TOPOLOGY PROCEEDINGS
Volume 28, Summer 2004
Pages 541-567

LIE THEORY AND THE STRUCTURE OF PRO-LIE GROUPS AND PRO-LIE ALGEBRAS

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Abstract. This text presents basic results from a projected monograph on "Lie Theory and the Structure of Pro-Lie groups and Locally Compact Groups" which may be considered a sequel to our book "The Structure of Compact Groups" [De Gruyter, Berlin, 1998]. In focus are the categories of projective limits of finite dimensional Lie groups and of projective limits of finite dimensional Lie algebras, their functorial relationship, and their intrinsic Lie theory. Explicit information on pro-Lie algebras, simply connected pro-Lie groups and abelian pro-Lie groups is given.

Introduction

There are two prime reasons for the success of the structure and representation theory of locally compact groups: the existence of Haar integral on a locally compact group G and the successful resolution of Hilbert's Fifth Problem with the proof that connected locally compact groups can be approximated by finite dimensional Lie groups. Lie groups themselves have a highly developed structure and representation theory.

Haar measure is the key to the representation theory of compact and locally compact groups on Hilbert space, and the wide field of harmonic analysis with ever so many ramifications (including e.g. abstract probability theory on locally compact groups).

2000 *Mathematics Subject Classification.* 22D, 22E.

Key words and phrases. Lie group, projective limit, Lie algebra, exponential function, locally compact group, pro-Lie algebra, simply connected group, abelian pro-Lie group.

A theorem of A. Weil ([24], pp. 140–146) shows that, conversely, a complete topological group with a left- (or right-) invariant measure is locally compact (see also [7], pp. 266–289). Thus the category of locally compact groups is that which is exactly suited for real analysis resting on the existence of an invariant integral. One cannot expect to extend that aspect of locally compact groups easily to larger classes. [1]

However, from a category theoretical and from a Lie theoretical point of view the class of a locally compact groups has defects which go rather deep. Indeed while every locally compact group G has a Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp\colon \mathfrak{L}(G) \to G$, the additive group of the Lie algebra is never itself locally compact unless it is finite dimensional. Thus from the view point of Lie theory, the category of locally compact groups appears to have two major drawbacks:

—The topological abelian group underlying the Lie algebra $\mathfrak{L}(G)$ fails to be locally compact unless $\mathfrak{L}(G)$ is finite dimensional. In other words, the very Lie theory making the structure theory of locally compact groups interesting leads us outside the class.

—The category of locally compact groups is not closed under the forming of products, even of copies of $\mathbb{R}$; it is not closed under projective limits of projective systems of finite dimensional Lie groups, let alone under arbitrary limits. In other words, the category of locally compact groups is badly incomplete.

Let us denote the category of all (Hausdorff) topological groups and continuous group homomorphisms by $\mathbb{TOPGR}$. It turns out that the full subcategory pro$\mathbb{LIEGR}$ of $\mathbb{TOPGR}$ consisting of *all* projective limits of finite dimensional Lie groups avoids both of these

[1]The referee contributes the following interesting comment for which the authors are grateful: *"Strictly speaking, Weil's theorem … says that a complete group with a σ-*finite *left- (or right-) invariant measure is locally compact. σ-finiteness is essential and should be mentioned here, because in its absence there are some bizarre examples, e.g. the countable power of the reals with the product of Lebesgue measure."* Here the referee has pointed indeed into the direction of future research on the measure theory of pro-Lie groups, which are at the core of this investigation and whose definition we shall explain in Section 1 below. Such a theory would be well under way in view of [4], pp. 50–55, 70ff. But that is a different story, to be told elsewhere.
Referee's Report, May 12, 2004.

difficulties. This would perhaps not yet be a sufficient reason for advocating this category if it were not for two facts:

—Firstly, while not every locally compact group is a projective limit of Lie groups, every locally compact group has an open subgroup which is a projective limit of Lie groups, so that, in particular, every connected locally compact group is a projective limit of Lie groups. Lie theory itself does not carry far beyond the identity component of a topological group anyhow.

—Secondly, the category pro$\mathbb{LIEGR}$ is astonishingly well-behaved. Not only is it a complete category, it is closed under passing to closed subgroups and to those quotients which are complete, and it has a demonstrably good Lie theory.

It is therefore indeed surprising that this class of groups has been little investigated in a *systematic* fashion. The first to recognize that a Lie algebra can be attached to a locally compact group was Lashof [20] (1957). In a widely circulating set of Lecture Notes [9] (1966), one of the present authors offered a general Lie theory somewhat in the spirit of Chapter 2 below and applied it to compact groups for which a more full-fledged application is given in [11] (1998). Boseck, Czichowski and Rudolph wrote the first book on the topic of a Lie theory of locally compact groups [1] (1981) with applications to analysis on locally compact groups in mind. The idea of making one-parameter subgroups of a topological group the raw material of a Lie theory of topological groups is advocated Wojtyński [26, 27] (see his bibliography in these sources). In individual studies such as [5,12,14,15,17], the Lie theory of locally compact groups and of pro-Lie groups was considered. The foundations of a Lie theory of locally compact groups, of pro-Lie groups, or even of topological groups in general is no longer in need of being emphasized—it is displayed even in textbooks as well [22]. Infinite dimensional Lie group theory is topical and will soon be represented by a new monograph by Glöckner and Neeb [6] based on calculus on manifolds modelled on locally convex vector spaces. The emphasis in our effort, however, is twofold: firstly, we focus on the category of pro-Lie groups which contains everything that can be said on the structure of locally compact groups via Lie theory, and beyond, and, secondly, we *systematically* and seriously exploit the Lie theory for a structure theory of pro-Lie and locally compact groups. A good start was made in [11] for the purpose of a structure theory of *compact* groups.

At the Summer Conference on Topology in New York in 2001 we announced a program of presenting a comprehensive Lie and structure theory of locally compact groups by reducing such a theory to Lie groups on the one hand and compact groups on the other [10]. And indeed such an investigation is made in [13] where it is submitted that a general structure theory of locally compact groups should be based on a good understanding of the category pro$\mathbb{LIEGR}$. An exploitation of the Lie theory of pro-Lie groups for a structure theory of locally compact groups involves, in particular, a serious investigation of the structure theory of pro-Lie algebras, that is, topological Lie algebras which are projective limits of their finite dimensional quotient algebras.

This article presents a crisp overview of some of the central results whose proofs will be detailed and whose background and applications will be discussed in [13].

1. Core results on pro-Lie groups

For a description of some basic results on the theory of projective limits of Lie groups some technical background information appears inevitable.

Definition 1.1. A *projective system* D of topological groups is a family of topological groups $(C_j)_{j\in J}$ indexed by a directed set J and a family of morphisms $\{f_{jk}\colon C_k \to C_j \mid (j,k) \in J \times J,\ j \leq k\}$, such that f_{jj} is always the identity morphism and $i \leq j \leq k$ in J implies $f_{ik} = f_{ij} \circ f_{jk}$. Then the *projective limit of the system* $\lim_{j\in J} C_j$ is the subgroup of $\prod_{j\in J} C_j$ consisting of all J-tuples $(x_j)_{j\in J}$ for which the equation $x_j = f_{jk}(x_k)$ holds for all $j, k \in J$ such that $j \leq k$.

