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Transitive Actions of Compact Groups and Topological Dimension

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DEDICATED TO HELMUT WIELANDT ON THE OCCASION OF HIS 90TH BIRTHDAY

There are many dimension functions defined on arbitrary topological spaces taking either a finite value or the value infinity. This paper defines a cardinal valued dimension function, dim. The Lie algebra $\mathfrak{L}(G)$ of a compact group G is a weakly complete topological vector space. Quotient spaces of weakly complete spaces are weakly complete; the dimension of a weakly complete vector space is the linear dimension of its dual. Assume that a compact group G acts transitively on a given space X and that H is the isotropy group of the action at an arbitrary point; let $\mathfrak{L}(G)$ and $\mathfrak{L}(H)$ denote the Lie algebras of G, respectively, H. It is shown that $\dim X = \dim \mathfrak{L}(G)/\mathfrak{L}(H)$. Moreover, such an X contains a space homeomorphic to $[0,1]^{\dim X}$; conversely, if X contains a homeomorphic copy of a cube $[0,1]^{\aleph}$, then $\aleph \leq \dim X$. En route one establishes a good deal of information on the quotient spaces G/H; such information is of independent interest. Finally, these results are generalized to quotient spaces of locally compact groups. A generalization of a theorem of Iwasawa is instrumental; it is of independent interest as well. © 2000 Academic Press

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INTRODUCTION

The dimension of a topological space is an invariant, taking nonnegative integral values or the value ∞. The dimension is invariant under homeomorphism; spaces with different dimensions cannot be homeomorphic. Several definitions for topological dimension are in use; for sufficiently well behaved classes of topological spaces they agree. Paradigmatically, the dimension of a *cube* $\mathbb{I}^X$, $\mathbb{I} \stackrel{\text{def}}{=} [0, 1]$ for a set X is card X if X is finite and ∞ otherwise, regardless of the choice of a traditional topological dimension function. Another example of a class of spaces having "unique" topological dimension is the class of underying spaces of compact groups (cf. [6, pp. 381ff, 481ff]). Two cubes $\mathbb{I}^X$ and $\mathbb{I}^Y$ are homeomorphic iff $\operatorname{card} X = \operatorname{card} Y$. If a cube C is homeomorphic to $\mathbb{I}^X$, then card X can be recovered from C as follows:

$$\operatorname{card} X = \begin{cases} \dim C & \text{if } C \text{ is finite dimensional,} \\ w(C) & \text{otherwise.} \end{cases}$$

Here $w(X)$ is the weight of a topological space, the smallest of all cardinals of bases of the topology of X. This suggests that we define a topological dimension function taking arbitrary cardinals (and not only finite ones) as values. That this idea works for spaces underlying compact groups was shown in [6]. For a topological space X we consider the Lebgesgue covering dimension cdim (see [3, p. 52]; in [6, pp. 386–388] there is a survey on topological dimension theory in relation to compact groups) and define

$$\dim X = \begin{cases} \operatorname{cdim} X & \text{if } \operatorname{cdim} X \text{ is finite,} \\ \sup\{w(Y) : Y \text{ is a connected} \\ \qquad \text{component of } X\} & \text{otherwise.} \end{cases}$$

If X is a connected homogeneous space and $\operatorname{cdim} X$ is infinite, then $\dim X = w(X)$.

In linear algebra there was never any question that the dimension of a vector space V over a field K takes arbitrary cardinals as values. Indeed one shows that between any two bases of V there is a bijection and thus unambiguously one defines $\dim_K V$ to be the cardinal card B of a basis of V. The dimension of a K-vector space is in fact the only isomorphy invariant; indeed V is isomorphic to the direct sum $K^{(\dim_K V)}$.

The dual $V^* = \operatorname{Hom}_K(V, K)$ of V is isomorphic to the product $K^{\dim_K V}$. If K is the field $\mathbb{R}$ of real numbers, then V^* is a locally convex topological vector space with respect to the topology of pointwise convergence of functionals. A locally convex topological vector space which arises in this fashion is called a *weakly complete* vector space; these vector spaces can be characterized abstractly without making reference to a predual, i.e., a vector space whose duals they are. Since weakly complete vector spaces have a perfect duality theory (cf. [6, p. 319ff]), the predual and the topological dual E' of a weakly complete vector space E are one and the same thing, and we can define the *dimension of a weakly complete vector space* E to be $\dim E = \dim_{\mathbb{R}} E'$. Then $E \cong \mathbb{R}^{\dim E}$, where $\cong$ denotes the relation of being isomorphic as weakly complete vector spaces, and thus $w(E) = \aleph_0 \cdot \dim E$. Cleary, $\mathbb{I}^{\dim E} \subseteq \mathbb{R}^{\dim E} \cong E$. Assume that $C \subseteq E$ and C is cube. If E is finite dimensional then by invariance of domain, $\dim C \leq \dim E$. If not, then $\dim C \leq w(C) \leq w(E) = \dim E$. Thus $\dim E$, an invariant defined via linear algebra, is a good measurement of a "topological dimension" of E as well.

The considerable significance of weakly complete vector spaces is due to the fact that the underlying topological vector space of the Lie algebra $\mathfrak{L}(G)$ of a compact (or locally compact) group is weakly complete. Thus weakly complete vector spaces are basic to the exponential function of locally compact groups (cf. [6, pp. 334ff, 355, 379, 474ff]). In particular, $\dim \mathfrak{L}(G)$ is an isomorphy invariant of G in the spirit of linear algebra, taking cardinals as values.

In [6] is was shown that for a compact connected group G we have $\dim G = \dim \mathfrak{L}(G)$. (See [6, p. 607, 12.25, p. 483ff]). This equation expresses the equality of a topological invariant with a linear algebra invariant. The motivation of this article is the desire to prove a more general result which applies to homogeneous spaces of compact groups.

It follows from the duality theory of weakly complete vector spaces that for any closed vector subspace V of a weakly complete vector space W the topological quotient vector space W/V is also a weakly complete vector space. Thus if G is a compact group and H is a closed subgroup, we can associate with the pair (G, H) the linear algebra invariant $\dim \mathfrak{L}(G)/\mathfrak{L}(H)$.

A compact space X is said to be a *homogeneous space of a compact group* if there is a compact group G acting transitively on it. If $x \in X$ then G_x denotes the isotropy subgroup $\{g \in G : g \cdot x = x\}$ of G at x (cf., e.g., [6, pp. 6, 518]). We shall prove the following

THEOREM. *Let X be a homogeneous space of a compact group. Then*

$$\dim X = \dim \mathfrak{L}(G)/\mathfrak{L}(G_x)$$

for any compact group G acting transitively on X and any $x \in X$.

The space X contains a cube of dimension $\dim X$, *and if C is a cube contained in X, then* $\dim C \leq \dim X$.

Let us note that this theorem, in the case of infinite dimensions, overcomes two obstructions. First, unlike the case of homogeneous spaces of compact Lie groups, where G/G_x is a manifold with a tangent bundle, in the general case there does not appear to be any natural way to attach a tangent space to the base point in G/G_x. Second, infinite cardinals cannot be subtracted; thus, in the case of a finite dimensional compact group G, the natural number $\dim G - \dim G_x$ makes perfectly good sense, yielding $\dim X$ (see, e.g., [12, p. 632, 96.10]); in the case of infinitely many dimensions we have to make reference to the weakly complete quotient vector space $\mathfrak{L}(G)/\mathfrak{L}(G_x)$ which is well defined and has a well defined dimension. Finally, let us observe that $\dim X$ is a topological invariant of the compact space X. On the other hand many compact groups G may act transitively on X; the theorem tells us that the dimension of the weakly complete vector space $\mathfrak{L}(G)/\mathfrak{L}(G_x)$ does not depend on G or on x, but only on X.

Our main result is proved for homogeneous spaces of compact groups. It allows us, however, to extend the result, in a concluding section, to homogeneous spaces of locally compact groups. In the process of doing so we had to extend a classical result of Iwasawa's; our generalisation is of independent interest: *every locally compact group is locally isomorphic to the direct product of a compact group and a Lie group.*

Dedication. We dedicate this text to Helmut Wielandt. The first author is profoundly grateful having had him as a teacher in the years of 1952–1961 and as a coreferee for his Habilitationsschrift in 1962. Wielandt's influence on his outlook on mathematics has been profound.

Helmut Wielandt's group theoretical legacy has greatly influenced the second author as well; it contributed significantly to the second author's continuous attachment to the structure theory of groups.

1. CLOSED SUBGROUPS OF COMPACT GROUPS

Let G be a compact connected group and let $\mathfrak{L}(G)$ denote the Lie algebra of G. (See [6, p. 474ff].) Then $\mathfrak{L}(G)$ is a weakly complete Lie algebra. The underlying topological vector space of the Lie algebra $\mathfrak{L}(G)$ of a compact group G is a weakly complete vector space V. (See [6, Definition 7.27, p. 323].) The dimension $\dim V$ of V as a weakly complete topological vector space is defined to the the algebraic dimension $\dim_{\mathbb{R}} V'$ of the topological dual of V (see [6, Definition 9.53(i), p. 483].) We now collect some known information on weakly complete vector spaces.

LEMMA 1.1. (i) *The category $\mathcal{C}$ of weakly complete vector spaces and continuous linear maps is dual to the category of vector spaces.*

(ii) *If W is a closed vector subspace of a weakly complete vector space V, then V/W is a weakly complete vector space and $V \cong W \oplus \frac{V}{W}$, where $\cong$ denotes isomorphism in the category $\mathcal{C}$, and* $\dim V = \dim W + \dim \frac{V}{W}$.

(iii) *If* $\{V_j : j \in J\}$ *is a family of weakly complete nonzero vector spaces then*

$$\dim\left(\prod_{j\in J} V_j\right) = \sum_{j\in J} \dim V_j = \sup\big\{\{\operatorname{card} J\} \cup \{\dim V_j : j \in J\}\big\}.$$

Proof. (i) See [6, p. 319ff., notably Theorem 7.30 on p. 325].

(ii) This follows from (i), notably loc. cit. 7.30(iv).

(iii) is a consequence of (i) and the definition of the dimension of a weakly complete vector space. ∎

We shall have occasion to apply these facts to the weakly complete Lie algebras of a compact connected group G and a closed subgroup H.

If G is a compact connected group, then by [6, p. 458, 9.24] there is a family $\{S_j : j \in J\}$ of simple, simply connected compact Lie groups S_j and a quotient morphism $\mu : G^* \stackrel{\text{def}}{=} Z_0(G) \times \prod_{j\in J} S_j \to G$ with a totally disconnected central kernel. Set $H^* = \mu^{-1}(H)$.

LEMMA 1.2. $G^*/H^* \cong G/H$, $\mathfrak{L}(G^*)/\mathfrak{L}(H^*) \cong \mathfrak{L}(G)/\mathfrak{L}(H)$.

