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LIMIT LAWS FOR WIDE VARIETIES OF TOPOLOGICAL GROUPS II

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ABSTRACT. A class of topological groups is a wide variety if it is closed under the formation of subgroups, products and continuous homomorphic images. Walter Taylor introduced limit laws as analogues for topological groups of algebraic laws for abstract groups, and proved a Birkhoff-style characterization: a class is a wide variety if and only if it is the class of models for some set of algebraic laws and some perhaps proper class of limit laws. The class of wide varieties $T(\kappa)$, for infinite cardinals κ, has played a central role in the theory to date. A group is in $T(\kappa)$ if and only if each neighbourhood of its identity contains a normal subgroup of index strictly less than κ. This paper contributes to our knowledge of the $T(\kappa)$, and of the relation of other wide varieties to the $T(\kappa)$. In particular, it is shown that the $T(\kappa)$ are definable by a set (rather than a proper class) of limit laws; indeed, the same is true of any wide subvariety of any $T(\kappa)$. Further, the class of wide varieties lying in each $T(\kappa)$ is a set. On the other hand, it is also shown that there exists a proper class of wide varieties which do not lie in any $T(\kappa)$, and constructions are given of certain families of such varieties, each defined by sets of particularly simple limit laws.

1. INTRODUCTION

A *wide variety* of topological groups is a class of topological groups closed under the formation of subgroups, products, and continuous homomorphic images. This notion was introduced by Taylor [14], following the introduction by Morris [8, 9, 10] of the notion of a *variety* of topological groups, in which the third condition above is replaced by closure under quotients. Taylor showed that wide varieties may be characterized by a Birkhoff-style theorem, as those classes

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which satisfy a collection, possibly a proper class, of *limit laws*. (See below for the definition of a limit law.) This characterization holds, indeed, for wide varieties of general topological algebras, given appropriate changes in the relevant definitions.

This paper continues the study of wide varieties and limit laws begun in [7], solving some problems posed there, and others. Extensive use will be made of material from [7], and it will be assumed that the reader is familiar with this, though some definitions and results from [7] will be quoted here.

The family of wide varieties $T(\kappa)$, for infinite cardinals κ, has proven fundamental in the theory to date. A group G is in $T(\kappa)$ if and only if each neighbourhood of the identity in G contains a normal subgroup of index strictly less than κ. Two easily proved assertions (Propositions 3.2 and 3.3 of [7]) give some indication of the central role played by the $T(\kappa)$: a wide variety is generated by a set of topological groups if and only if it is a subvariety of some $T(\kappa)$; and $T(\kappa)$ is precisely the wide variety generated by all topological groups of cardinality less than κ, for each $\kappa > \aleph_0$. In addition, the $T(\kappa)$ play a major role as a tool in the proofs of numerous results in the area (see [7] again, and, for example, [8, 9, 10]).

The division of the class of wide varieties into those which lie in some $T(\kappa)$ and those which do not is therefore a natural one. This paper contributes to knowledge of both these subclasses. In the first case, we show that $T(\kappa)$ is defined by a set, rather than necessarily a proper class, of limit laws. It follows by a theorem of [7] that every wide variety contained in a $T(\kappa)$ is also defined by a set of limit laws. We also show that the class of wide varieties lying in $T(\kappa)$ is a set, of cardinality at most 2^{2^κ}. Our second group of results deal with those varieties which lie inside no $T(\kappa)$. We show that there is a proper class of such wide varieties, and we exhibit general constructions which allow us to manufacture certain such varieties. All these are defined by sets of laws, and we know of no example of a wide variety which is not definable by a set of laws.

Definitions and notation.

Suppose that D is a directed set and V is any set. Then a *limit law* (with respect to D and V) is a formal expression $[\tau_d]$, where d runs through the elements of D, and each τ_d is a term in the first order theory of groups which has V as its (not necessarily countable) set of variables. Given a topological group G and a variable valuation $\phi : V \to G$, such a law *is satisfied* (or *holds*) *with respect to* ϕ *in* G if the net $\tau_d[\phi]$ converges to the identity e of G, where for any term τ, $\tau[\phi]$ is the term assignment of τ in G with respect to ϕ; we may then write $G \models [\tau_d][\phi]$.

A law $[\tau_d]$ *is satisfied* (or *holds*) *in* G (or G *models* the law) if $[\tau_d]$ is satisfied with respect to every valuation ϕ; we may then write $G \models [\tau_d]$.