Every cartesian product of topological groups may be considered as a projective limit. Indeed, if $(G_\alpha)_{\alpha\in A}$ is an arbitrary family of topological groups indexed by an infinite set A, one obtains a projective system by considering J to be the set of finite subsets of A directed by inclusion, by setting $C_j = \prod_{a\in j} G_a$ for $j \in J$, and by letting $f_{jk}\colon C_j \to C_k$ for $j \leq k$ in J be the projection onto the partial product. The projective limit of this system is isomorphic to $\prod_{a\in A} G_a$.

There are a few sample facts one should recall about the basis properties of projective limits (see e.g. [2], [5], [18], or [13] 1.27 and 1.33):

Let $G = \lim_{j\in J} G_j$ be a projective limit of a projective system

$$\mathcal{P} = \{f_{jk} \colon G_k \to G_j \mid (j,k) \in J \times J,\, j \le k\}$$

of topological groups with limit morphisms $f_j \colon G \to G_j$, and let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then $\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J,\, U \in \mathcal{U}_k\}$ and $\mathcal{N}$ is a filter basis of closed normal subgroups converging to 1. If all bonding maps $f_{jk} \colon G_j \to G_k$ are quotient morphisms and all limit maps f_j are surjective, then the limit maps $f_j \colon G \to G_j$ are quotient morphisms. The limit G is complete if all G_j are complete.

Definition 1.2. For a topological group G let $\mathcal{N}(G)$ denote the set of all closed normal subgroups N such that the quotient group G/N is a finite dimensional real Lie group. Then $G \in \mathcal{N}(G)$, and G is said to be *a proto-Lie group* if

(1) every identity neighborhood of G contains a member of $\mathcal{N}(G)$.

If, furthermore, the following condition (2) is satisfied it is called a *pro-Lie group*:

(2) G is a complete topological group, that is, every Cauchy filter converges.

One can show (see [13], Chapter 3, Definition 3.25 and the discussion leading to it) that if condition (1) is satisfied, then $\mathcal{N}(G)$ is a filter basis, so that (1) is equivalent to saying that $\mathcal{N}(G)$ is a filterbasis converging to the identity.

The full subcategory of the category $\mathbb{TOPGR}$ of topological groups and continuous homomorphisms consisting of all pro-Lie groups and continuous homomorphisms between them is called pro$\mathbb{LIEGR}$.

Every product of a family of finite dimensional Lie groups $\prod_{j\in J} G_j$ is a pro-Lie group. In particular, $\mathbb{R}^J$ is a pro-Lie group for any set J which is locally compact if and only if the set J is finite. The subgroup

$$\bigl\{(g_j)_{j\in J} \in \prod_{j\in J} G_j : \{j \in J : g_j \neq 1\} \text{ is finite}\bigr\}$$

is a proto-Lie group which is not a pro-Lie group if J is infinite and the G_j are nonsingleton. Every proto-Lie group has a completion which is a pro-Lie group. A topological group G is called *almost connected* if the factor group G/G_0 modulo the connected component G_0 of the identity is compact. In the middle of the last century it was proved that every almost connected locally compact group is a pro-Lie group [28, 29].

Every pro-Lie group G gives rise to a projective system

$$\{p_{NM}\colon G/M \to G/N : M \supseteq N \text{ in } \mathcal{N}(G)\}$$

whose projective limit it is (up to isomorphism). The converse is a difficult issue, but it is true.

Theorem 1.3. *Every projective limit of Lie groups is a pro-Lie group. Every closed subgroup of a pro-Lie group is a pro-Lie group. Every quotient group of a pro-Lie group is a proto-Lie group and has a completion which is a pro-Lie group.*

Proof. [13], 3.34, 3.35, 4.1; [14]. □

It is important to have a firm grasp of the concept of a pro-Lie group, and the information available at this point allows to formulate the

Scholium. *A topological group G is a pro-Lie group if and only if it satisfies one, hence all of the following equivalent conditions*

(1) *G is complete and each identity neighborhood contains a normal subgroup such that G/N is a Lie group.*
(2) *G is the projective limit of Lie groups* (*see* 1.1).
(3) *G is algebraically and topologically isomorphic to a closed subgroup of a product of Lie groups.*

In a topological Lie algebra $\mathfrak{g}$ the filterbasis of closed ideals $\mathfrak{j}$ satisfying $\dim \mathfrak{g}/\mathfrak{j} < \infty$ is denoted by $\mathcal{I}(\mathfrak{g})$.

Definition 1.4. A topological Lie algebra $\mathfrak{g}$ is called a *pro-Lie algebra* (short for *profinite dimensional Lie algebra*) if $\mathcal{I}(\mathfrak{g})$ converges to 0 and if $\mathfrak{g}$ is a complete topological vector space.

Under these circumstances, $\mathfrak{g} \cong \lim_{\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{j}$, and the underlying vector space is a weakly complete topological vector space, that is, it is the dual of a real vector space with the weak star topology. For a systematic treatment of the duality of vector spaces and weakly complete topological vector spaces we refer to [11], pp. 319ff. The category of pro-Lie algebras and continuous vector space morphisms is denoted pro$\mathbb{LIEALG}$.

Theorem 1.5. (i) *The category* pro$\mathbb{LIEGR}$ *of pro-Lie groups is closed in* $\mathbb{TOPGR}$ *under the formation of all limits and is therefore complete. It is the smallest full subcategory of* $\mathbb{TOPGR}$ *that contains all finite dimensional Lie groups and is closed under the formation of all limits.*

(ii) *The category* pro$\mathbb{LIEALG}$ *of pro-Lie algebras is closed in the category of topological Lie algebras under the formation of all limits and is therefore complete. It is the smallest category that contains all finite dimensional Lie algebras and is closed under the formation of all limits.*

Proof. [13], 3.3, 3.36; [14]. □

2. The Lie Theory of Topological Groups: Lie's Third Theorem

Definition 2.1. A topological group G is said to *have a Lie algebra* $\mathfrak{L}(G)$ if the space $\mathrm{Hom}(\mathbb{R}, G)$ of all continuous one parameter subgroups (i.e. morphisms of topological groups) $X\colon \mathbb{R} \to G$ with the topology of uniform convergence on compact subsets of $\mathbb{R}$ has a continuous addition and bracket multiplication making it into a topological Lie algebra in such a fashion that

$$(X+Y)(r) = \lim_{n\to\infty} X(\frac{r}{n})Y(\frac{r}{n})$$

and

$$[X,Y](r2) = \lim_{n\to\infty} X(\frac{r}{n})Y(\frac{r}{n})X(\frac{r}{n})^{-1}Y(\frac{r}{n})^{-1}.$$

If G has a Lie algebra, set $\exp X = X(1)$ and $\exp(r{\cdot}X) = X(r)$ and call

$$\exp\colon \mathfrak{L}(G) \to G$$

the exponential function of G.

The full subcategory $\mathbb{LIEALGGR}$ of $\mathbb{TOPGR}$ containing all topological groups with Lie algebra is complete and there is a limit preserving functor $\mathfrak{L}\colon \mathbb{LIEALGGR} \to \mathrm{top}\mathbb{LIEALG}$ assigning to a topological group with Lie algebra its topological Lie algebra.

Remarkably, this functor has a left adjoint.