Proof. The morphism $\mathfrak{L}(\mu)$: $\mathfrak{z} \times \prod_{j\in J} \mathfrak{s}_j \to \mathfrak{L}(G)$ with $\mathfrak{z} = \mathfrak{L}(Z_0(G))$ and $\mathfrak{s}_j = \mathfrak{L}(S_j)$ is an isomorphism of weakly complete Lie algebras (see [6, p. 478, 9.49]). Let μ_H: $H^* \to H$ be the induced morphism. Then $\mathfrak{L}(\mu_H)$: $\mathfrak{L}(H^*) \to \mathfrak{L}(H)$ is an isomorphism by [6, p. 477, 9.48]. Then $G/H \cong G^*/H^*$ and $\mathfrak{L}(G^*/H^*) \cong \mathfrak{L}(G/H)$ as we deduce from the following diagram in which the lines are exact and the two left vertical maps are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{L}(H^*) \arrow[r, "\mathfrak{L}(\mathrm{incl})"] \arrow[d, "\mathfrak{L}(\mu_H)"'] & \mathfrak{L}(G^*) \arrow[r, "\mathrm{quot}"] \arrow[d, "\mathfrak{L}(\mu)"'] & \mathfrak{L}(G^*)/\mathfrak{L}(H^*) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathfrak{L}(H) \arrow[r, "\mathfrak{L}(\mathrm{incl})"'] & \mathfrak{L}(G) \arrow[r, "\mathrm{quot}"'] & \mathfrak{L}(G)/\mathfrak{L}(H) \arrow[r] & 0.
\end{tikzcd}$$ ∎

Thus $\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{L}(G)$ is of the form $\mathfrak{z} \oplus \prod_{j\in J} \mathfrak{s}_j$, where $\mathfrak{z} = \mathfrak{L}(Z_0(G))$, $\mathfrak{s}_j = \mathfrak{L}(S_j)$. We set $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{L}(H)$.

DEFINITION 1.3. In addition we extend the index set J by an element $\# \notin J$ and set $J_\# = J \cup \{\#\}$. We define and $\mathfrak{s}_\# \stackrel{\text{def}}{=} \mathfrak{z}$ so that $\mathfrak{g} = \prod_{j\in J_\#} \mathfrak{s}_j$. For $j \in J_\#$ we let p_j: $\mathfrak{g} \to \mathfrak{s}_j$ denote the projection. We allow the abuse of language involved in identifying $\mathfrak{s}_k$ or any partial product $\prod_{j\in I} \mathfrak{s}_j$, $I \subseteq J$, with its canonical isomorphic copy in the product $\prod_{j\in J}\mathfrak{s}_j$. Define

$$J_= = \{j \in J_\# : p_j(\mathfrak{h}) = \mathfrak{s}_j\}, \tag{1}$$

$$J_\neq = \{j \in J_\# : p_j(\mathfrak{h}) \neq \mathfrak{s}_j\}, \tag{2}$$

$$J_\circ = \{j \in J_\# : \mathfrak{s}_j \subseteq \mathfrak{h}\}, \tag{3}$$

$$\mathfrak{h}_p = \prod_{j\in J_\neq} p_j(\mathfrak{h}), \tag{4}$$

$$\mathfrak{h}^\circ = \prod_{j\in J_\circ} p_j(\mathfrak{h}) = \prod_{j\in J_\circ} \mathfrak{s}_j. \tag{5}$$

We identify the algebra $\mathfrak{h}_p$ with $p_\#(\mathfrak{h}) \oplus \prod_{j\in J} p_j(\mathfrak{h})$. Obviously, we have

$$\mathfrak{h}^\circ \subseteq \mathfrak{h} \subseteq \mathfrak{h}_p \subseteq \mathfrak{g}. \tag{6}$$

We write $\mathfrak{a} \trianglelefteq \mathfrak{b}$ when $\mathfrak{a}$ is an ideal of $\mathfrak{b}$.

LEMMA 1.4. (i) $\mathfrak{h}^\circ \trianglelefteq \mathfrak{g}$.

(ii) $\mathfrak{h}$ *is a subdirect product in* $\prod_{j\in J_\#} p_j(\mathfrak{h}) = \mathfrak{h}_p$.

(iii) $\mathfrak{g}/\mathfrak{h}_p$ *and* $\prod_{j\in J_\neq} \mathfrak{s}_j/p_j(\mathfrak{h})$ *are isomorphic as weakly complete vector spaces*.

(iv) $\dim \mathfrak{g}/\mathfrak{h}_p = \dim \mathfrak{z}/p_\#(\mathfrak{h}) + \sum_{j\in J_\neq\setminus\{\#\}} \dim \mathfrak{s}_j/p_j(\mathfrak{h})$.

(v) $\mathfrak{g}/\mathfrak{h} \cong (\mathfrak{g}/\mathfrak{h}^\circ)/(\mathfrak{h}/\mathfrak{h}^\circ)$.

(vi) $\dim \mathfrak{g}/\mathfrak{h} = \dim(\mathfrak{g}/\mathfrak{h}^\circ)/(\mathfrak{h}/\mathfrak{h}^\circ) = \dim \mathfrak{g}/\mathfrak{h}_p + \dim \mathfrak{h}_p/\mathfrak{h}$.

(vii) *If* $J_\neq$ *is infinite*, *then*

$$\dim \mathfrak{g}/\mathfrak{h}_p = \max\left\{\dim \frac{\mathfrak{z}}{p_\#(\mathfrak{h})}, \operatorname{card} J_\neq\right\}.$$

(viii) *If* $\dim \mathfrak{g}/\mathfrak{h}$ *is finite*, *then the following cardinals are finite*:

$$\operatorname{card} J_\neq, \qquad \dim \mathfrak{z}/p_\#(\mathfrak{h}), \qquad \dim \mathfrak{h}_p/\mathfrak{h}.$$

Proof. Assertions (i)–(vi) are immediate from the definitions. In view of the fact that for $\# \neq j \in J_\neq$, the cardinal $\dim \mathfrak{s}_j/p_j(\mathfrak{h}) \leq \dim \mathfrak{s}_j$ is finite, (vii) follows from (iv) with a simple cardinality calculation; (viii) is straightforward. ∎

Parallel to the Lie algebra situation we have the group situation. Let $G = Z \times \prod_{j\in J} S_j$ with a compact connected abelian group Z and simple simply connected Lie groups S_j, and let H be a closed subgroup of G; then $\mathfrak{h} = \mathfrak{L}(H) = \mathfrak{L}(H_0)$.

We set $S_\# = Z_0(G)$, identify G with $\prod_{j\in J_\#} S_j$, and define $H_p = \prod_{j\in J_\#} p_j(H)$ and $H^\circ = \prod_{j\in J_\circ} S_j$. From [6, p. 460, 9.26, p. 476, 9.47] we conclude that $p_j(H) = S_j$ iff $j \in J_=$. We write $A \trianglelefteq B$ when A is a normal subgroup of B. In analogy with Lemma 1.4 we have

LEMMA 1.5. (i) $H^\circ \trianglelefteq G$.

(ii) *H is a subdirect product in* $\prod_{j\in J_=} p_j(H) = H_p$.

(iii) *The homogeneous spaces* G/H_p *and* $\prod_{j\in J_{\neq}} S_j/p_j(H)$ *are naturally homeomorphic.*

(iv) $\dim \mathfrak{L}(G)/\mathfrak{L}(H_p) = \dim \mathfrak{L}(Z)/\mathfrak{L}(p_\#(H)) + \sum_{j\in J_{\neq}\setminus\{\#\}} \dim S_j/\mathfrak{L}(p_j(H))$.

(v) $\dim \mathfrak{L}(G)/\mathfrak{L}(H) = \dim \mathfrak{L}(G/H^\circ)/\mathfrak{L}(H/H^\circ)$.

(vi) $\dim \mathfrak{L}(G)/\mathfrak{L}(H) = \dim \mathfrak{L}(G)/\mathfrak{L}(H_p) + \dim \mathfrak{L}(H_p)/\mathfrak{L}(H)$.

(vii) *If $J_{\neq}$ is infinite, then*

$$\dim \mathfrak{L}(G)/\mathfrak{L}(H_p) = \max\left\{\dim \frac{\mathfrak{L}(Z)}{\mathfrak{L}(p_\#(H))}, \operatorname{card} J_{\neq}\right\}.$$

The dimension of G/H_p, if infinite, is controlled by the abelian group $Z/p_\#(H)$ and $\operatorname{card} J_{\neq}$, whichever is bigger. We therefore turn to the situation of subdirect products in order to deal with H_p/H and $\mathfrak{h}/\mathfrak{h}_p$.

We let $\mathcal{S}$ denote a set of representatives of the class of all simple centerfree connected compact Lie groups and let $\mathfrak{S}$ denote a set of representatives of the class of all real compact simple Lie algebras.

LEMMA 1.6. (i) *Let* $G = A \times \prod_{S\in\mathcal{S}} S^{J(G,S)}$ *be a product of a compact connected abelian group A with a product of simple, centerfree connected compact groups S, and assume that H is a* connected *compact subdirect product of G. Then* $H = A \times \prod_{S\in\mathcal{S}} H_S$ *with H_S subdirect in* $G_S \stackrel{\text{def}}{=} S^{J(G,S)}$.

(ii) *Let* $\mathfrak{g} = \mathfrak{a} \oplus \prod_{\mathfrak{s}\in\mathfrak{S}} \mathfrak{s}^{J(\mathfrak{g},\mathfrak{s})}$ *be the direct sum of a weakly complete abelian Lie algebra and a product of compact simple real Lie algebras. Assume that $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$ which is a subdirect product. Then* $\mathfrak{h} = \mathfrak{h}_\mathfrak{a} \oplus \prod_{\mathfrak{s}\in\mathfrak{S}} \mathfrak{h}_\mathfrak{s}$, *where $\mathfrak{h}_\mathfrak{s}$ is subdirect in* $\mathfrak{g}_\mathfrak{s} \stackrel{\text{def}}{=} \mathfrak{s}^{J(\mathfrak{g},\mathfrak{s})}$.

Proof. (i) Let p_X, $X \in \{A\} \cup \mathcal{S}$ be the projection. Then p_A maps H' trivially because $H' \subseteq G' = \ker p_A$. Each projection onto one of the simple factors $S \in \mathcal{S}$ kills the center $Z(H)$ of H by [6, 9.28, p. 461]. Hence the center is killed by all p_S. Since the family of the p_X separates the points of H, and since $H = Z(H)H'$ with $Z(H') \subseteq Z(H)$ (see [6, p. 458, 9.24]) we conclude that

$$H = Z(H) \times H' \subseteq Z(G) \times G' = A \times G',$$

and that $Z(H) = A$; furthermore H' is subdirect in $\prod_{S\in\mathcal{S}} S^{J(G,S)} = G'$ and is itself centerfree. Then by [6, p. 450, Theorem 9.19], $H' = \prod_{S\in\mathcal{S}} S^{J(H,S)}$. The projection p_S kills $\prod_{S\neq T\in\mathcal{S}} T^{J(H,T)}$ and maps $S^{J(H,S)}$ faithfully. We thus get the assertion of the lemma with $H_S = S^{J(H,S)}$

(ii) The proof of (ii) either can be deduced from (i), since for each $\mathfrak{g}$ we find G as in (i) with $\mathfrak{L}(G) = \mathfrak{g}$, $\mathfrak{L}(A) = \mathfrak{a}$, $\mathfrak{L}(S) = \mathfrak{s}$, etc., or else can be carried out directly in analogy to that of (i). ∎

The subgroups G_S in Lemma 1.8 are called the *isotypic components* of G (cf. [6, p. 454]).

