cap N) \rightarrow HN/N$  is an isomorphism of topological groups.*

Glöckner noticed in [1] that (\*) is false; the error in the proof was the claim that for  $N \subseteq M$  in  $\mathcal{N}$  the natural morphism  $HN/N \rightarrow HM/M$  had to be open.

In circumstances where (a)  $G$  is a pro-Lie group, (b)  $\mathcal{N} = \mathcal{N}(G)$ , and (c)  $H$  is almost connected and normal, then Theorem 1.2 shows that (\*), while possibly not true for *all*  $N \in \mathcal{N}$ , is true for  $N \in \mathcal{N}$  time and again for smaller and smaller  $N$ .

In [4], (\*) is used in three proofs under the hypotheses (a), (b), and (c) and indeed in these applications the cofinal validity of the conclusion is perfectly sufficient. We explain how in these instances the correction is implemented.

*Correcting the proof of [4], 5.17.* We refer to [4], Proposition 5.17 and its proof on pp. 225, 226. We replace lines 3, 4 and 5 on page 226 by the following:

- (1) ...  $G_1 \cap U$ . Now let  $M \in \mathcal{N}(G)$  be contained in  $U$ . Then  $M \cap G_1 \subseteq U \cap G_1 = \{0\}$ . Let  $N \stackrel{\text{def}}{=} M^* \subseteq M$  be the subgroup attached to  $M$  by Theorem 1.2 of this paper. Since  $G$  is abelian,  $N$  is normal. Then  $G/N$  is a Lie group and  $(G_1 + N)/N$  is isomorphic to  $G_1/(G_1 \cap N) \cong G_1$  by Theorem 1.2 of this paper. So  $(G_1 + N)/N \dots$

*Correcting the proof of [4], 11.15.* We replace lines 2 and 3 of page 475 of [4] by the following:

- (2) ... so, since  $N$  is finite dimensional, there is a  $P \in \mathcal{N}(G)$ , such that  $N \cap P = \{1\}$ . Now let  $M = P^* \subseteq P$  be the member of  $\mathcal{N}(G)$  attached to  $P$  according to Theorem 1.2 of this paper. Then by Theorem 1.2 of this paper,  $n \mapsto nM : N \rightarrow NM/M$  is an isomorphism, and ...

*Correcting the proof of [4], 3.29(iii).* We replace section (iii) of [4], 3.29 by the following

- (3) (iii) *The set  $I = \{j \in J : G_0/(G_0 \cap K_j) \rightarrow (G_0 K_j)/K_j \text{ is an isomorphism of topological groups}\}$  is cofinal in  $J$ . For  $j \in I$ , the group  $(G_0 K_j)/K_j$  is a Lie group and a closed subgroup of  $G/K_j$ , and  $G_0 = \lim_{j \in J} (G_0 K_j)/K_j$ .*

We replace the last sentence of the proof of 3.29(iii) in [4] by the following:

Theorem 1.2 of this paper shows that the set  $I$  is cofinal in  $J$ , and it establishes the other statements of (iii) as well.

This correction reinstates our original proof of the Pro-Lie Group Theorem [4], 3.34, which was independently proved in [5] and [1]. The revised version of 3.29 corroborates the first sentence of the proof of 3.34.

However, this comment requires additional inspection in order that the argument is not circular. We did take care of this inspection in the Remark following Lemma 1.3.

*Correcting the proof of [4], 9.54.* This occurrence of 1.34(iv) has nothing to do with an application of Theorem 1.2(b) of this paper and is easily remedied directly. Indeed in lines 8 and 7 from below on page 406 of [4], replace “By Theorem 1.34(iv) ... as topological groups” by

- (4) “We know that  $C/(C \cap N) \cong CN/N$  as groups”.

The point is that the algebraic isomorphism suffices in this case.

### 3. Concluding Remarks

Theorem 1.2 secures another instance of the validity of the so-called Second Isomorphism Theorem  $H/(H \cap N) \cong HN/N$  in the category of topological groups. Usually, if it is true at all in the sense that the isomorphism holds algebraically *and topologically*, some application of the Open Mapping Theorem is involved (see e.g. [4], Corollary 9.62, p. 413) which is not the case in this instance. Theorem 1.2 pertains to the theory of projective limits. Normally, hypotheses on the projective system lead to conclusions on the limit. It is rare that, as in Theorem 1.2, assumptions on the limit entail conclusions on the projective system.

**Acknowledgments.** We gratefully acknowledge the referee's careful study of the text, his listing of our typos, and his appropriate comments pertaining to the mathematics of the paper.

### References

- [1] Glöckner, H., *Simplified Proofs for the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma*, J. of Lie Theory **17** (2007), 899–902.
- [2] Gotô, M., *On an arcwise connected subgroup of a Lie group*, Proc. Amer. Math. Soc. **20** (1969), 157–162.
- [3] Hofmann, K. H., and S. A. Morris, “The Structure of Compact Groups,” Walter DeGruyter, Berlin, 2006<sup>2</sup>, xvii+858pp.
- [4] —, “The Lie Theory of Connected Pro-Lie Groups,” EMS Publishing House, Zürich, 2007, xv+678 pp.
- [5] George Michael, A. A., *On Inverse Limits of Finite-Dimensional Lie Groups*, J. Lie Theory **16** (2006), 221–224.
- [6] Yamabe, H., *On an arcwise connected subgroup of a Lie group*, Osaka Math. J. **2** (1950), 13–14.

Karl H. Hofmann  
 Fachbereich Mathematik  
 Technische Universität Darmstadt  
 Schloßgartenstraße 7  
 64289 Darmstadt  
 Germany  
 hofmann@mathematik.tu-darmstadt.de

Sidney A. Morris  
 School of Information Technology  
 and Mathematical Sciences  
 University of Ballarat  
 P.O. Box 663, Ballarat, Vic. 3353  
 Australia  
 s.morris@ballarat.edu.au

Received November 30, 2007  
 and in final form April 8, 2008