Theorem 2.2. *The functor* $\mathfrak{L}\colon \mathbb{LIEALGGR} \to \mathrm{top}\mathbb{LIEALG}$ *has a left adjoint* $\Gamma\colon \mathrm{top}\mathbb{LIEALG} \to \mathbb{LIEALGGR}$. *Thus there is a natural morphism of topological Lie algebras* $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}\Gamma(\mathfrak{g})$ *such that for each topological group* H *with a Lie algebra and each morphism* $f\colon \mathfrak{g} \to \mathfrak{L}(H)$ *of topological Lie algebras there is a unique morphism* $f'\colon \Gamma(\mathfrak{g}) \to H$ *such that* $f = \mathfrak{L}(f') \circ \eta_{\mathfrak{g}}$*:*

$$\begin{array}{ccc|c}
\multicolumn{3}{c}{\mathrm{top}\mathbb{LIEALG}} & \mathbb{LIEALGGR}\\
\hline
\mathfrak{g} & \xrightarrow{\eta_{\mathfrak{g}}} & \mathfrak{L}\big(\Gamma(\mathfrak{g})\big) & \Gamma(\mathfrak{g})\\
{\scriptstyle\forall f}\big\downarrow & & \big\downarrow{\scriptstyle\mathfrak{L}(f')} & \big\downarrow{\scriptstyle\exists! f'}\\
\mathfrak{L}(H) & \xrightarrow[\mathrm{id}_{\mathfrak{L}(H)}]{} & \mathfrak{L}(H) & H\\
\hline
\end{array}$$

The universal property can also be expressed dually as follows: For each topological group G *with a Lie algebra there is a natural morphism of topological groups* $\epsilon_G\colon \Gamma\mathfrak{L}(G) \to G$ *such that for each morphism* $f\colon \Gamma(\mathfrak{h}) \to G$ *there is a unique morphism* $f'\colon \mathfrak{h} \to \mathfrak{L}(G)$ *of topological Lie algebras such that* $f = \epsilon_G \circ \Gamma(f')$*:*

$$\begin{array}{c|ccc}
\mathrm{top}\mathbb{LIEALG} & \multicolumn{3}{c}{\mathbb{LIEALGGR}}\\
\hline
\mathfrak{L}(G) & \Gamma\big(\mathfrak{L}(G)\big) & \xrightarrow{\epsilon_G} & G\\
{\scriptstyle\exists! f'}\big\uparrow & {\scriptstyle\Gamma(f')}\big\uparrow & & \big\uparrow{\scriptstyle\forall f}\\
\mathfrak{h} & \Gamma(\mathfrak{h}) & \xrightarrow[\mathrm{id}_{\Gamma(\mathfrak{h})}]{} & \Gamma(\mathfrak{h})\\
\hline
\end{array}$$

The proof follows from the Adjoint Functor Existence Theorem (see for instance [11], Appendix 3, p. 728, Theorem A3.60), for which the Solution Set Condition has to be verified (see for instance [11], p. 728, Definition A3.58). The adjunction provides a natural bijection

$$f \mapsto f' : \mathrm{Hom}\,\big(\mathfrak{g}, \mathfrak{L}(H)\big) \to \mathrm{Hom}(\Gamma(\mathfrak{g}), H).$$

If $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}\Gamma(\mathfrak{g})$ happens to be an isomorphism, then the universal property represents *Lie's Third Theorem* because $\Gamma\mathfrak{g}$ is a topological group with Lie algebra whose Lie algebra is the given Lie algebra $\mathfrak{g}$. If G is a finite dimensional Lie group and $\mathfrak{g} = \mathfrak{L}(G)$ then $\Gamma(\mathfrak{g})$ is none other than the universal covering group $\widetilde{G}$ of the identity component G_0 of G, and $\epsilon_G\colon \widetilde{G} \to G$ is the universal covering morphism onto this component. If $\mathfrak{g}$ is one of the notorious Banach Lie algebras failing the conclusion of Lie's Third Theorem, then $\Gamma(\mathfrak{g})$ may well be singleton or otherwise degenerate in the sense that $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}\Gamma(\mathfrak{g})$ may fail to be an isomorphism. Quite generally, it is clear that the natural morphism $\epsilon_G\colon \Gamma\mathfrak{L}(G) \to G$ is the natural generalisation of the universal covering morphism of general Lie theory and we do define

$$\widetilde{G} \stackrel{\text{def}}{=} \Gamma\big(\mathfrak{L}(G)\big).$$

If G is a compact connected abelian group, then $\widetilde{G}$ is just the underlying topological abelian group of the Lie algebra $\mathfrak{L}(G) = \operatorname{Hom}(\mathbb{R}, G)$, and if $\widetilde{G}$ is identified with $\mathfrak{L}(G)$, then $\epsilon_G\colon \widetilde{G} \to G$ is to be identified with the very exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ which we studied at length in [11] and [17]. This example shows that ϵ_G in general is rather far from a covering morphism while retaining all the while its universal property.

It is somewhat surprising that the adjunction theorem and the existence of the functor Γ has not been formulated before in spite of the high level of development of Lie group theory; it seems as if the theory of finite dimensional Lie groups was not particularly conducive to reveal the full functorial set-up of Lie's Third Theorem.

Be that as it may, on the category pro$\mathbb{LIEGR}$ of pro-Lie groups, the adjunction theorem works with astonishing perfection. Indeed as a first step we have

Theorem 2.3. *Every pro-Lie group G has a pro-Lie algebra as Lie-algebra and thus is an object of $\mathbb{LIEALGGR}$. The assignment $\mathfrak{L}$ which associates with a a pro-Lie group G its pro-Lie algebra is a limit preserving functor $\mathfrak{L}\colon \mathrm{pro}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEALG}$.*

Proof. Chapters 2 and 3. $\square$

From Theorem 2.2 we get a left adjoint Γ of $\mathfrak{L}$, and in [13], Chapter 6, we show that for each $\mathfrak{g}$ the topological group $\Gamma(\mathfrak{g})$ is simply connected and arcwise connected. Taking this and the preceding two theorems together, for pro-Lie groups we obtain the following result:

Theorem 2.4. (Lie's Third Theorem for pro-Lie groups) *The Lie algebra functor $\mathfrak{L}\colon \mathrm{pro}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEALG}$ has a left adjoint Γ. It associates with every pro-Lie algebra $\mathfrak{g}$ a unique simply connected and arcwise pro-Lie group $\Gamma(\mathfrak{g})$ and a natural isomorphism $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}\big(\Gamma(\mathfrak{g})\big)$ of topological Lie algebras such that for every morphism $\varphi\colon \mathfrak{g} \to \mathfrak{L}(G)$ there is a unique morphism $\varphi'\colon \Gamma(\mathfrak{g}) \to G$ such that $\varphi = \mathfrak{L}(\varphi') \circ \eta_{\mathfrak{g}}$. For each pro-Lie group G there is a functorially associated simply connected and arcwise connected group $\widetilde{G} = \Gamma\mathfrak{L}(G)$ and a natural morphism $\epsilon_G\colon \widetilde{G} \to G$ whose image is dense in the identity component G_0 of G; for each morphism $f\colon \Gamma(\mathfrak{h}) \to G$ there is a unique morphism $f'\colon \mathfrak{h} \to \mathfrak{L}(G)$ of topological Lie algebras such that $f = \epsilon_G \circ \Gamma(f')$.*

Proof. [13], Chapter 6. $\square$

Let us denote by $\widetilde{\mathrm{pro}}\mathbb{LIEGR}$ the full subcategory of $\mathrm{pro}\mathbb{LIEGR}$ consisting of all *simply connected* pro-Lie groups and all morphisms between them. Then we get

Corollary 2.5. *The adjoint functors $\mathfrak{L}$ and Γ via restriction and corestriction induce functors*

$$\mathfrak{L}\colon \widetilde{\mathrm{pro}}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEALG} \ \textit{and}\ \Gamma\colon \mathrm{pro}\mathbb{LIEALG} \to \widetilde{\mathrm{pro}}\mathbb{LIEGR},$$

which implement an equivalence of categories.