We often adopt the equivalent view in which, given D and V as before, a limit law is a formal expression $[\tau_d]$, where the τ_d are elements of the (abstract) free group $F(V)$ on V, and in which the place of valuations ϕ is taken by maps ϕ of the free basis V of $F(V)$ into G and their canonical extensions to homomorphisms from $F(V)$ to G.

If Σ is a set of algebraic laws and Θ is a class of limit laws, then we denote by $\sin(\Sigma \cup \Theta)$ the wide variety of topological groups which model all the laws in Σ and Θ.

For other definitions and notations, we refer to [7] and others of the references cited above. In particular, we refer the reader to [7] for a discussion of free topological groups in wide varieties.

2. Inside the Wide Varieties $T(\kappa)$

We recall the definitions of three classes of wide varieties studied in [7].

The first is the family of wide varieties $T(\kappa)$, which were discussed briefly above. For κ an infinite cardinal, $T(\kappa)$ is the class of topological groups in which every neighbourhood of the identity e contains a normal subgroup of index less than κ.

The second is a family of wide varieties $B(\kappa)$ related in an obvious fashion to the varieties $T(\kappa)$. We recall (see [6]) that if κ is an infinite cardinal, then a uniform space $(X, \mathcal{U})$ is said to be κ*-precompact* if for each $U \in \mathcal{U}$ there is a set $\{x_\alpha\}$ of fewer than κ points in X such that $X = \bigcup_\alpha U[x_\alpha]$. We now define $B(\kappa)$ be the class of topological groups which are κ-precompact in their left uniformity.

Third, for any infinite cardinal κ, we define $S(\kappa)$ to be the class of topological groups in which each neighbourhood of e contains a (not necessarily normal) subgroup of index less than κ.

It is straightforward to verify that each of these classes is a wide variety. Clearly, we have $T(\kappa) \subseteq S(\kappa) \subseteq B(\kappa)$ for all κ. Some relations between these varieties were derived in [7]. (The assertion that (4) on page 324 of [7] follows from the theorem of [5], however, is incorrect; but see Theorem 2.3 below.) We begin by extending the work of [7]. This yields results which are of interest in themselves, and which also lead to our proof that $T(\kappa)$ is defined by a set of limit laws.

For technical reasons (see the proofs of Theorems 2.3 and 2.5, for example), it is convenient to consider, in addition to $T(\kappa)$, $S(\kappa)$ and $B(\kappa)$, the wide varieties $T^+(\kappa) = T(\kappa^+)$, $S^+(\kappa) = S(\kappa^+)$ and $B^+(\kappa) = B(\kappa^+)$.

An explicit description of the topology of the free topological group $F(X)$ on any topological space X was given by Tkachenko [15], and a simplified description, generalized to the free topological group $F(X,\mathcal{U})$ on any uniform space $(X,\mathcal{U})$ was given by Pestov [13]. A case of interest to us here is that of a uniform space $(X,\mathcal{U})$ with the property that $\mathcal{U}$ is closed under countable intersections. In the abstract free group $F(X)$, and for any $U \in \mathcal{U}$, let $j(U)$ denote the set of elements $\{xy^{-1}, x^{-1}y : (x,y) \in U\}$. If we choose, for each $w \in F(X)$, an arbitrary element $U_w \in \mathcal{U}$, we can represent the set of choices as a function $\Phi\colon F(X) \to \mathcal{U}$, where $\Phi(w) = U_w$. For any such function Φ, we denote by $N(\Phi)$ the subgroup of $F(X)$ generated by $\bigcup_{w\in F(X)} w\, j(\Phi(w))\, w^{-1}$. Then the result below follows easily from the description of the topology of $F(X,\mathcal{U})$ given in [13].

Theorem 2.1. *Let $(X,\mathcal{U})$ be a uniform space such that $\mathcal{U}$ is closed under countable intersections. Then the free topological group $F(X,\mathcal{U})$ has as an open basis at the identity the collection of subgroups $N(\Phi)$ of $F(X)$, as Φ varies over all functions from $F(X)$ to $\mathcal{U}$.*

We recall a definition and a result from [7].

Let S be a set, and κ any infinite cardinal. Then following [7], we let $\mathcal{U}^+_\kappa(S)$ denote the coarsest uniformity on S which makes every map from S into all uniform spaces of cardinality less than or equal to κ uniformly continuous. Equivalently, $\mathcal{U}^+_\kappa(S)$ is the coarsest uniformity on S which makes every map from S into all topologically discrete uniform spaces of cardinality less than or equal to κ uniformly continuous (where we call a uniform space *topologically discrete* if its induced topology is discrete). It is easy to see that $\mathcal{U}^+_\kappa(S)$ has a basis consisting of all equivalence relations on S with κ or fewer equivalence classes. (The topology induced on S by $\mathcal{U}^+_\kappa(S)$ is discrete, though $\mathcal{U}^+_\kappa(S)$ is only the discrete uniformity if the cardinality of S is less than or equal to κ.) We note the following result from [7] (a proof from the relevant definitions is straightforward).