LEMMA 1.7. (i) *Let S be a connected simple compact centerfree Lie group and let J be an arbitrary set. Let H be a* connected *subdirect product of $G \overset{\text{def}}{=} S^J$.*

Then there is a partial product $N \overset{\text{def}}{=} S^K \leq G$ (in the obvious notation) for a subset K of J such that G is the semidirect product $N \rtimes H$ of the closed normal subgroup N and the given subgroup H, and that $H \cong S^{J \setminus K}$

(ii) *Let $\mathfrak{s}$ be a real simple compact Lie algebra and let J be an arbitrary set. Let $\mathfrak{h}$ be a subdirect product of $\mathfrak{g} \overset{\text{def}}{=} \mathfrak{s}^J$.*

Then there is a partial product $\mathfrak{n} \overset{\text{def}}{=} \mathfrak{s}^K \leq \mathfrak{g}$ (in the obvious notation) for a subset K of J such that $\mathfrak{g}$ is the semidirect product $\mathfrak{n} \rtimes \mathfrak{h}$ of the closed ideal $\mathfrak{n}$ and the given subalgebra $\mathfrak{h}$, and that $\mathfrak{h} \cong \mathfrak{s}^{J \setminus K}$.

Proof. Again we prove (i) and derive (ii) from (i).

We begin by invoking [6, p. 579, E11.5] in order to find (a) a surjective function $\sigma\colon J \to I$ and (b) a function $j \mapsto \alpha_j\colon J \to \operatorname{Aut} S$ such that the morphism

$$\varphi\colon S^I \to G, \qquad \varphi((x_i)_{i \in I}) = (\alpha_j(x_{\sigma(j)}))_{j \in J}$$

induces an isomorphism from S^I onto H.

Now, using the axiom of choice, we pick an arbitrary cross-section $\tau\colon I \to J$ for σ and define

$$N \overset{\text{def}}{=} \prod_{j \in J} S_j, \qquad S_j = \begin{cases} \{1\} & \text{if } (\exists i \in I)\; j = \tau(i), \\ S & \text{otherwise.} \end{cases}$$

We claim that this N, depending on our choice of τ, satisfies the requirements. First, N is a partial product and thus is a closed normal subgroup. We set $K = J \setminus \tau(I)$. Then I and $J \setminus K$ have the same cardinality, and $N = S^K$, $H \cong S^I \cong S^{J \setminus K}$.

Second, we claim $N \cap H = \{1\}$. Let $g = (s_j)_{j \in J} \in N \cap H$. Then, since $g \in H$ there is a unique $(x_i)_{i \in I}$ such that $y_j = \alpha_j(x_{\sigma(j)})$ for all $j \in J$. Since $g \in N$, for each $i \in I$ we have $y_{\tau(i)} = 1$. Hence $x_i = x_{\sigma(\tau(i))} = 1$ for all i and thus $g = 1$

Third, we must show that $NS = G$, equivalently, that $NS/N = G/N$. Let $g = (y_j)_{j \in J}$ be given. We must find an element $(x_i)_{i \in I} \in S^I$ such that $gN = (\alpha_j(x_{\sigma(j)}))_{j \in J} N$; that is, $(\alpha_j(x_{\sigma(j)}) y_j^{-1})_{j \in J} \in N$. By the definition of N this means precisely that for each $i \in I$ we have $\alpha_{\tau(i)}(x_{\sigma(\tau(i))}) = y_{\tau(i)}$. Thus if for a given $i \in I$ we set $x_i = \alpha_{\tau(i)}^{-1}(y_{\tau(i)})$, the element $(x_i)_{i \in i}$ satisfies the requirements. ∎

LEMMA 1.8. *Assume that the compact group G is a semidirect product $N \rtimes_\alpha B$ (for a morphism α: $B \to \operatorname{Aut} N$) and that H is a closed subgroup containing $\{1\} \times B$. Then H is the semidirect product $M \rtimes \alpha_M B$ with a subgroup M of N such that $\alpha_M(b)(m) \overset{\text{def}}{=} \alpha(b)(m) \in M$ for all $b \in B$, $m \in M$.*

If in addition $\{1\} \times B$ is the identity component H_0 of H, then H and $M \times B$ are isomorphic as topological groups.

Proof. We set $M = \{n \in N : (\exists b \in B)(n,b) \in H\}$. Since $(n,b) = (n,1)(1,b)$, then (n,b), $(1,b) \in H$ implies $(n,1) \in H$. We know $M \times \{1\} \subseteq H$ and thus $M \times B = (M \times \{1\})(\{1\} \times B) \subseteq H$. Conversely, let $(n,b) \in H$. Then $n \in M$ and $(1,b) \in H$, whence $(n,b) = (n,1)(1,b) \in M \times B$. Thus $H = M \times B$ and $M \times \{1\} = H \cap (N \times \{1\})$ is a subgroup of G, whence M is a subgroup of N. If $m \in M$ and $b \in B$, then $(\alpha(b)(m), 1) = (1,b)(m,1)(1,b)^{-1} \in H \cap (N \times \{1\}) = M \times \{1\}$, and thus $\alpha(b)(m) \in M$. The continuous morphism α: $B \to \operatorname{Aut} N$ induces a continuous morphism α_M: $B \to \operatorname{Aut} M$ given by $\alpha_M(b)(m) = \alpha(b)(m)$, and H is the semidirect product $M \rtimes \alpha_M B$.

Now assume that $\{1\} \times B = H_0$. Then M is homeomorphic to $\frac{M \times B}{\{1\} \times B} \cong H/H_0$ under the map $m \mapsto \{m\} \times H = (m,1)(\{1\} \times B)$: $M \to H/H_0$. Thus, as H/H_0 is totally disconnected, M is totally disconnected and thus $\operatorname{Aut} M$ is totally disconnected (see [6, p. 505, Theorem 9.82]; the statement of that theorem erroneously included the word "Lie"). Since $B \cong H_0$ is connected, the morphism α_M: $B \to \operatorname{Aut} M$ is constant, and thus the the semidirect product $M \rtimes_{\alpha_M} B$ is direct. ∎

From the two preceding lemmas we get the following one:

LEMMA 1.9. *Let S be a connected simple compact centerfree Lie group and let J be an arbitrary set. Let H be a subdirect product of $G \overset{\text{def}}{=} S^J$.*

Then there is a partial product $N \overset{\text{def}}{=} S^K \leq G$ (in the obvious notation) for a subset K of J such that G is the semidirect product $N \rtimes H_0$ of the closed normal subgroup N and the identity component H_0 of the given subgroup H, and that $H \cong S^{J \setminus K}$. Moreover, N contains a totally disconnected closed subgroup M such that $H = M \times H_0$.

Proof. This is a direct consequence of Lemmas 1.7 and 1.8. ∎

PROPOSITION 1.10. *Let $\mathfrak{g} = \prod_{j \in J_\#} \mathfrak{s}_j$ as in* Definition 1.3 *and let $\mathfrak{h}$ be a closed subalgebra such that* $\dim \mathfrak{g}/\mathfrak{h} < \infty$. *Then* $\dim \mathfrak{g}/\mathfrak{h}^\circ < \infty$ *and thus the vector space $\mathfrak{g}/\mathfrak{h}$ is isomorphic to the quotient vector space space of the two finite dimensional Lie algebras $\mathfrak{g}/\mathfrak{h}^\circ$ by $\mathfrak{h}/\mathfrak{h}^\circ$.*

Proof. The last assertion is a consequence of the first by 1.4(iv). In view of $\mathfrak{h}^\circ \trianglelefteq \mathfrak{h} \subseteq \mathfrak{h}^p \subseteq \mathfrak{g}$ it suffices to show that $\dim \mathfrak{h}_p/\mathfrak{h} < \infty$ implies $\dim \mathfrak{h}/\mathfrak{h}^\circ < \infty$. Therefore, by 1.4(ii) we assume now that $\mathfrak{h}$ is a subdirect product in $\mathfrak{g}$ and by 1.6(ii) we assume $\mathfrak{g} = \mathfrak{a} \oplus \prod_{\mathfrak{s} \in \mathcal{S}} \mathfrak{s}^{J(\mathfrak{g},\mathfrak{s})}$ and $\mathfrak{h} = \mathfrak{a} \oplus \prod_{\mathfrak{s} \in \mathcal{S}} \mathfrak{h}_{\mathfrak{s}}$, where $\mathfrak{h}_s$ is subdirect in $\mathfrak{s}^{J(\mathfrak{g},\,\mathfrak{s})}$. Since $\dim \mathfrak{g}/\mathfrak{h}_s = \sum_{\mathfrak{s}\in\mathcal{S}} \dim \mathfrak{s}^{J(\mathfrak{g},\,\mathfrak{s})}/\mathfrak{h}_{\mathfrak{s}} < \infty$ we conclude that $\mathfrak{h}_{\mathfrak{s}} = \mathfrak{s}^{J(\mathfrak{g},\,\mathfrak{s})}$ for all but finitely many $\mathfrak{s}$. So we finally assume that $g = \mathfrak{s}^J$ and $\mathfrak{h}$ is a subdirect subalgebra. Then $\mathfrak{h}^\circ = \mathfrak{s}^{J_\circ}$. Now we invoke Lemma 1.7(ii) and first conclude from $\dim \mathfrak{g}/\mathfrak{h} < \infty$ that K is finite. Hence the dimensions of $\mathfrak{n}$ and thus of its derivation algebra $\operatorname{Der}\mathfrak{n}$ (cf. [6, p. 122]) are finite. Hence the representation $\operatorname{ad}|\mathfrak{h} : \mathfrak{h} \to \operatorname{Der}\mathfrak{n}$ has a finite dimensional image. Hence its kernel is cofinite dimensional in $\mathfrak{g}$. This kernel is an ideal of $\mathfrak{h}$ and, since it commutes elementwise with $\mathfrak{n}$, also an ideal of $\mathfrak{g}$; it is therefore a partial product of $\mathfrak{g}^J$ and thus belongs to $\mathfrak{h}^\circ$. This completes the proof. ∎

For the following Theorem, recall from [6, p. 483ff] the concept of dimension. In [6] we have defined the dimension of a compact group G to be $\dim \mathfrak{L}(G)$ (see [6, Definition 9.53(ii), p. 483]), and we have shown that this number, if finite, agrees with the topological dimension of G with respect to any of the common concepts of dimension (see [6, 9.54, p. 483]).