The functor

$$G \mapsto \widetilde{G} : \mathrm{pro}\mathbb{LIEGR} \to \widetilde{\mathrm{pro}}\mathbb{LIEGR}$$

is a retraction functor and is left adjoint to the inclusion $\widetilde{\mathrm{pro}}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEGR}$.

In this regard, the category of simply connected and arcwise connected pro-Lie groups is a faithful image of the category of pro-Lie algebras. We do have a rather satisfactory body of information on the structure of pro-Lie algebras; therefore Corollary 2.5 helps us to have a good grip on the structure of simply connected pro-Lie groups. How well this works we shall see soon. Since every pro-Lie group G carries with it the natural morphism $\epsilon_G\colon \widetilde{G} \to G$ whose image is dense in G_0, we have an effective hold on the structure theory of connected pro-Lie groups. The morphism ϵ_G, in a loose but well understood sense may be regarded as a group theoretical substitute of the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$. For abelian pro-Lie groups, ϵ and exp may be identified.

Just as the Bohr compactification of a topological group provides a functorial left reflection from the category of topological groups into the category of compact groups, we have left reflections $\mathbb{TOPGR} \to \mathbb{LIEALGGR} \to \mathrm{pro}\mathbb{LIEGR}$. This in fact a consequence of a rather general lemma on full subcategories of the category $\mathbb{TOPGR}$ of topological groups:

Lemma 2.6. (Retraction Lemma for Full Closed Subcategories of $\mathbb{TOPGR}$)
For any full subcategory $\mathcal{G}$ of the category $\mathbb{TOPGR}$ of topological groups and continuous morphisms that is closed in $\mathbb{TOPGR}$ under the formation of all limits and the passage to closed subgroups, there is a left adjoint functor $F\colon \mathbb{TOPGR} \to \mathcal{G}$ which on $\mathcal{G}$ agrees (up to a natural isomorphism) with the identity functor on $\mathcal{G}$. In particular given any topological group G, there is a topological group FG in $\mathcal{G}$ and a morphism $\eta_G\colon G \to FG$ with dense image such that for every morphism $f\colon G \to H$ into a $\mathcal{G}$-group H there is a unique morphism $f'\colon FG \to H$ such that $f = f' \circ \eta_G$.

Proof. Corollary 1.41 of [13]. $\square$

In particular, taking for $\mathcal{G}$ the category $\mathrm{pro}\mathbb{LIEGR}$, for each topological group G we get a pro-Lie group PG and a natural morphism $\eta_G\colon G \to PG$ such that every morphism from G into a proLie group factors through η_G. Without too much additional effort one shows the existence of free pro-Lie groups over pointed topological spaces. In fact, every pointed completely regular space may be considered as a subspace of a pro-Lie group such that the subgroup algebraically generated by the subspace is a free group and that the universal property holds; we do not dwell here on this subject; for free compact groups instead of free pro-Lie groups see for instance [11], Chapter 11.

We noted prominently that the functor $\mathfrak{L}$ preserves all limits. It it, however, remarkable that $\mathfrak{L}$ preserves some colimits as well:

Theorem 2.7. *The functor $\mathfrak{L}$ preserves quotients. Specifically, assume that G is a pro-Lie group and N a closed normal subgroup and denote by $q\colon G \to G/N$ the quotient morphism. Then G/N is a proto-Lie group whose Lie algebra $\mathfrak{L}(G/N)$ is a pro-Lie algebra and the induced morphism of pro-Lie algebras $\mathfrak{L}(q)\colon \mathfrak{L}(G) \to \mathfrak{L}(G/N)$ is a quotient morphism. The exact sequence*

$$0 \to \mathfrak{L}(N) \to \mathfrak{L}(G) \to \mathfrak{L}(G/N) \to 0$$

induces an isomorphism $X + \mathfrak{L}(N) \mapsto \mathfrak{L}(f)(X) : \mathfrak{L}(G)/\mathfrak{L}(N) \to \mathfrak{L}(G/N)$. $\square$

The core of Theorem 2.7 is proved by showing that for every quotient morphism $f\colon G \to H$ of topological groups, where G is a pro-Lie group, every one parameter subgroup $Y\colon \mathbb{R} \to H$ lifts to one of G, that is, there is a one parameter subgroup σ of G such that $Y = f \circ \sigma$. ([13], 4.19, 4.20.) This requires the Axiom of Choice.

Corollary 2.8. *Let G be a pro-Lie group. Then $\{\mathfrak{L}(N) \mid N \in \mathcal{N}(G)\}$ converges to zero and is cofinal in the filter $\mathcal{I}(\mathfrak{L}(G))$ of all ideals $\mathfrak{i}$ such that $\mathfrak{L}(G)/\mathfrak{i}$ is finite dimensional.*

Furthermore, $\mathfrak{L}(G)$ is the projective limit $\lim_{N\in\mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N)$ of a projective system of bonding morphisms and limit maps all of which are quotient morphisms, and there is a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\gamma_G)} & \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \frac{G}{N}) & \cong & \lim_{n\in\mathcal{N}(G)} \frac{\mathfrak{L}(G)}{\mathfrak{L}(N)} \\
{\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \exp_{G/N})} & & \\
G & \xrightarrow[\gamma_G]{} & \lim_{N\in\mathcal{N}(G)} G/N. & &
\end{array}$$

Proof. [13], 4.21. $\square$

Theorem 2.7 expresses a version of exactness of $\mathfrak{L}$. But there is also an exactness theorem for Γ.

Theorem 2.9. *If $\mathfrak{h}$ is a closed ideal of a pro-Lie algebra $\mathfrak{g}$, then the exact sequence*

$$0 \to \mathfrak{h} \xrightarrow{\quad i \quad} \mathfrak{g} \xrightarrow{\quad q \quad} \mathfrak{g}/\mathfrak{h} \to 0$$

induces an exact sequence

$$1 \to \Gamma(\mathfrak{h}) \xrightarrow{\Gamma(j)} \Gamma(\mathfrak{g}) \xrightarrow{\Gamma(q)} \Gamma(\mathfrak{g}/\mathfrak{h}) \to 1,$$

in which $\Gamma(j)$ is an algebraic and topological embedding and $\Gamma(q)$ is a quotient morphism.

Proof. [13], 5.7, 5.8, and 5.9. □

There are some noteworthy consequences of Theorem 2.7.