Theorem 2.2. *For any set S and any κ^+-precompact uniform space $(X,\mathcal{U})$, every map from $(S,\mathcal{U}^+_\kappa(S))$ to $(X,\mathcal{U})$ is uniformly continuous.*

One of the results of [7] is a converse to this result: If S is of cardinality κ^+, and if every map from $(S,\mathcal{U}^+_\kappa(S))$ to $(X,\mathcal{U})$ is uniformly continuous, then $(X,\mathcal{U})$ is κ^+-precompact.

Theorem 2.3. *If κ is an infinite cardinal such that $\kappa^{\aleph_0} = \kappa$, then*

$$S^+(\kappa) = B^+(\kappa).$$

Proof. Fix κ satisfying $\kappa^{\aleph_0} = \kappa$. We show first that $F(S,\mathcal{U}_\kappa^+(S)) \in S^+(\kappa)$, for any set S.

We noted above that the basis elements of $\mathcal{U}_\kappa^+(S)$ are equivalence relations $U \subseteq S \times S$ on S with at most κ equivalence classes. Let $\{U_1, U_2, \dots\}$ be a sequence of such basis elements. Then clearly $\bigcap_{n=1}^\infty U_n$ is again an equivalence relation on S, and has at most $\kappa^{\aleph_0}$ equivalence classes. Since by hypothesis $\kappa^{\aleph_0} = \kappa$, we therefore have $\bigcap_{n=1}^\infty U_n \in \mathcal{U}_\kappa^+(S)$. It follows that $\mathcal{U}_\kappa^+(S)$ is closed under countable intersections. Therefore, by Theorem 2.1, $F(S,\mathcal{U}_\kappa^+(S))$ has a basis at the identity of open subgroups. Since $(S,\mathcal{U}_\kappa^+(S))$ is κ^+-precompact, it follows by a theorem of Guran [2] that $F(S,\mathcal{U}_\kappa^+(S))$ is also κ^+-precompact, and we conclude that $F(S,\mathcal{U}_\kappa^+(S))$ has a basis at the identity consisting of open subgroups of index less than or equal to κ. Hence, as claimed, $F(S,\mathcal{U}_\kappa^+(S)) \in S^+(\kappa)$.

For any set S, we let S_d denote S equipped with the discrete topology. We claim that $F(S,\mathcal{U}_\kappa^+(S))$ can be naturally identified with $F_{B^+(\kappa)}(S_d)$. In fact, given a (continuous) map $\phi : S_d \to G$, for any $G \in B^+(\kappa)$, Theorem 2.2 shows that ϕ is uniformly continuous with respect to $\mathcal{U}_\kappa^+(S)$, and the definition of $F(S,\mathcal{U}_\kappa^+(S))$ then yields an extension of ϕ to a continuous homomorphism on $F(S,\mathcal{U}_\kappa^+(S))$. Since $F(S,\mathcal{U}_\kappa^+(S)) \in S^+(\kappa) \subseteq B^+(\kappa)$, we can therefore make the claimed identification.

But for any $G \in B^+(\kappa)$, the natural continuous homomorphism

$$F(G,\mathcal{U}_\kappa^+(G)) \to G$$

is surjective (where G on the left represents the underlying set of the topological group G). Since $F(G,\mathcal{U}_\kappa^+(G)) \in S^+(\kappa)$, we therefore have $G \in S^+(\kappa)$, by the definition of a wide variety. Therefore $B^+(\kappa) \subseteq S^+(\kappa)$, completing the proof. $\square$

The main result of [5] is the following general structure theorem for topological groups: If G is a topological group of density κ, then each neighbourhood of the identity in G contains a subgroup of index less than or equal to $\kappa^{\aleph_0}$. Clearly, if G has density κ, then also $G \in B^+(\kappa)$ (but not necessarily conversely), and the following corollary is therefore a (strict) generalization of the structure theorem.