THEOREM 1.11. *Let G be a compact group and let H be a closed subgroup such that* $\dim \mathfrak{L}(G)/\mathfrak{L}(H) < \infty$. *Let $N = \bigcap\{gHg^{-1} : g \in G\}$ be the largest compact normal subgroup of G contained in H. Then* $\dim \mathfrak{L}(G)/\mathfrak{L}(N) < \infty$ *and $G/H \cong \frac{G}{N}\big/\frac{H}{N}$ with both G/N and H/N finite dimensional compact groups. In particular, $G_0N/N \cong G_0/(G_0 \cap N)$ is metric.*

Proof. Set $\mathfrak{n} \stackrel{\text{def}}{=} \mathfrak{L}(N) = \mathfrak{L}(N_0)$. Then $\mathfrak{n}$ is an ideal of $\mathfrak{g} = \mathfrak{L}(G)$ (cf. [6, p. 476, 9.47]) which is contained in $\mathfrak{h} = \mathfrak{L}(H)$. Let $\mathcal{N}$ denote the filter basis of compact normal subgroups M of G such that G/M is a Lie group. Then $\{\mathfrak{L}(M) : M \in \mathcal{N}\}$ is a filterbasis of ideals of $\mathfrak{g} = \mathfrak{L}(G)$ and since $\mathfrak{L}(\cdot)$ preserves limits, $\lim_{M\in\mathcal{N}} \mathfrak{g}/\mathfrak{L}(M) = \mathfrak{g}$ as weakly complete Lie algebras. Hence $\lim\{\mathfrak{L}(M) : M \in \mathcal{N}\} = 0$ in $\mathfrak{g}$. Recall $\mathfrak{h} = \mathfrak{L}(H)$. Since $\dim \mathfrak{g}/\mathfrak{h} < \infty$, there is an open identity neighborhood U in $\mathfrak{g}$ such that $U + \mathfrak{h} = U$ and $U/\mathfrak{h}$ has no nonsingleton vector subspaces. Since the filter basis of the $\mathfrak{L}(M)$ converges to 0, there is an $M \in \mathcal{N}$ such that $\mathfrak{L}(M) \subseteq U$. Then $\mathfrak{L}(M) \subseteq \mathfrak{h}$ and thus $M_0 = \overline{\exp \mathfrak{L}(M)} \subseteq \overline{\exp \mathfrak{h}} = H_0 \subseteq H$. Since $gM_0g^{-1} = M_0$ we know that $M_0 \subseteq N$, and consequently $\mathfrak{L}(M) = \mathfrak{L}(M_0) \subseteq \mathfrak{L}(N)$. Since G/M is a Lie group, $\dim \mathfrak{g}/\mathfrak{L}(M) < \infty$ and therefore $\dim \mathfrak{g}/\mathfrak{L}(N) < \infty$.

Now

$$\mathfrak{L}(G)/\mathfrak{L}(H) \cong \frac{\mathfrak{L}(G)}{\mathfrak{L}(N)}\Big/\frac{\mathfrak{L}(H)}{\mathfrak{L}(N)} \cong \mathfrak{L}(G/N)/\mathfrak{L}(H/N)$$

in view of [6, p. 476, 9.47] again. By the remark preceding the theorem, both G/N and H/N are finite dimensional.

For the fact that every finite dimensional compact connected group is metric see [6, p. 482, 9.52(vi)]. ∎

Note that any nonmetrizable profinite group G and $H = \{1\}$ shows that G/H need not be metrizable if $\dim \mathfrak{L}(G)/\mathfrak{L}(H) = 0 < \infty$. However, 1.11 shows that all connected components of G/H are metrizable if $\dim \mathfrak{L}(G)/\mathfrak{L}(H) < \infty$.

Theorem 1.11 is to be compared with [11, p. 239, Theorem 6.2.2 ff]. The ascent from compact G to locally compact G is rather immaterial. The claim that G need not be connected, however, is relevant. In [11] it is postulated that $G = \lim_{M\in\mathcal{N}} G/M$ (plus the additional hypothesis that $\mathcal{N}$ has a countable basis).

2. THE WEIGHT OF A HOMOGENEOUS SPACE

The goal of this section is a proof of the fact that for a compact *connected* group G and any closed proper subgroup H we have

$$w(G/H) = \max\{\aleph_0, \dim \mathfrak{L}(G)/\mathfrak{L}(H)\}. \qquad (*)$$

(See Theorem 2.6 below.) Therefore, whenever the weight of G/H is uncountable, $\dim \mathfrak{L}(G)/\mathfrak{L}(H) = w(G/H)$, and thus this cardinal depends only on the underlying topological space of G/H—given, of course, the knowledge that the space in question is the homogeneous space of a compact connected group. If G is an arbitrary compact group and H is a proper subgroup such that G/H is connected, then the natural homeomorphism $g(G_0 \cap H) \mapsto gH\colon G_0/(G_0 \cap H) \to G/H$ reduces the issue to $(*)$.

LEMMA 2.1. *Let X be the projective limit of an infinite inverse system*

$$\{f_{jk}\colon X_k \to X_j \mid (j,k) \in J \times J,\ j \le k\}$$

of compact manifolds X_j. Then $w(X) \le \operatorname{card} J$.

Proof. Let $\mathcal{B}_j$ be a countable basis of the topology of X_j and let $f_j\colon X \to X_j$ denote the limit maps. Then $\mathcal{B} \overset{\text{def}}{=} \{f_j^{-1}(U) : U \in \mathcal{B}_j,\ j \in J\}$ is a basis for the topology of X. Since $\operatorname{card} \mathcal{B}_j \le \aleph_0$ and J is infinite, $w(X) \le \operatorname{card} \mathcal{B} \le \operatorname{card} J$. ∎

LEMMA 2.2. *Let G be a compact Lie group. Then the set of closed normal subgroups of G is countable.*

Proof. If G is abelian, then the lattice of closed subgroups is antiisomorphic to the set of subgroups of the character group $\widehat{G}$ (cf. [6, p. 353, 7.64(v)]). Since $\widehat{G}$ is of the form $\mathbb{Z}^n \oplus E$ with a finite group ([6, p. 50, 2.42]), the assertion is true in this case. Since G_0 has finite index in G, we may assume that G is connected. If $N = \overline{N} \trianglelefteq G$, then N_0, the identity component, being characteristic in G, is normal in G. Now N/N_0 is a finite normal subgroup of the connected group G/N_0; it is therefore central (cf. [6, p. 777, A4.27]). The center of G is a compact abelian Lie group, and there are therefore at most countably many finite subgroups by the first part of the proof. Now let N be a connected compact subgroup of G. Then $\mathfrak{L}(N) = \mathfrak{a} \oplus \mathfrak{s}$, where $\mathfrak{a}$ is a vector subspace of $\mathfrak{L}(Z(G))$ such that $\exp \mathfrak{a}$ is a closed subgroup of the center, and where $\mathfrak{s}$ an ideal of the finite direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ of simple real compact Lie algebras $\mathfrak{s}_m$ (cf. [6, p. 479, 9.50, p. 190, 6.4(vii)]). There are only countably many possibilities for $\mathfrak{a}$ and finitely many possibilities for $\mathfrak{s}$. ∎

We note that 2.2 fails in the noncompact case, for instance even for the abelian Lie group $\mathbb{R}^2$; the set of closed connected subgroups consists of $\{0\}$, $\mathbb{R}^2$, and all one-dimensional real vector subspaces of $\mathbb{R}^2$ and thus has the cardinality of the continuum $2^{\aleph_0}$. If the restriction of the normality of the subgroups were removed, then 2.2 would break down at once as the example of the group SO(3) shows in which there are uncountably many circle groups, since every one-dimensional subspace of $\mathfrak{so}(3)$ generates a circle group.

PROPOSITION 2.3. *Let G be a compact group and let H be a closed subgroup. Let $\mathcal{N}$ denote the filterbasis of all closed normal subgroups N such that G/N is a Lie group. Let $\mathcal{F}$ denote the filterbasis $\{NH : N \in \mathcal{N}\}$. If G/H is not a manifold, then $w(G/H) = \operatorname{card} \mathcal{F}$.*

Proof. The homogeneous space G/H is the projective limit of the manifolds $G/NH \cong \frac{G}{N}\big/\frac{NH}{N}$, $N \in \mathcal{N}$ with respect to the obvious quotient maps. Since G/H is not a manifold, the inverse system is infinite. Then from 2.1 it follows that $w(G/H) \leq \operatorname{card} \mathcal{F}$.

We have to prove the reverse inequality. Now let $\mathcal{U}$ be an open neighborhood filter of $H \in G/H$ of cardinality $\leq w(G/H)$. Since $\bigcap \mathcal{F} = H$ and since G/H is compact we know that $\lim_{K\in\mathcal{F}} K/H = H$ in G/H. For $K \in \mathcal{F}$ let $p_K \mapsto G/H \to G/K$ be the quotient map. Then for each $U \in \mathcal{U}$ we select a $K(U) \in \mathcal{F}$ such that we claim that $\{K(U) : U \in \mathcal{U}\}$ is cofinal in $\mathcal{F}$. That is, for each $L \in \mathcal{F}$ there is a $U \in \mathcal{U}$ such that $K(U) \subseteq L$. There is an $N \in \mathcal{N}$ such that $L = NH$. Since G/N is a compact connected Lie group, there is a $V \in \mathcal{U}$ such that every subalgebra A of G containing NH such that $A/N \subseteq p_L(V)$ agrees with L/N (for it is contained in a conjugate of L/N [11], and contains L/N, and G/N is a Lie group). Now let $U \in \mathcal{U}$ be such that $U \subseteq V$. Then $p_L(K(U)) \subseteq p_L(V)$ and thus $K(U)N = L$; i.e., $K(U) \subseteq L$, as asserted. Next we claim that for each $L \in \mathcal{F}$, the set $\{K \in \mathcal{F} : L \subseteq K\}$ is countable. Indeed let $L = NH$ with $N \in \mathcal{N}$. Then the set in question is in bijective correspondence with the set of normal subgroups $M \in \mathcal{N}$ containing N such that $NH/N \subseteq MH$. Since by 2.2 there are countably many closed normal subgroups M containing N (giving closed normal subgroups M/N of G/N), the claim follows. Now $\mathscr{F}$ contains the cofinal subset $\mathscr{S} \overset{\text{def}}{=} \operatorname{im} K(\cdot)$ and for each $L \in \mathscr{S}$ the set of predecessors is countable. Hence $\operatorname{card} \mathscr{F} \leq \aleph_0 \cdot \operatorname{card} \mathscr{S} \leq \aleph_0 \cdot \operatorname{card} \mathscr{U} \leq \aleph_0 \cdot w(G/H) = w(G/H)$. This completes the proof of the proposition. ∎

COROLLARY 2.4. (i) *Let G be a compact group, let H be a closed subgroup, and let $\mathscr{N}$ denote the filter basis of normal subgroups N such that G/N is a Lie group. Assume that D is a normal subgroup of G such that*

$$\operatorname{card}\{NDH : N \in \mathscr{N}\} = \operatorname{card}\{NH : N \in \mathscr{N}\}.$$

Then $w(G/H) = w(G/DH)$.