Proposition 2.10. *Any quotient morphism $f\colon G \to H$ of pro-Lie groups onto a finite dimensional Lie group is a locally trivial fibration.*

Proof. [13], 4.22 (iv). □

For a topological group let $\mathbb{E}(G)$ denote the subgroup generated by all one parameter subgroups, that is

$$\mathbb{E}(G) \stackrel{\text{def}}{=} \langle \exp \mathfrak{L}(G) \rangle.$$

Proposition 2.11. (i) *For a pro-Lie group G, the subgroup $\mathbb{E}(G)$ is dense in G_0, i.e. $\overline{\mathbb{E}(G)} = G_0$. In particular, a connected nonsingleton pro-Lie group has nontrivial one parameter subgroups.*

Proof. [13], 4.22(i). □

Corollary 2.12. *For a pro-Lie group G the following statements are equivalent:*

(a) *G is totally disconnected.*
(b) $\mathfrak{L}(G) = \{0\}$.
(c) *The filter basis of open normal subgroups of G converges to 1.*

Proof. [13], 4.22. $\square$

We noted that for any pro-Lie group G, the additive group of its Lie algebra $\mathfrak{L}(G)$ is also a pro-Lie group. So we noted that for an abelian pro-Lie group G, the exponential function $\exp\colon \mathfrak{L}(G) \to G$ is in fact a morphism of pro-Lie groups, and the underlying additive group of $\mathfrak{L}(G)$ is the group $\widetilde{G} = \Gamma\big(\mathfrak{L}(G)\big)$. All of this applies, in particular, to locally compact abelian groups and, in particular, to compact abelian groups. In [11], Chapters 7 and 8, one finds the information that for a compact abelian group G, the kernel of the exponential function $\exp\colon \mathfrak{L}(G) \to G$ is naturally isomorphic to the fundamental group $\pi_1(G)$, and that the image of exp is the arc component G_a of 1 in G. Thus there is a bijective morphism $\mathfrak{L}(G)/\pi_1(G) \to G_a$. It is proved in [17] and in [13], 4.10ff. that for a compact connected abelian group G this morphism is an isomorphism iff in the character group $\widehat{G}$ of G every finite rank pure subgroup is a free direct summand. Whenever this condition is satisfied, G_a is a quotient of the pro-Lie group $\mathfrak{L}(G)$ and this quotient is incomplete if G is not arcwise connected. The simplest such example is the character group G of the discrete group $\mathbb{Z}^{\mathbb{N}}$. In this case $\mathfrak{L}(G) \cong \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{2^{\aleph_0}}$, and this vector group is a simple example of a pro-Lie group with an incomplete quotient group.

Quotients of pro-Lie groups, after all of this, are a somewhat delicate matter. It is therefore good to have sufficient conditions for quotients to be complete, such as for instance in the following theorem.

Theorem 2.13. *The quotient of an almost connected pro-Lie group modulo an almost connected closed normal subgroup is a pro-Lie group.*

Proof. [13], 4.28. $\square$

3. Core results on pro-Lie algebras

In view of the functorial correspondence set up between the categories $\mathrm{pro}\mathbb{LIEGR}$, $\widetilde{\mathrm{pro}\mathbb{LIEGR}}$, and $\mathrm{pro}\mathbb{LIEALG}$ every piece of information on pro-Lie algebras translates at once into information on pro-Lie groups; this translation process is often referred to as

Lie Theory. Because of Corollary 2.5 this works especially well for the translation between pro-Lie algebras and simply connected pro-Lie groups. Chapter 7 of [13] gives details on the workings of a Lie theory of pro-Lie groups. It is this Lie theory of pro-Lie groups that calls for a thorough understanding of the fine structure of pro-Lie algebras in the first place.

Definition 3.1. A pro-Lie algebra $\mathfrak{g}$ is called *semisimple* if it is isomorphic to a product $\prod_{j\in J}\mathfrak{s}_j$ of a family of finite dimensional simple real Lie algebras $\mathfrak{s}_j$. A pro-Lie algebra $\mathfrak{g}$ is called *reductive* iff it is isomorphic to a product of an abelian algebra $\mathbb{R}^I$ for a set I and a semisimple algebra $\mathfrak{s}$.

Definition 3.2. For subsets $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ let $[\mathfrak{a},\mathfrak{b}]$ denote the linear span of all commutator brackets $[X,Y]$ with $X\in\mathfrak{a}$ and $Y\in\mathfrak{b}$. Inductively, define $\mathfrak{g}^{(1)}=\mathfrak{g}^{[1]}=[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}^{(n+1)}=[\mathfrak{g}^{(n)},\mathfrak{g}^{(n)}]$, $\mathfrak{g}^{[n+1]}=[\mathfrak{g},g^{[n]}]$. A Lie algebra $\mathfrak{g}$ is said to be *countably solvable* if $\bigcap_{n=1}^{\infty}\mathfrak{g}^{(n)}=\{0\}$ and *countably nilpotent* if $\bigcap_{n=1}^{\infty}\mathfrak{g}^{[n]}=\{0\}$. If a Lie algebra $\mathfrak{g}$ has a unique largest countably solvable ideal, then it will be called the radical $\mathfrak{r}(\mathfrak{g})$, and if it has a largest countably nilpotent ideal, then it will be called the nilradical $\mathfrak{n}(\mathfrak{g})$.

This information suffices for pro-Lie groups. However we mention that it is more satisfactory from an algebraic point of view to proceed to a transfinite definition of the commutator series of a Lie algebra by transfinite recursion, forming the intersection of all prior terms in the sequence at a limit ordinal. The transfinite commutator series has to become stable for cardinality reasons; if stability is attained at $\{0\}$ we call the Lie algebra transfinitely solvable. However, in [13] we prove a theorem that says that a pro-Lie algebra is transfinitely solvable iff it is countably solvable iff every finite dimensional Hausdorff quotient algebra is solvable. Similar comments apply to nilpotency in place of solvability.

If the pro-Lie algebra $\mathfrak{g}$ happens to have a unique smallest member among the family of all closed ideals $\mathfrak{i}$ such that $\mathfrak{g}/\mathfrak{i}$ is reductive, then it is called the *coreductive radical* $\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$.

Theorem 3.3. *Every pro-Lie algebra* $\mathfrak{g}$ *has a closed radical, a closed nilradical and a coreductive radical such that the following properties are satisfied:*

(i) $\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g}) \subseteq \mathfrak{n}(\mathfrak{g}) \subseteq \mathfrak{r}(\mathfrak{g})$.

(ii) $\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g}) = \overline{[\mathfrak{g},\mathfrak{g}]} \cap \mathfrak{r}(\mathfrak{g}) = \overline{[\mathfrak{g},\mathfrak{r}(\mathfrak{g})]}$

(iii) $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ *is semisimple and* $\mathfrak{g}/\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *is reductive.*

$$\begin{array}{c} \mathfrak{g} \\ | \\ \mathfrak{r}(\mathfrak{g}) \\ | \\ \mathfrak{n}(\mathfrak{g}) \\ | \\ \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g}) \\ | \\ \{0\}. \end{array}$$

Proof. [13], 6.48, 6.66, 6.67. □

For finite dimensional Lie algebras, these are standard facts, but for pro-Lie algebras, a lot is to be proved here. Solvability for infinite dimensional Lie algebras is really a transfinite concept involving ordinals, and for topological Lie algebras we must also consider the closed commutator series. As we remarked, for pro-Lie algebras one never has to go beyond the commutator sequence indexed by natural numbers, and that the algebraic and topological concepts of solvability agree. Similar comments apply to nilpotency. An effective treatment of semisimplicity and reductivity involves the duality of weakly complete topological vector spaces applied to $\mathfrak{g}$-modules.

But indeed more is true.

Definition 3.4. For a a pro-Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{s}$ is called a *Levi summand* if the function

$$(X, Y) \mapsto X + Y : \mathfrak{r}(\mathfrak{g}) \times \mathfrak{s} \to \mathfrak{g}$$

is an isomorphism of topological vector spaces.

For each X in a pro-Lie algebra $\mathfrak{g}$, a derivation $\operatorname{ad} X$ and an automorphism of topological Lie algebras $e^{\operatorname{ad} X}$ are defined by $(\operatorname{ad} X)(Y) = [X, Y]$ and $e^{\operatorname{ad} X}(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot (\operatorname{ad} X)^n(Y)$, where the infinite series is summable (that is, the net of finite partial sums converges for all X and Y).