(ii) *If G is a compact connected group and H is a closed subgroup, and if D is a totally disconnected normal, hence central, subgroup of G then $w(G/H) = w(G/HD)$.*

Proof. (i) is an immediate consequence of Proposition 1.12

(ii) We claim that the function $F \mapsto FD$: $\{NH : N \in \mathscr{N}\} \to \{NHD : N \in \mathscr{N}\}$ is injective; since it is clearly surjective, it is bijective, and the assertion will follow from (i) above. Thus let $N_1H \neq N_2H$, say $n_1h \notin N_2H$; that is, $n_1 \notin N_2H$. Then $n_1N_2 \notin N_2H/N_2$, and since N_2HD/N_2 is finite and G/N_2 is infinite (by moving n_1N_2) we find an $n_1' \in N_1$ such that $n_1' \notin N_2HD$. Then N_1DH is not contained in N_2DH, i.e., $N_1HD \neq N_2HD$, and this proves the claim. ∎

LEMMA 2.5. *In the notation of* Lemma 1.5, *the following statements hold*:

(i) *If $G \neq H_p$, then* $w(G/H_p) = \max\{\aleph_0, \dim \mathfrak{L}(Z)/\mathfrak{L}(p_\#(H)), \operatorname{card} J_{\neq}\}$,

(ii) $w(G/H) = w(G/H_p) + w(H/H_p)$.

Proof. (i) is simply the calculation of the weight of a product $\prod_{j \in J_{\neq}} S_j/p_j(H)$ in view of [6, p. 764, EA4.3].

(ii) The group H acts on the space G/H_p with stable isotropy H_p and G/H as orbit space (see [6, p. 519, Definition 10.5]). The assertion then follows from [7, 4.9]. ∎

For a compact group Γ its dimension is defined to be $\dim \mathfrak{L}(\Gamma)$ (see [6, p. 483, Definition 9.53(ii)]).

LEMMA 2.6. *If H is a closed normal subgroup of a compact group G, then G/H is a group and*

$$\dim G/H = \dim \mathfrak{L}(G/H) = \dim \mathfrak{L}(G)/\mathfrak{L}(H).$$

Proof. If H is normal, then $\mathfrak{L}(H)$ is an ideal, and the weakly complete factor Lie algebra $\mathfrak{L}(G)/\mathfrak{L}(H)$ and the Lie algebra $\mathfrak{L}(G/H)$ are naturally isomorphic by [6, p. 467, 9.47]. ∎

If, in Lemma 2.6, the group G is abelian, this applies to all closed subgroups.

We now prove the main theorem of this section as announced in $(*)$

THEOREM 2.7. *Let G be a compact connected group and let H be a proper closed subgroup. Then*

$$w(G/H) = \max\{\aleph_0, \dim \mathfrak{L}(G)/\mathfrak{L}(H)\}. \qquad (*)$$

Proof. (a) If H is normal, the claim is a consequence of [6, p. 607, 12.25]. In particular, the assertion is clearly true if G is abelian.

(b) By Lemma 1.2. we may assume $G = A \times \widetilde{P}$, $\widetilde{P} = \prod_{S\in\mathcal{S}} \widetilde{S}^{J(G,S)}$ with a compact connected abelian group A and the universal covering group $\widetilde{S}$ of S. Accordingly, $\mathfrak{g} = \mathfrak{a} \oplus \prod_{\mathfrak{s}\in\mathfrak{S}} \mathfrak{s}^{J(\mathfrak{g},\mathfrak{s})}$, where $J(\mathfrak{g},\mathfrak{s}) = J(G,S)$.

(c) Let $D = \prod_{S\in\mathcal{S}} Z(\widetilde{S})^{J(G,S)}$. Then $\{1\} \times D$ is a totally disconnected central subgroup of G and thus

$$w(G/H) = w(G/HD) = w\left(\frac{G}{D}\bigg/\frac{HD}{D}\right).$$

Because $\mathfrak{L}(HD) = \mathfrak{L}(H)$ we have

$$\dim \mathfrak{L}(G)/\mathfrak{L}(H) = \dim \mathfrak{L}(G)/\mathfrak{L}(HD) = \dim \mathfrak{L}\left(\frac{G}{D}\right)\bigg/\mathfrak{L}\left(\frac{HD}{D}\right).$$

We may therefore assume that $G = A \times P$, $P = \prod_{S\in\mathcal{S}} \widetilde{S}^{J(G,S)}$.

(d) Assume $H = H_p$; that is, $H = \prod_{j\in J_\#} H_j$, $H_j \overset{\text{def}}{=} p_j(H)$. Then by 1.5(iii) we know

$$G/H = \prod_{j\in J_\#} S_j/Hj = A/Z(H) \times \prod_{j\in J} S_j/H_j.$$

(Here we have identified $H_\# \cong Z(H)$ with $Z(H)$.) Then in view of [6, p. 607, 12.25]

$$\begin{aligned} w(G/H) &= w(A/Z(H)) + \max\{\aleph_0, \operatorname{card} J\} \\ &= \max\{\aleph_0, \dim A/Z(H), \operatorname{card} J\}. \end{aligned}$$

We note that by 1.5(iv) and 2.6 we have

$$\dim \mathfrak{L}(G)/\mathfrak{L}(H) = \dim A/Z(H) + \sum_{j\in J} \dim \mathfrak{L}(S_j)/\mathfrak{L}(H_j),$$

and thus

$$\max\{\aleph_0, \dim \mathfrak{L}(G)/\mathfrak{L}(H)\} = \max\{\aleph_0, \dim A/Z(H), \operatorname{card} J\}.$$

Thus the theorem holds in the present case.

(e) By 1.5(v) and 2.5(ii) the assertion of the theorem holds for the pair (G, H) if it holds for both of the pairs (G/H_p) and (H_p/H). Since it holds for (G, H_p) by (d) above, we must now establish it in the case that H is a subdirect product of $\prod_{j\in J_\#} S_j$.

Then Lemma 1.6(i) permits us to assume that G is either abelian or is of the form S^J with a simple centerfree compact connected Lie group. Since for abelian G the Theorem is true, we are left with the latter case. This allows us to apply Lemma 1.9. Then $G = S^K \rtimes H_0$ and $H = M \times H_0$ with a totally disconnected subgroup M of S^K. Then $G/H \cong S^K/M$ as homogeneous spaces. Thus $w(G/H) = w(S^K/M)$. Now let $\mathcal{N}$ be the set of cofinite partial products of S^K. Then $N \mapsto NM\colon \mathcal{N} \to \{NM\colon \ \in \mathcal{N}\}$ is a bijection by an argument emulating that of the proof of 2.4(ii). Hence by 2.4(i) we have $w(S^K) = w(S^K/M)$ and if G/H is not singleton, then $w(S^K) = \max\{\aleph_0, \operatorname{card} K\}$. Also $\dim S^K/M = \dim \mathfrak{L}(S^K)/\mathfrak{L}(M) = \dim \mathfrak{L}(S^K) = \dim \mathfrak{s}^K = (\operatorname{card} K)\cdot \dim \mathfrak{s}$ since M is totally disconnected (cf. [6, p. 477, 9.48]). Hence $\max\{\aleph_0, \dim \mathfrak{L}(S^M)/\mathfrak{L}(M)\} = \max\{\aleph_0, \operatorname{card} K\}$. Therefore $w(G/H) = \max\{\aleph_0, \operatorname{card} K\}$. But by Lemma 1.9, in the category of weakly complete Lie algebras, $\mathfrak{h}$ is a retract of $\mathfrak{g}$; that is, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as weakly complete vector spaces, where $\mathfrak{n} = \mathfrak{s}^K$. Hence $\dim \mathfrak{L}(G)/\mathfrak{L}(H) \cong \mathfrak{n} = \mathfrak{s}^K = (\operatorname{card} K) \times \dim \mathfrak{s}$. Therefore the theorem holds in this case and the proof is complete. ∎

3. THE DIMENSION OF A HOMOGENEOUS SPACE

We need the following lemma in various parts of this article.

LEMMA 3.1. *Let G be a compact group acting on a compact space X with stable isotropy, and let $p\colon X \to X/G$ be the orbit projection.*

(i) *If C is a contractible subspace of the orbit space X/G, and $p(x_0) \in C$, then the arc component of x_0 in X contains a subspace C' such that p maps C' homeomorphically onto C.*

(ii) *If C is a neighborhood of $p(x_0)$, then C' is a local cross-section. If the action is free, then $(g, c') \mapsto g{\cdot}c'\colon G \times C' \to G{\cdot}C'$ is a homeomorphism onto a neighborhood of $p^{-1}(p(x_0))$.*

(iii) *Conclusions* (i) *and* (ii) *apply, in particular, to any C which is homeomorphic to a cube $\mathbb{I}^A$, $\mathbb{I} = [0, 1]$.*

Proof. If C is a compact contractible subspace of X/G (e.g., a cube), then we consider the space $X' = p^{-1}(C)$ and the action of X' on X which has stable isotropy. The space C is the orbit space X'/G. The orbit map $p'\colon X' \to X'/G = C$ then has a cross section $s\colon X'/G \to X'$ by Theorem 10.42. Since G acts transitively on the orbit $G{\cdot}x_0$, we may assume $s(p(x_0)) = x_0$. Then $C' \stackrel{\text{def}}{=} s(C)$ satisfies the requirement. ∎

More general sufficient conditions in statement (ii) are available in [6, p. 539ff].

DEFINITION 3.2. We say that a function DIM on the class of locally compact Hausdorff spaces with values in $\{0, 1, \ldots\} \cup \{\infty\}$ is an *admissible topological dimension* function if it satisfies the [6, axioms (Da), (Db), (Db*), (Dc), (Dd), p. 385, 8.25, p. 483, 9.54].

In [6, p. 386ff]. it is shown that the standard topological dimensions, the small inductive dimension, local large inductive dimension, Lebesque covering dimension, cohomological dimension, sheaf theoretical dimension, are admissible topological dimension functions.

If DIM is an admissible topological dimension, then the series of conclusions in 3.1 can be complemented by the following one

(iv) If C is as in 3.1(ii), (iii) and DIM denotes an admissible topological dimension, then DIM $X = \text{DIM } G + \text{DIM } \mathbb{I}^A$.

Lemma 3.1 applies, in particular, in the following situation.

PROPOSITION 3.3. *Let G be a compact connected group and let H be a closed subgroup such that* $\dim \mathfrak{L}(G)/\mathfrak{L}(H) < \infty$. *Then the following conclusions hold*:

(i) *If the largest normal subgroup $\bigcap_{g\in G} gHg^{-1}$ of G contained in H is trivial, then there is a totally disconnected central subgroup D of G such that G/D is a Lie group and that the base point H of G/H has a neighborhood homeomorphic to $D \times \mathfrak{L}(G)/\mathfrak{L}(H)$.*

(ii) *Let* DIM *denote any admissible topological dimension function on the class of locally compact spaces. Then*

$$\mathrm{DIM}(G/H) = \dim \mathfrak{L}(G)/\mathfrak{L}(H) \tag{7}$$

Proof. Assume the hypothesis of (i). Then by Theorem 1.11 we know that $\dim \mathfrak{L}(G)$ is finite. Then by [6, the structure theorem of finite dimensional compact groups, p. 481] the group D exists as asserted. Since G/D is a Lie group, and HD/D is a closed subgroup, $G/HD \cong \frac{G}{D}\Big/\frac{HD}{D}$ is a manifold of dimension

$$\dim \mathfrak{L}(G/D)/\mathfrak{L}(HD/D) \stackrel{\text{def}}{=} m.$$

Thus there is a neighborhood C of $HD \in G/HD$ which is homeomorphic to $\mathbb{I}^m$. Now the group $D/D \cap H$ acts on G/H via $(d(D \cap H), gH) \mapsto dgH\colon \frac{D}{D\cap H} \times G/H \to G/H$. The isotropy group at H is $\{d(D \cap H) : d \in D : dH = H\} = D \cap H$; therefore the action is free. The orbit of gH is $DgH = gDH$; thus the orbit space is $G/DH \cong \frac{G}{D}\Big/\frac{HD}{D}$. Let C be a cube neighborhood of $DH \in G/DH$ of dimension $\mathfrak{L}(G/D)/\mathfrak{L}(HD/D) \cong \mathfrak{L}(G)/\mathfrak{L}(H)$.

Now from 3.1 we know that $H \in G/H$ has a neighborhood homeomorphic to $D/(D \cap H) \times C$. The interior of C is homeomorphic to $\mathfrak{L}(G)/\mathfrak{L}(H)$, and thus (i) is proved.

(ii) Let $N = \bigcap_{g \in G} gHg^{-1}$. If DIM is any admissible topological dimension function we apply (i) to $\frac{G}{N}\Big/\frac{H}{N}$. Since this homogeneous space is naturally homeomorphic to G/H and

$$\mathfrak{L}(G)/\mathfrak{L}(H) \cong \mathfrak{L}(G/N)/\mathfrak{L}(H/N),$$

conclusion (ii) follows from (i) and [6, axioms (Db), (Dc), (Dd)] for an admissible topological dimension function. ∎

Before we formulate the following definition, we note the following facts: The cube $\mathbb{I}^{\aleph_0}$ of dimension $\aleph_0$ is homogeneous in the sense that its homeomorphism group acts transitively. (See [1, p. 104, Theorem 4.1; 4; 10].) On the other hand, if $X = G/H$ for a compact group G and a closed subgroup H and if X is cohomologically acyclic over $\mathbb{Q}$ and over $\mathbb{Z}/2\mathbb{Z}$, then X is singleton. (See [8, p. 310, 4.3.]) Therefore, no compact group can act transitively on the cube of dimension $\aleph_0$ (also called the *Hilbert cube*). In any homogeneous space, all connected components of X are translates of a fixed one, say X_0.

DEFINITION 3.4. (i) A compact Hausdorff space X is said to be a *homogeneous space of a compact group* if there is a compact group acting transitively on X.

(ii) For such a homogeneous space X we let cdim denote the Lebesgue covering dimension (see [3, p. 52]), fix an arbitrary connected component X_0, and set

$$\dim X = \begin{cases} \operatorname{cdim} X & \text{if } \operatorname{cdim} X < \infty, \\ w(X_0) & \text{if } \operatorname{cdim} X = \infty. \end{cases}$$

∎

Obviously, $\dim X$ is a topological invariant. If X is a homogeneous space of a compact group, by 3.3, any admissible topological dimension funcion on the class of locally compact spaces in place of cdim yields the same function dim.

As before we shall write $\mathbb{I} \stackrel{\text{def}}{=} [0, 1]$. If $\aleph$ is any (finite or infinite) cardinal we say that a space C homeomorphic to $\mathbb{I}^{\aleph}$ is a *cube of dimension* $\aleph$ and write $\dim C = \aleph$.

PROPOSITION 3.5. *If a homogeneous space X of a compact group contains a cube of a dimension $\aleph$, then $\aleph \leq \dim X$.*

Proof. Let $C \subseteq X$ be a cube of dimension $\aleph$.

(i) Case $\operatorname{cdim} X < \infty$. By [6, axiom (Dc) p. 385] we know $\operatorname{cdim} C \leq \operatorname{cdim} X$. Since $\operatorname{cdim} C = \aleph$ by [6, Axiom (Db)], we have $\aleph \leq \operatorname{cdim} X = \dim X$.

(ii) $\operatorname{cdim} X = \infty$. Then $w(C) \leq w(X)$ by the monotonicity of the weight. Now $\aleph \leq \max\{\aleph_0, \aleph\} = w(C)$. Since X is infinite dimensional, by the definition of $\dim X$ we have $w(X) = \dim X$. Thus $\aleph \leq \dim X$. ∎

THEOREM 3.6. *Assume that G is a compact group and H is a closed subgroup. Then G contains a cube of dimension* $\dim \mathfrak{L}(G)/\mathfrak{L}(H)$ *which is mapped homeomorphically under the quotient map $G \to G/H$.*

Proof. (i) First reduction: Any cube containing the identity will have to be in G_0. After replacing G by G_0, and H by $G_0 \cap H$, since $G_0H/H \cong G_0/(G_0 \cap H)$ and $\mathfrak{L}(G) = \mathfrak{L}(G_0)$ as well as $\mathfrak{L}(H) = \mathfrak{L}(H_0) = \mathfrak{L}(G_0 \cap H)$ we may assume that G is connected.

(ii) Second reduction: There is a central totally disconnected subgroup D of G such that $G/D \cong \mathbb{T}^{J(G/D,\mathbb{T})} \times \prod_S S^{J(G/D,S)}$ where the product ranges through a set of representatives of all simple centerfree compact connected Lie groups S and where the exponents $J(G/D, \mathbb{T})$ and $J(G/D, S)$ are suitable cardinals. (See [6, p. 376, 8.15, p. 459, 9.25].) Then $(G/H)/(HD/H) \cong G/HD \cong (G/D)/(HD/D)$ and $G/H \to (G/H)/(HD/H)$ is the orbit map of the action of the compact group HD/H on G/H. Note that $\mathfrak{L}(G) \cong \mathfrak{L}(G/D)$ and $\mathfrak{L}(HD/D) \cong \mathfrak{L}(H/(H \cap D)) \cong \mathfrak{L}(H)$ (see [6, p. 476, Proposition 9.47]). Lemma 3.1 then shows that if $(G/D)/(HD/H)$ contains a cube of dimension $\mathfrak{L}(G/D)/\mathfrak{L}(HD/H) \cong \mathfrak{L}(G)/\mathfrak{L}(H)$, then G/H contains a homeomorphic copy of this cube. We may therefore assume that $G = \prod_{j\in J} S_j$ where S_j is either a circle group or a simple centerfree compact connected Lie group. If $I \subseteq J$ and $j \in J$, we shall identify $\prod_{j\in I} S_j$, respectively, S_j in a natural fashion with a subgroup of G.

(iii) Third reduction: Let $p_j\colon G \to S_j$ denote the projection and set $G_j = p_j(G)$. Then H is a subdirect product of $\Gamma \stackrel{\text{def}}{=} \prod_{j\in J} G_j$. The group G/Γ is isomorphic to $\prod_{j\in I} S_j/G_j$. Set $I = \{j \in J : G_j \neq S_j\}$. Since S_j/G_j for $j \in I$ is a nondegenerate manifold of dimension $d_j \stackrel{\text{def}}{=} \mathfrak{L}(S_j)/\mathfrak{L}(G_j)$ and thus contains a cube of dimension d_j, we have in G/Γ a subspace homeomorphic to $\prod_{i\in I} \mathbb{I}^{d_j}$, i.e., a cube of dimension $\dim \mathfrak{L}(G)/\mathfrak{L}(\Gamma)$, and thus by Lemma 3 we find a cube C_1 of this dimension in G which is mapped faithfully into G/Γ. If we find a cube C_2 in Γ of dimension $\dim \mathfrak{L}(\Gamma)/\mathfrak{L}(H)$ being mapped faithfully into Γ/H, then $C_1C_2 \cong C_1 \times C_2$ is a cube in G of dimension $\mathfrak{L}(G)/\mathfrak{L}(H)$ mapped faithfully into G/H. Therefore we may now assume that H is a subdirect product in G.

(iv) Fourth reduction: Let $p_S\colon G \to S^{J(G,S)}$ be the projection onto the isotypic components G_S of G (cf. [6, p. 454]) and set $H_S = p_S(H)$;

then $p_S(H_0) = (H_S)_0$ by [6, 9.26(i)]. Also, H_0 is subdirect in G by 9.26(i), and thus $H_0 = \prod_{S\in\mathcal{S}}(H_0)_S$ with $(H_0)_S$ being subdirect in $G_S = S^{J(G,S)}$ by Lemma 4. Set $H^* \stackrel{\text{def}}{=} \prod_S p_S(H)$; since $p_S(H)_0 = p_S(H_0) = (H_0)_S$ by [6, p. 460, 9.26], we get $H_0^* = H_0$. From $H_0 \subseteq H \subseteq H^*$ we get $\mathfrak{L}(H) = \mathfrak{L}(H_0) = \mathfrak{L}(H^*)$; thus by Lemma 3.1, if G contains a cube of the appropriate size mapping homeomorphically onto G/H^*, then it maps homeomorphically into G/H. Thus we may assume that $H = H^*$; this allows us to assume that G is of the form $G = S^J$ and H is subdirect. If $S = \mathbb{T}$, then we are in the abelian case and the theorem is true by [6, p. 382, 8.21].

Now we prove the theorem in the case that $G = S^J$, S is centerfree simple connected compact, and H is a subdirect product. Since the theorem is true for Lie groups we shall assume that J is infinite.

First, by Lemma 1.7 there is a closed normal subgroup $N \cong S^K$ for some subset $K \subseteq J$ such that $G = NH_0$ is a semidirect product. By Lemma 1.9 we know that N contains a totally disconnected subgroup M such that $H = MH_0$ and this product is direct. Now $G/H = NH_0/MH_0 \cong N/M$ and $\dim \mathfrak{L}(G)/\mathfrak{L}(H) = \dim \mathfrak{L}(M)$.

Thus in order to simplify notation once more, we assume that $G = S^J$ and H is totally disconnected. If p_j is the projection on any factor S of the power, then $p_j(H)$ is a compact totally disconnected (cf. [6, p. 460, 9.26(i)]) subgroup of the Lie group S and is therefore finite. Then $H^* \stackrel{\text{def}}{=} \prod_{j\in J} p_j(H)$ is a totally disconnected subgroup containing H. By Lemma 3.1, if we find a cube of dimension $\dim \mathfrak{L}(G)$ in G mapping faithfully into G/H^* then we find a cube of the same dimension in G mapping faithfully into G/H. Therefore we assume a final time that $H = \prod_{j\in J} H_j$ for H_j a finite subgroup of S for each j. Then $G/H \cong \prod_{j\in J} S/H_j$ as homogeneous spaces. Since S_j/H_j is a manifold of dimension $\dim S$ this product contains a cube $\prod_{j\in J} \mathbb{I}^{\dim \mathfrak{L}(S)}$ which is a cube of dimension $\dim \mathfrak{L}(G)$. In view of Lemma 3.1 again, this completes the proof of the theorem. ∎

Several of our results such as 2.7, 3.3 are proved under the hypothesis that G is a *connected* compact group. We want to free ourselves from the assumption that the compact homogeneous space X is connected. We first note that a relatively small modification of [6, proof of Theorem 10.35, p. 539] yields the following sharper result.