Theorem 3.5. (The Levi-Mal'cev-Theorem for Pro-Lie Algebras) *Every pro-Lie algebra* $\mathfrak{g}$ *has Levi summands* $\mathfrak{s}$ *so that* $\mathfrak{g}$ *is algebraically and topologically the semidirect sum* $\mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$*. Each Levi summand* $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ *is semisimple. For two Levi-summands* $\mathfrak{s}$ *and* $\mathfrak{s}_*$ *there is an element* $X \in \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *in the coreductive radical such that* $\mathfrak{s}_* = e^{\operatorname{ad} X}\mathfrak{s}$.

Proof. [13], 6.52., 6.76. □

These results translate into structural information concerning simply connected pro-Lie groups. In order to see this we first record the fact that we have explicit information on $\Gamma(\mathfrak{g})$ for a countably nilpotent pro-Lie algebra $\mathfrak{g}$ such as it occurs as the nilradical and the coreductive radical of a pro-Lie algebra.

We need to recall some background on the algebra of the Baker-Campbell-Dynkin-Hausdorff series.

Lemma 3.6. *If x and y are two elements then in the $\mathbb{Q}$-algebra of formal power series, the formal power series $x*y = \log(\exp x \exp y)$ is of the form $x * y = \sum_{r,s\geq 0} H_{r,s}(x,y)$ where $H_{r,s}$ is a Lie polynomial of degree r in x and s in y which is computed as follows, $H_{0,0}(x,y) = 0$, $H_{1,0}(x,y) = x$, $H_{0,1}(x,y) = y$, $H_{1,1}(x,y) = [x,y]$, where the higher terms are computed as follows. For each pair of nonnegative integers r and s with $r+s = 1$, let $J'(r,s)$ be the set of tuples $(r_1,\ldots,r_m,s_1,\ldots,s_{m-1})$ such that $r_1+\cdots+r_m = r$, $s_1+\cdots+s_{m-1} = s-1$, $r_1+s_1, r_2+s_2,\ldots,r_{m-1}+s_{m-1} \geq 1$ for $m \in \mathbb{N}$, and let $J''(r,s)$ be the set of tuples $(r_1,\ldots,r_{m-1},s_1,\ldots,s_{m-1})$ such that $r_1+\cdots+r_{m-1} = r-1$, $s_1+\cdots+s_{m-1} = s$, $r_1+s_1, r_2+s_2,\ldots,r_{m-1}+s_{m-1} \geq 1$ for $m \in \mathbb{N}$. Then $H_{r,s}(x,y) = H'_{r,s}(x,y) + H''_{r,s}(x,y)$ where*

$$(r+s)\cdot H'_{r,s}(x,y) =$$

$$\sum_{J'} \frac{(-1)^{m-1}}{m\prod_{j=1}^{m} r_j!\prod_{k=1}^{m-1} s_k!}(\operatorname{ad} x)^{r_1}(\operatorname{ad} y)^{s_1}\cdots(\operatorname{ad} x)^{r_{m-1}}(\operatorname{ad} y)^{s_{m-1}}(\operatorname{ad} x)^{r_m} y,$$

where $\sum_{J'}$ is extended over all $m\in\mathbb{N}$ and $(r_1,\ldots,r_m,s_1,\ldots,s_{m-1}) \in J'(r,s)$, and

$$(r+s)\cdot H''_{r,s}(x,y) =$$

$$\sum_{J''} = \frac{(-1)^{m-1}}{m\prod_{j=1}^{m-1} r_j!\prod_{k=1}^{m-1} s_k!}(\operatorname{ad} x)^{r_1}(\operatorname{ad} y)^{s_1}\cdots(\operatorname{ad} x)^{r_{m-2}}(\operatorname{ad} y)^{s_{m-2}}(\operatorname{ad} x)^{r_{m-1}} y,$$

where $\sum_{J''}$ is extended over all $m\in\mathbb{N}$ and $(r_1,\ldots,r_{m-1},s_1,\ldots,s_{m-1}) \in J''(r,s)$.

Proof. See [3], Chapitre II, §6, n$^{\rm o}$ 4, Theorème 2. □

Given any Lie algebra L and elements $X, Y \in L$, the elements $H_{r,s}(X,Y) \in L$ are well defined, and thus $\big(H_{r,s}(X,Y)\big)_{r,s\in\mathbb{N}}$ is a family of elements in L. If L is a topological Lie algebra, then it may or may not be summable.

Lemma 3.7. *Let $\mathfrak{g}$ be a countably topologically nilpotent pro-Lie algebra. Then the family $\big(H_{r,s}(X,Y)\big)_{r,s\in\mathbb{N}}$ is summable for all $X, Y \in \mathfrak{g}$. Therefore the element $X * Y$ is well defined.*

Proof. See [13], Chapter 8, Lemma 8.4. □

Theorem 3.8. (Theorem on Pro-Lie groups with Pronilpotent Lie Algebra)
(i) *Let $\mathfrak{g}$ be a pronilpotent pro-Lie algebra. Then $\Gamma(\mathfrak{g}) \cong (\mathfrak{g}, *)$ and*

$$\begin{array}{ccc} (\mathfrak{g}, *) & \xrightarrow{\cong} & \Gamma(\mathfrak{g}) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\Gamma(\mathfrak{g})}} \\ \mathfrak{g} & \xrightarrow[\mathrm{id}]{} & \mathfrak{g} \end{array}$$

In particular, $\Gamma(g)$ is homeomorphic to to $\mathbb{R}^J$ for some set J and thus is arcwise connected and simply connected.

(i) $X * Y * (-Y) = e^{\mathrm{ad}\, X} Y$ *for all* $X, Y \in \mathfrak{g}$.

(ii) $Z\big((\mathfrak{g}, *)\big) = (\mathfrak{z}(\mathfrak{g}), *) = (\mathfrak{z}, +)$.

Proof. See [13], Chapter 8, Theorem 8.5. □

There are examples of countably topologically nilpotent pro-Lie algebras with trivial center. A theorem that would be similar to Theorem 3.8 fails in the prosolvable case. However, we can show

Proposition 3.9. (Theorem on the Topological Structure of Simply Connected Pro-Lie Groups with Prosolvable Lie Algebras) *Let G be a simply connected pro-Lie group whose Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is prosolvable, that is, which is its own radical. Let $\mathfrak{n}$ denote its nilradical or its reductive radical, as the case may be. Then the following statements hold:*

(i) $\Gamma(\mathfrak{n}) \cong (\mathfrak{n}, *)$ *may be considered as a closed normal subgroup N of G such that $\mathfrak{L}(G/N)$ is naturally isomorphic to $\mathfrak{g}/\mathfrak{n}$.*

(ii) $\exp_{G/N} \mathfrak{g}/\mathfrak{n} \to G/N$ *is an isomorphism of weakly complete vector groups.*

(iii) *The quotient morphism* $q\colon G \to G/N$ *admits a continuous cross section* $\sigma\colon G/N \to G$ *such that* $\sigma(N) = 1$
(iv) *There is an* N*-equivariant homeomorphism* $\varphi\colon G \to N \times (G/N)$ *such that* $\varphi(n) = (n, N)$ *for all* $n \in N$, *and* $\mathrm{pr}_2 \circ \varphi = q$.
(v) G *is homeomorphic to* $\mathbb{R}^J$ *for some set* J.
(vi) G *is simply connected in any sense for which the additive group of a weakly complete topological vector space is simply connected.*

Proof. [13], 8.13. □

From the Levi-Mal'cev Theorem for Pro-Lie Algebras 3.5 we know that every pro-Lie algebra is algebraically and topologically the semidirect sum of the radical $\mathfrak{r}(\mathfrak{g})$ and a semisimple Levi summand $\mathfrak{s}$. One deduces for a connected pro-Lie group G the existence of a closed radical $R(G)$ which is simply connected, if G is simply connected, and whose structure is given in the previous proposition.

The categorical equivalence of the category $\mathrm{pro}\mathbb{LIEALG}$ of pro-Lie algebras and of $\widetilde{\mathrm{pro}\mathbb{LIEGR}}$ in Corollary 2.5 allows us to conclude that for a simply connected Lie group G the radical $R(G)$ is a semidirect factor with a semisimple *Levi complement*. This all results in the following

Theorem 3.10. (Structure Theorem for Simply Connected Pro-Lie Groups)
Let G *be a simply connected pro-Lie group with Lie algebra* $\mathfrak{g}$. *Then*
(i) G *is the semidirect product of a closed normal subgroup* $R(G)$ *whose Lie algebra* $\mathfrak{L}\big(R(G)\big)$ *is the radical* $\mathfrak{r}(\mathfrak{g})$ *and a closed subgroup* S *whose Lie algebra* $\mathfrak{s}$ *is a Levi summand of* $\mathfrak{g}$.
(ii) *There is a family of simple simply connected Lie groups* S_j, $j \in J$ *such that* $S \cong \prod_{j\in J} S_j$.
(iii) *There is a closed normal subgroup* $N = N_{\mathrm{cored}}(G)$ *of* G *contained in* $R(G)$ *such that the pro-Lie algebra* $\mathfrak{L}(N) = \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *is the coreductive radical of* $\mathfrak{g}$ *and that there is an* N*-equivariant isomorphism* $\varphi\colon R \to N \times (R/N)$, *where* $N \cong (\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g}),)$ *and where* $R/N \cong \mathfrak{r}(\mathfrak{g})/\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *is a vector group.*
(iv) R *is homeomorphic to* $\mathbb{R}^J$ *for some set* J.
(v) G *is homeomorphic to a product of copies of* $\mathbb{R}$ *and of a family of simple, simply connected real finite dimensional Lie groups.*

Proof. [13] Theorem 8.14. □

We say that a topological group G with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is *exponentially generated* if G is algebraically generated by the image of the exponential function, that is, if $G = \langle \exp_G \mathfrak{g} \rangle$.

Corollary 3.11. *If G is a simply connected pro-Lie group, then G is exponentially generated.*

Proof. [13], Corollary 8.17. □

In particular, a simply connected pro-Lie group is arcwise connected.

According to a classical theorem ([8], p. 180, Theorem 3.1) every one of the Lie groups S_j in Theorem 3.10 is homeomorphic to $\mathbb{R}^{n_j} \times C_j$ with a natural number n_j and a maximal compact connected subgroup C_j of S_j. As a consequence of 3.10 we therefore have:

Theorem 3.12. (Topological Structure of Simply Connected Pro-Lie Groups) *If a simply connected pro-Lie group G is written as a semidirect product of $R(S)$ and a subgroup isomorphic to $\prod_{j \in J} S_j$ according to* Theorem 3.10, *and if we denote a maximal compact subgroup of S_j by C_j, then for a suitable set P, the group G is homeomorphic to $\mathbb{R}^P \times \prod_{j \in J} C_j$.*

In particular, G is homotopy equivalent to a a compact connected semisimple group, and its entire algebraic topology (homotopy, cohomology) is that of a simply connected semisimple compact group.

Proof. This is an immediate consequence of Theorem 3.10. □

4. The Structure of Abelian Pro-Lie Groups

In order to understand what we have to face when we leave the terrain of simply connected pro-Lie groups we turn to the subcategory of commutative pro-Lie groups; we shall see that in the connected case we completely understand their structure and that in the totally disconnected domain, some questions remain.

Irrespective of our earlier listing of examples, let us record what examples we are looking at. A topological group G which is complete and has a filterbasis of identity neighborhoods which are open normal subgroups is called *prodiscrete*. By Definition 1.2 and Corollary 2.12,

a topological group is prodiscrete if and only if it is a totally disconnected pro-Lie group.

Example 4.1. Let $\mathbb{Z}(p^\infty) = \left(\bigcup_{n=1}^\infty \frac{1}{n}\cdot\mathbb{Z}\right)/\mathbb{Z}$ denote the discrete Prüfer group of all elements of p-power order in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, further $\mathbb{Z}$ and $\mathbb{Q}$ the discrete group of integers, respectively, rationals. Let $\mathbb{Q}_p$ be the field of p-adic rationals for some prime p with its p-adic topology. The following examples are abelian pro-Lie groups.
(i) All locally compact abelian groups. (In particular, all abelian Lie groups.)
(ii) All products of locally compact abelian groups,
specifically:
(iii) the groups $\mathbb{R}^J$;
(iv) the groups $\mathbb{Q}_p^J$;
(v) the groups $\mathbb{Q}^J$;
(vi) the groups $\mathbb{Z}^J$;
(vii) the groups $\mathbb{Z}(p^\infty)^J$.
Another noteworthy example is
(viii) The free abelian group $\mathbb{Z}^{(\mathbb{N})}$ in countably infinitely many generators supports a nondiscrete nonmetric pro-Lie topology.

None of the groups in (iii)–(vii) is locally compact if J is infinite; but if J is countable, all are Polish, that is, completely metrizable and second countable. A countable product of discrete infinite countable sets in the product topology is homeomorphic to the space $\mathbb{R}\setminus\mathbb{Q}$ of the irrational numbers in the topology induced by $\mathbb{R}$. Thus the space of irrational numbers in its natural interval topology can be given a prodiscrete group topology in many ways. The examples (iv)—(viii). Example (viii) illustrates that a countable group can very well be a nondiscrete pro-Lie group; we learn that a pro-Lie group may very well be a countable union of closed indeed compact subsets with empty interior and thus may fail to satisfy the conclusion of the Baire category theorem.

Recall that a subgroup of a topological group is said to be *fully characteristic*, if it is mapped into itself by every continuous endomorphism of G. We start off with defining three fully characteristic closed subgroups of an abelian pro-Lie group.

For the first we refer to André Weil's Lemma in the domain of of locally compact groups which says

Let g be an element of a locally compact group and $\langle g\rangle$ the subgroup generated by it. Then one (and only one) of the two following cases occurs

(i) $n \mapsto g^n : \mathbb{Z} \to \langle g\rangle$ *is an isomorphism of topological groups.*

(ii) $\overline{\langle g\rangle}$ *is compact.*

This important tool generalizes to pro-Lie groups:

Theorem 4.2. *Let E be either $\mathbb{R}$ or $\mathbb{Z}$ and $X\colon E \to G$ a morphism of topological groups into a pro-Lie group G. Then one and only one of the two following cases occurs*

(i) $r \mapsto X(r) : E \to X(E)$ *is an isomorphism of topological groups.*

(ii) $\overline{X(E)}$ *is compact.*

Proof. [13], Chapter 5 or [15]. □

Definition 4.3. Let $\mathrm{comp}(G)$ denote the set $\{x \in G : \overline{\langle x\rangle}$ is compact$\}$.