THEOREM 3.7. *Let G be a compact group acting on a compact space X with stable isotropy such that the orbit space X/G is totally disconnected. Then the action is trivial.*

If a group G acts on X and Y then a function $f\colon X \to Y$ is called *equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$ (cf.[6, p. 521, Definition 10.8]).

COROLLARY 3.8. *Let G be a compact group and let H be a closed subgroup. Then there is a G_0-equivariant homeomorphism*

$$\frac{G_0}{G_0 \cap H} \times \frac{G}{G_0 H} \to \frac{G}{H},$$

where G_0 acts on $G_0/(G_0 \cap H)$ and on G/H by multiplication on the left. The space G/G_0H is totally disconnected compact.

Proof. Since G acts transitively on X, all connected components are homeomorphic under actions from G. Since the quotient map $G \to G/H$ is continuous, open and closed, clopen sets go to clopen sets. The intersection of all clopen neighborhoods of 1 in G is G_0. Since G is compact, $X_0 \stackrel{\text{def}}{=} G_0H/H$ is the intersection of all clopen neighborhoods of H in X and thus is the component of H in X.

The group G_0 acts on X under the restriction of the action of G on X. The orbit of $H \in X$ is $G_0 \cdot H = G_0H/H = X_0$. The element $g_0 \in G_0$ is in the isotropy group $(G_0)_{gH}$ iff $g_0 gH = gH$ iff $g_0 \in gHg^{-1} \cap G_0 = g(H \cap G_0)g^{-1}$. Thus this action of G_0 on X has stable isotropy, and the orbit space is $X/G_0 = \{G_0 gH/H = gG_0H : g \in G\} = G/G_0H$, a compact totally disconnected space. Hence by Theorem 10.35, the action is trivial; that is, X is G_0-equivariantly homeomorphic to $G_0/G_0 \cap H \times G/G_0H$. ∎

From this corollary it follows that for every admissible topological dimension DIM we have

$$\dim G/H = \dim G_0H/H = \dim G_0/(G_0 \cap H). \tag{8}$$

Let us summarize our results in the following main theorem, where for any group G acting on a space X the isotropy subgroup of G at x is denoted G_x:

THEOREM 3.9 (Dimension Theorem for Homogeneous Spaces). *Let X be a homogeneous space of a compact group. Then for any compact group G acting transitively on X, and for any $x \in X$,*

$$\dim X = \dim \mathfrak{L}(G)/\mathfrak{L}(G_x). \tag{9}$$

The homogeneous space X contains a cube of dimension $\dim X$, *and the dimension of any cube contained in X does not exceed* $\dim X$.

Proof. Set $H = G_x$. Note that $\mathfrak{L}(G) = \mathfrak{L}(G_0) = \mathfrak{L}(G_0H)$, whence

$$\mathfrak{L}(G_0H/H)/\mathfrak{L}(H) = \mathfrak{L}(G)/\mathfrak{L}(H).$$

If we set $X_0 = G_0/(G_0 \cap H) \cong G_0H/H$ then $\dim X = \dim X_0$ by Definition 3.4(ii) and Corollary 3.8. Recall $\mathfrak{L}(G_0 \cap H) = \mathfrak{L}(G_0) \cap \mathfrak{L}(H)$,

$\mathfrak{L}(H) = \mathfrak{L}(H_0) \subseteq \mathfrak{L}(G_0)$, whence $\mathfrak{L}(G_0H) = \mathfrak{L}(G_0) = \mathfrak{L}(G_0) + \mathfrak{L}(H)$. If $\dim X < \infty$, then Proposition 3.3(ii) shows

$$\dim X = \dim X_0 = \dim \frac{\mathfrak{L}(G_0)}{\mathfrak{L}(G_0 \cap H)} = \dim \frac{\mathfrak{L}(G_0H)}{\mathfrak{L}(H)} = \dim \frac{\mathfrak{L}(G)}{\mathfrak{L}(H)}.$$

This proves (9) in the present case. If $\dim X$ is infinite, then by the definition of dim we have $\dim X = w(X_0)$. Now Theorem 2.7(∗) proves

$$w(X_0) = \dim \mathfrak{L}(G_0)/\mathfrak{L}(G_0 \cap H) = \dim \mathfrak{L}(G_0H)/\mathfrak{L}(H) = \dim \mathfrak{L}(G)/\mathfrak{L}(H)$$

and this proves (8) in the present case. The remainder of the theorem pertains to the component $X_0 = G_0/(G_0 \cap H)$ and thus follows from Theorem 3.6 and Proposition 3.5. ∎

By our main Theorem 3.9 we have for homogeneous spaces of a compact group a transfinite topological invariant $\dim X$ which agrees with the "linear algebra" invariant $\dim \mathfrak{L}(G)/\mathfrak{L}(G_x)$ for any compact group G acting transitively on X. In view of Theorem 3.6, it reflects the geometrically intuitive property of such homogeneous spaces as to the containment of cubes of the right dimension. Whereas topological dimension in general is a complicated matter, the dimension of a cube is of exemplary simplicity.

4. THE LOCALLY COMPACT CASE

In [9, p. 547, Theorem 11], Iwasawa proved

IWASAWA'S LOCAL SPLITTING THEOREM. *Let G be a locally compact* connected *group. Then G has arbitrarily small neighbohoods which are of the form NC such that N is a compact normal subgroup and C is an open n-cell which is a local Lie group commuting elementwise with N and is such that* $(n, c) \mapsto NC$: $N \times C \to NC$ *is a homeomorphism.*

Iwasawa had to assume that G is a projective limit of Lie groups. However, in the process of settling Hilbert's Fifth Problem (see [11, p. 184]), Yamabe showed that every locally compact group has an open subgroup which is a projective limit of Lie groups (see [11, p. 175]). Thus if G is an arbitrary locally compact *connected* group we may apply Iwasawa's local splitting theorem. We now show how to extend Iwasawa's splitting theorem to not necessarily connected locally compact groups.

THEOREM 4.1. *Let G be a locally compact group. Then for every identity neighborhood U there is a compact subgroup N contained in U, a (simply) connected Lie group L, and an open and continuous morphism* φ: $N \times L \to G$ *with discrete kernel such that* $\varphi(n, 1) = n$ *for all* $n \in N$.

Proof. (a) First assume that G is connected. We apply Iwasawa's local splitting theorem. The fact that C is a local Lie group on an open n-cell means that there is a Lie group L, an n-cell identity neighborhood W of L, and a homeomorphism $f\colon W \to C$ for which $x, y, xy \in W$ implies $f(xy) = f(x)f(y)$. We may assume L to be simply connected. Then f extends to a unique morphism of groups $F\colon L \to G$ (see [6, Corollary A2.26]). Since C is in the centralizer of N so is the subgroup $F(L)$ generated by C. Hence the definition $\varphi\colon N \times L \to G$, $\varphi(n, x) = nF(x)$ yields a continuous morphism which maps $N \times W$ homeomorphically onto the the identity neighborhood NC of G. Thus $\ker \varphi$ is discrete and φ is locally open and hence open. Clearly $\varphi(n, 1) = n$. The assertion follows in this special case.

(b) Now let G be an arbitrary locally compact group and let U be an identity neighborhood. Let G_1 be an open subgroup of G which has arbitrarily small compact normal subgroups N_1 such that G_1/N_1 is a Lie group. Since it obviously suffices to prove the theorem for G_1, we may assume that $G = G_1$.

By (a) let G_0 have an open identity neighborhood $NC \cong N \times C$ with a compact normal subgroup N of G_0 contained in U and an open n-cell local Lie group C, and let $\varphi_0\colon N \times L \to G_0$ be the surjective morphism guaranteed by part (a) of the proof. We may and will assume that $C = C^{-1}$. Let $\mathcal{N}$ be the filter basis of compact normal subgroups M of G such that G/M is a Lie group. Since the filter basis $\mathcal{N}$ converges to 1, there is an $M \in \mathcal{N}$ such that $M \subseteq U$ and $M \cap G_0 \subseteq NC$. Since N is the unique largest compact normal subgroup of G_0 contained in NC we conclude $M \cap G_0 \subseteq N$. Since G/M is a Lie group, G/MG_0 is finite. Thus MG_0 is open in G, and we may as well assume that $G = MG_0$.

Now there is a morphism $\alpha\colon L \to \operatorname{Aut} M$, $\alpha(x)(m) = F(x)mF(x)^{-1}$ with $F(x) = \varphi_0(1, x)$. We form the semidirect product $\widetilde{G} \overset{\text{def}}{=} M \rtimes \alpha L$ and define $\varphi\colon \widetilde{G} \to G$ by $\varphi(m, x) = mF(x)$. Then φ is a morphism of topological groups. Its image contains M and the group generated by $C = F(W)$, that is, $F(L)$. Hence it contains $NF(L) = G_0$ and thus $MG_0 = G$. Hence φ is surjective. Since $\widetilde{G}$ is the countable union of compact subsets and G is locally compact, by the open mapping theorem for locally compact groups (see, e.g., [6, p. 650]), φ is open. Now let $(m, x) \in (M \times U) \cap \ker \varphi$. Then $1 = mF(x)$ and $F(x) \in C$, whence $m = F(x)^{-1} \in M \cap G_0 \cap C^{-1} \subseteq N \cap C = \{1\}$. Since $(n, c) \mapsto nc\colon N \times C \to NC$ is a homeomorphism, we have $m = F(x) = 1$. Thus $x \in \ker F$. Hence $(M \times U) \cap \ker \varphi \subseteq \{1\} \times \ker F$ which is a discrete subgroup of $\widetilde{G}$. Thus $\ker \varphi$ itself is a discrete subgroup of $\widetilde{G}$. Since $M_0 \subseteq M \cap G_0 \subseteq N$ and N centralizes C and thus $\langle C \rangle = F(L)$, the subgroup $M_0 \times L \subseteq \widetilde{G}$ is a *direct* product. In particular, M_0 and L commute elementwise.

(c) We will finish the proof by showing that in semidirect products like $M \rtimes \alpha L$ the semidirect factor L can actually be replaced by a *direct* factor which is isomorphic to L.

The following lemma emulates an argument from [9, p. 515].