Theorem 4.4. *In an abelian pro-Lie group,* $\mathrm{comp}\, G$ *is a fully characteristic closed subgroup such that $G/\mathrm{comp}(G)$ is an abelian pro-Lie group which does not contain any nondegenerate compact subgroup.*

Proof. For the proof which is rather straightforward from Weil's Lemma for pro-Lie groups, see [13], [15]. □

The identity component G_0 of any topological group is a fully characteristic closed subgroup. In the case of abelian pro-Lie groups we can say:

If G is a commutative pro-Lie group then G/G_0 is a pro-Lie group which has arbitrarily small open subgroups.

We define $G_1 \stackrel{\text{def}}{=} G_0 \operatorname{comp}(G)$ and note

If G is a commutative pro-Lie group then G_1 is a closed fully characteristic subgroup and G/G_1 is a pro-Lie group with arbitrarily small open subgroups and no nontrivial compact subgroups. G_1 is the smallest closed subgroup containing all connected and all compact subgroups.

Theorem 4.5. (Main Theorem on Abelian Pro-Lie Groups) *Let G be an abelian pro-Lie group. Then the following conclusions hold:*
(i) *There is a closed subgroup V of G which is isomorphic to a vector group $\mathbb{R}^J$ for some set J such that $(v,c) \mapsto v+c : V \times \operatorname{comp}(G) \to G_1$ is an isomorphism of topological groups.*
(ii) *There is a closed subgroup H of G containing $\operatorname{comp}(G)$ such that $(v,h) \mapsto v+h : V \times H \to G$ is an isomorphism of topological groups, that is $G \cong \mathbb{R}^J \times H$.*
(iii) *H_0 is compact and equals $\operatorname{comp} G_0$, and $\operatorname{comp}(H) = \operatorname{comp}(G)$.*
(iv) *$H/H_0 \cong G/G_0$, and this group a pro-discrete group.*

Proof. [13], Theorem 5.30, or [15]. □

Corollary 4.6. *Let G be an abelian pro-Lie group. Then the exponential function $\exp_G$ of $G = V \oplus H$ decomposes as*

$$\exp_G = \exp_V \oplus \exp_H$$

where $\exp_V : \mathfrak{L}(V) \to V$ is an isomorphism of weakly complete vector groups, and where $\exp_H$ may be identified via restriction with the exponential function $\exp_{\operatorname{comp}(G_0)} : \mathfrak{L}\big(\operatorname{comp}(G_0)\big) \to \operatorname{comp}(G_0)$ of the unique largest compact connected subgroup.

As a topological group, the group $\widetilde{G} = \Gamma(\mathfrak{L}(G))$ is isomorphic $V \times \mathfrak{L}(H) \cong \mathbb{R}^I$ for some I.

The arc component G_a of zero in G is $V \oplus H_a = V \oplus \operatorname{comp}(G_0)_a = \exp_G \mathfrak{L}(G)$.

Proof. [13], Theorem 5.20. □

Moreover, if $\mathfrak{h}$ is a closed vector subspace of $\mathfrak{L}(G)$ such that $\exp \mathfrak{h} = G_a$, then $\mathfrak{h} = \mathfrak{L}(G)$.

The most instructive special case is the connected case.

Corollary 4.7. *A connected abelian pro-Lie group is isomorphic to* $\mathbb{R}^J \times C$ *where* C *is unique largest compact subgroup of* G.

The group G *is simply connected iff* $C = \{0\}$.

Proof. This is an immediate consequence of Theorem 4.5. □

This illustrates very clearly how the structure of abelian pro-Lie groups decomposes into vector groups of arbitrary dimension and compact groups of arbitrary dimension. We have seen in the structure theorems contained in 3.8, 3.9, and 3.10 that in a topological description of simply connected pro-Lie group, the vector groups $\mathbb{R}^J$ play a central role. The absence of simple connectivity in the abelian case is due to the presence of compact subgroups. In noncommutative groups, other causes may account for a possible failing of simple connectivity, such as the factoring of a closed central subgroup of a semisimple pro-Lie group of the form $\prod_{j\in J} S_j$ for simply connected simple Lie groups S_j in Theorem 3.10.

We say that a topological abelian group G *has duality* if the natural evaluation morphism $G \to \widehat{\widehat{G}}$ into its Pontryagin bidual is an isomorphism of topological groups.

Corollary 4.8. *A connected abelian pro-Lie group has duality, and its dual is isomorphic to* $\mathbb{R}^{(J)} \oplus A$ *for a real vector space* $\mathbb{R}^{(J)}$ *equipped with its finest locally convex topology and for a torsion free discrete group* A.

Proof. This is an immediate consequence of the fact that the product of two groups with duality has duality (see e.g. [11], p. 306, Proposition 7.10). □

In order to test the distance between abelian pro-Lie groups and locally compact abelian groups it is instructive to consider those abelian pro-Lie groups which are compactly generated. Indeed a topological group G is said to be *compactly generated* if there is a compact subset K such that $G = \langle K \rangle$, that is, G is *algebraically* generated by K.

Theorem 4.9. (Structure of Compactly Generated Abelian pro-Lie groups).

(i) *For a compactly generated abelian pro-Lie group G the characteristic closed subgroup* $\operatorname{comp}(G)$ *is compact and the characteristic closed subgroup G_1 is locally compact.*

(ii) *In particular, every vector group complement V is isomorphic to a euclidean group $\mathbb{R}^m$.*

(iii) *The factor group G/G_1 is a compactly generated pro-Lie group which has arbitrarily small open subgroups but no nontrivial compact subgroups.*

If G/G_1 is Polish, then G is locally compact and

$$G \cong \mathbb{R}^m \times \operatorname{comp}(G) \times \mathbb{Z}^n.$$

Proof. See [13], Theorem 5.32. □

A compactly generated pro-Lie group with arbitrarily small open subgroups but no nontrivial compact subgroups is isomorphic to a closed subgroup of a group $\mathbb{Z}^J$. If it is not of finite rank, then it is not isomorphic to a subgroup of $\mathbb{Z}^{\mathbb{N}}$. There is an example of a pro-Lie group which is algebraically isomorphic to $\mathbb{Z}^{(\mathbb{N})}$, the free abelian group of countably many generators, but it fails to be compactly generated.

What is unsatisfactory here is that we do not know whether there is an example of a compactly generated pro-Lie group with arbitrarily small open subgroups and no nondegenerate compact subgroups which is *not* isomorphic to $\mathbb{Z}^n$.

For connected abelian pro-Lie groups we have presented a very satisfactory structure theory. The simply connected abelian pro-Lie groups are bland: they are the additive groups of weakly complete vector spaces and thus are isomorphic to $\mathbb{R}^J$ for some set. At the opposite end we have those abelian pro-Lie groups which have no infinite discrete cyclic subgroups; they are precisely the compact ones. The dichotomy between compact groups and groups that are homeomorphic to $\mathbb{R}^J$ was also evident in the topological structure theorem of arbitrary, not necessarily abelian, connected pro-Lie groups 3.12. Due to the presence of a semisimple pro-Lie group factor, the product decomposition with one factor being a compact connected group and the other factor being homeomorphic to $\mathbb{R}^J$ is just a topological one. For locally compact connected groups such a decomposition persists in the absence of simple connectivity, but whether an arbitrary connected pro-Lie group is homeomorphic to a product of a maximal compact subgroup and a space homeomorphic to $\mathbb{R}^J$ is not established for pro-Lie groups in general. (Added in proof May 14, 2004:) On the other hand, we were able to establish, by developing a systematic theory of Cartan subalgebras of pro-Lie algebras, that *each pro-Lie group does have maximal compact connected subgroups and that these are conjugate under inner automorphisms.*

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