LEMMA A. *Let G be a locally compact group which is of the form $G = MG_0$ with a compact normal subgroup M. Let $Z(M, G)$ denote the centralizer of M in G. Then $G_0 = M_0 Z(M, G)_0$ and $G = MZ(M, G)_0$.*

Proof. Let the morphism $\pi\colon G_0 \to \operatorname{Aut} M$ be defined by $\pi(g)(x) = gxg^{-1}$. Then $\pi(G_0) \subseteq (\operatorname{Aut} G)_0$. Now $(\operatorname{Aut} G)_0$ is the group of inner automorphisms of M implemented by elements of $M_0 \subseteq M \cap G_0 = N$ (see [9, p. 514, Theorem 1′; 6, p. 505, Theorem 9.82], in whose formulation the word "Lie" should be omitted). Accordingly, for every $g \in G_0$ there is an $m_0 \in M_0$ such that $gmg^{-1} = m_0^{-1} m m_0$ for all $m \in M$; thus the element $m_0 g$ is in the centralizer $H \stackrel{\text{def}}{=} Z(M, G)$ that is, $G_0 \subseteq M_0 H$. We claim that $G_0 = M_0 H_0$. Since M_0 is connected, we have $M_0 \subseteq G_0$ and thus the compact normal subgroup $N \stackrel{\text{def}}{=} M_0 \cap H$ of $M_0 H$ is contained in $G_0 = (M_0 H)_0$. Now the group G/N is algebraically the direct product $(M_0/N)(H/N)$ and since M_0/N is compact, G/N and $(M_0/N) \times (H/N)$ are isomorphic as topological groups. It follows that $G_0/N = (G/N)_0 = (M_0/N)(H/N)_0$. Since G is locally compact, so is H. Hence $p\colon H \to H/N$ is a quotient morphism of locally compact groups, and therefore $p(H)/\overline{p(H_0)}$, as a quotient of the totally disconnected locally compact group H/H_0, is totally disconnected. Hence $(NH_0)/N = p(H_0) \subseteq (H/N)_0 \subseteq \overline{p(H_0)} = \overline{NH_0}/N$. But since N is compact, $\overline{NH_0} = NH_0$, and thus $(H/N)_0 = NH_0/N$, whence $G_0/N = (M_0/N)(NH_0/N) = M_0 H_0/N$. This establishes the claim $G_0 = M_0 H_0$. Therefore, $G = MG_0 \subseteq MM_0 H_0 = MH_0$. ∎

Note that local compactness of G was only used in the last part of the proof when we argued that the quotient group of a totally disconnected locally compact group is still totally disconnected.

Now we apply this lemma to prove

LEMMA B. *Let $G = ML$ be a locally compact group with a compact normal subgroup M and a closed simply connected subgroup L that is a Lie group such that $M \cap L = \{1\}$. Assume that the identity component M_0 of M and L commute elementwise. Then there is a closed Lie subgroup $L^* \cong L$ commuting elementwise with M such that $M \cap L^* = \{1\}$ and $G = ML^*$. That is, G is the direct product of M and L^*.*

Proof. We apply Lemma A and set $H = Z(M, G)$. Then $G_0 = M_0 H_0$. The projection of H_0 along M into L is surjective. Since $H_0 \subseteq G_0 = M_0 L$ and this product is direct, and since H_0 and M_0 commute elementwise, the projection of H_0 into M_0 is in the identity component $A \stackrel{\text{def}}{=} Z_0(M_0)$

of the center of M_0. Thus $H_0 \subseteq AL$. Hence $H_0' = L' \subseteq L$. Since L is simply connected, we have $L/L' \cong \mathbb{R}^n$; set $D \stackrel{\text{def}}{=} H_0/L'$. Then we may write $D \subseteq A \times \mathbb{R}^n$ where D projects onto $\mathbb{R}^n$. On the level of Lie algebras we get $\mathfrak{L}(D) \subseteq \mathfrak{L}(A) \times \mathbb{R}^n$ (where we have identified $\mathbb{R}^n$ with its own Lie algebra). The weakly complete vector space $\mathfrak{L}(D)$ projects onto $\mathbb{R}^n$. Hence we find a linear map λ: $\mathbb{R}^n \to \mathfrak{L}(A)$ such that the graph $\{(\lambda(v), v) : v \in \mathbb{R}^n\}$ is contained in $\mathfrak{L}(D)$. Since $\mathbb{R}^n$ is simply connected we get a morphism Λ: $\mathbb{R}^n \to A$ such that $\{(\Lambda(v), v) : v \in \mathbb{R}^n\}$ is contained in D. Denote the composition

$$L \xrightarrow{\text{quot}} L/L' \xrightarrow{\cong} \mathbb{R}^n \xrightarrow{\Lambda} A$$

by β and consider the graph $L^* \stackrel{\text{def}}{=} \{\beta(x)x \in AL : x \in L\}$. Now $\beta(x)xL' \in H_0/L'$, whence $\beta(x)x \in H_0$. Clearly $L^* \cong L$, and $\beta(x)x \in M$ means $x \in \beta(x)^{-1}M = M$; since also $x \in L$ and $M \cap L = \{1\}$ we note $x = 1$. Also $ML^* = ML = G$. Since M and $L^* \subseteq H$ commute elementwise, the product ML^* is direct.

Now we apply Lemma B to $\widetilde{G} = M \rtimes_\alpha L$ and conclude that $\widetilde{G}$ is the direct product of $M \times \{1\}$ and a Lie subgroup L^* of $\widetilde{G}$ which is isomorphic to L. This completes the proof of Theorem 4.1. ∎

Observe that the normalizer of N contains the open subgroup $\varphi(N \times L)$ of G and thus is open; we do not say that N is normal in G.

With the aid of results like this one it is frequently possible to generalize results on compact groups to locally compact groups. This applies, in particular, to our main dimension theorem for homogeneous spaces 3.9.

THEOREM 4.2. *Let G be a locally compact group and H a closed subgroup. Then*

$$\dim G/H = \dim \mathfrak{L}(G)/\mathfrak{L}(H). \tag{9'}$$

The homogeneous space G/H contains a cube of dimension $\dim G/H$, *and the dimension of any cube contained in G/H does not exceed* $\dim G/H$.

Proof. We achieve the proof by various reductions. Let U denote be any open subgroup of G. We consider the continuous open map p: $U/(U \cap H) \to UH/H$, $p(u(U \cap H)) = uH$. Assume that $p(u(U \cap H)) = p(u'(U \cap H))$. Then $uH = u'H$, that is, $u' = uh$ for some $h \in H$. Then $h \in U \cap H$, and $u'(U \cap H) = uh(U \cap H) = u(U \cap H)$. This means that p is a homeomophism. In particular $\dim U/(U \cap H) = \dim UH/H$. On the homogenous space G/H dimension is local; that is, it agrees with the dimension of any open subset such as UH/H. Hence we have $\dim G/H = \dim U/(U \cap H)$. Any cube of G/H containing H lifts to a cube of G containing 1 and thus is contained in G_0H/H; it therefore is the image under p of a cube in

$G_0(U \cap H)/(U \cap H)$. Therefore, if U is any open subgroup of G, it suffices to prove Theorem 4.2 with U in place of G. As a first reduction, it is therefore no loss to assume that $G = U$.

Next we let $\varphi\colon N \times L \to G$ be the morphism whose existence is guaranteed by Theorem 4.1. Now we take $U = \varphi(N \times L)$ and set $H^* = \varphi^{-1}(H)$. Then H^* is a closed subgroup of $N \times L$ such that the homogeneous spaces $(N \times L)/H^*$ and G/H are naturally homeomorphic. We may therefore assume that $G = N \times L$.

Since N is compact, the subgroup $G_1 = (N \times \{1\})H$ is closed. Now any element of G_1 is of the form $(n, g) = (n', 1)h$, where $h = (n'', g'') \in H$; thus $n = n'n''$ and $g = g''$. Thus if H_L is the projection of H into L then $G_1 = N \times H_L$. The following diagram describes the situation:

$$\begin{array}{rcl}
N \times L \cong & G & \left.\begin{array}{l} \\ \end{array}\right\} \cong L/H_L \\
N \times H_L \cong & G_1 & \qquad\qquad\qquad \cong \quad G/H \\
& H & \left.\begin{array}{l} \\ \end{array}\right\} \cong G_1/H
\end{array}$$

Since L is a Lie group, the quotient $G/G_1 \cong L/H_L$ is a locally trivial principal bundle. In particular, the identity in G has a neighborhood of the form $G_1 \times B$, where B is a cell homeomorphic to $\mathbb{I}^{\dim G/G_1}$. Also $\dim G/G_1 = \dim L/H_L = \dim \mathfrak{L}(L) - \dim H_L$, and because $\mathfrak{L}(G) = \mathfrak{L}(N) \oplus \mathfrak{L}(L)$ and $\mathfrak{L}(G_1) = \mathfrak{L}(N) \oplus \mathfrak{L}(H_L)$, this number equals $\dim \mathfrak{L}(G)/\mathfrak{L}(G_1)$. Since $H \subseteq G_1$ we conclude that G/H is locally homeomorphic to $(G_1/H) \times B$ and thus $\dim(G/H) = \dim G_1/H + \dim G/G_1$. If we know that $\dim G_1/H = \dim \mathfrak{L}(G_1)/\mathfrak{L}(H)$, the claim $\dim G/H = \dim \mathfrak{L}(G)/\mathfrak{L}(H)$ follows. If G_1/H contains a cube of dimension $\dim \mathfrak{L}(G_1)/\mathfrak{L}(H)$, then $G_1/H \times B$ and thus G/H contains a cube of dimension $\dim \mathfrak{L}(G)/\mathfrak{L}(H)$. Conversely, if C is a cube in G/H we may assume that C is homeomorphic to $[-1, 1]^{\aleph}$, the midpoint corresponding to the base point $H \in G/H$. The sequence of cubes C_n corresponding to $[-\frac{1}{n}, \frac{1}{n}]^{\aleph}$ converges to H and thus eventually "is contained in" $G_1/H \times B$. If G_1/H is finite dimensional then H has in G_1/H a neighborhood which is homeomorphic to a Cantor space and a $\dim G_1/H$-cell B_1, and we may assume that C_n is contained in $B_1 \times B$ and thus has a dimension $\le \dim G_1/H + \dim G/G_1 = \dim \mathfrak{L}(G)/\mathfrak{L}(H)$. If G_1/H fails to be finite dimensional, then let $(G_1/H)_0$ denote the connected component of H in G_1/H and observe $\dim C = \dim C_n \le w(C_n) \le w((G_1/H)_0 \times B) = w((G_1/H)_0) = \dim G_1/H \le \dim G/H$.

Thus the assertion of the theorem is true if it is true for G_1/H. We may therefore assume $G = G_1$; that is, G contains a compact normal subgroup N such that $G = NH$. The canonical bijective continuous function $\alpha\colon N/(H \cap N) \to G/H$, $\alpha(n(H \cap N)) = nH$ is a homeomorphism because N is compact. Now our dimension theorem for homogeneous spaces 3.9

applies to $N/(N \cap H)$ and shows that

$$\begin{aligned}\dim G/H &= \dim N/(N \cap H) = \dim \mathfrak{L}(N)/\mathfrak{L}(N \cap H)\\ &= \dim \mathfrak{L}(N)/(\mathfrak{L}(N) \cap \mathfrak{L}(H)) = \dim(\mathfrak{L}(N) + \mathfrak{L}(H))/\mathfrak{L}(H)\\ &= \dim \mathfrak{L}(NH)/\mathfrak{L}(H) = \dim \mathfrak{L}(G)/\mathfrak{L}(H).\end{aligned}$$

The assertions about the cubes apply to $N/(N \cap H)$ by 3.9. ∎

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