Corollary 2.4. *Let G be a κ^+-precompact topological group. Then each neighbourhood of the identity in G contains a subgroup of index less than or equal to $\kappa^{\aleph_0}$.*

Proof. We have $G \in B^+(\kappa) \subseteq B^+(\kappa^{\aleph_0})$, and since $(\kappa^{\aleph_0})^{\aleph_0} = \kappa^{\aleph_0 \times \aleph_0} = \kappa^{\aleph_0}$, we have $G \in S^+(\kappa^{\aleph_0})$ by Theorem 2.3, and the conclusion follows. □

We note in particular that if G is a κ^+-precompact topological group with no small subgroups, then $|G| \leq \kappa^{\aleph_0}$.

The example of the circle group $\mathbb{T}$ shows that, at least in the case $\kappa = \aleph_0$, the index given in the theorem cannot be lessened.

Theorem 2.5. *Let κ be any infinite cardinal. Then $S^+(\kappa) \subseteq T^+(2^\kappa)$.*

Proof. If G is any abstract group, and H a subgroup of index at most κ, then a subgroup N, contained in H, and normal and of index at most 2^κ in G, may be constructed by an argument due essentially to Marshall Hall [3] (see also [4], 4.21(d)). In brief, let $S(G/H)$ be the symmetric group on the set of right cosets Hg of H in G. Then we define a homomorphism $\pi\colon G \to S(G/H)$, by setting $\pi(g)(Hg') = Hg'g$, for all $g, g' \in G$. It is straightforward to check that the kernel N of π lies in H, and that $[G : N] \leq |S(G/H)| \leq 2^\kappa$. (We can describe N explicitly as the intersection of all conjugates of H.) The theorem clearly follows. □

We can now prove the main result of this section.

Theorem 2.6. *If κ is any infinite cardinal, then $T(\kappa)$ is defined by a set of limit laws.*

Proof. Using the relevant definitions and earlier results, we have

$$B(\kappa) \subseteq B^+(\kappa) \subseteq B^+(\kappa^{\aleph_0}) = S^+(\kappa^{\aleph_0}) \subseteq T^+(2^{\kappa^{\aleph_0}}) = T((2^{\kappa^{\aleph_0}})^+).$$

Thus there exists a cardinal τ $(= (2^{\kappa^{\aleph_0}})^+)$ such that $T(\kappa) \subseteq B(\kappa) \subseteq T(\tau)$. By Theorem 2.3 of [7], to show that $T(\kappa)$ is defined by a set rather than a proper class of laws, it suffices to find a cardinal μ such that $G \in T(\kappa)$ if and only if every subgroup H of G satisfying $|H| \leq \mu$ is in $T(\kappa)$. We claim that this statement holds with $\mu = \tau$.

Thus, let G be a topological group such that every subgroup H of G satisfying $|H| \leq \tau$ is in $T(\kappa)$. In particular, each such H is in $B(\kappa)$. As noted in [7] (following Corollary 3.12), a topological group is κ-precompact if and only if each subgroup of cardinality exactly κ is κ-precompact, so it follows that G belongs to $B(\kappa)$, and hence to $T(\tau)$.

Let V be an arbitrary neighbourhood of the identity e in G. Then there exists a normal subgroup N of G such that $N \subseteq V$ and $|G/N| < \tau$. Consider the natural quotient homomorphism $\pi\colon G \to G/N$. Choose a complete set S

of coset representatives for N in G; clearly $|S| < \tau$ and $\pi(S) = G/N$. Let $H = \langle S \rangle$, the subgroup of G generated by S. It is clear that $|H| < \tau$, so by our assumption, $H \in T(\kappa)$. Since $\pi\colon H \to G/N$ is a continuous surjection, we also have $G/N \in T(\kappa)$. By Proposition 3.1 of [7], we therefore have $G \in T(\kappa)$, as required, and the proof is complete. □

Corollary 2.7. *Let $\mathcal{V}$ be a wide variety. If $\mathcal{V}$ is generated by a set of groups, or if $\mathcal{V}$ is contained in any variety $T(\kappa)$, then $\mathcal{V}$ is defined by a set of limit laws, together with a set of algebraic laws.*

Proof. By Proposition 3.2 of [7], a wide variety $\mathcal{V}$ is generated by a set of groups if and only if $\mathcal{V}$ lies in $T(\kappa)$, for some infinite cardinal κ. By Proposition 3.6 of [7], a wide variety with either of these properties requires (in addition to its set of algebraic laws) only a set of limit laws in addition to any chosen class of limit laws defining $T(\kappa)$; and the latter class, by the above theorem, may be taken to be a set. □

We note that the converse of this corollary is trivially false: the wide variety of all topological groups is defined by the empty set of limit laws, but is not generated by any set of topological groups.

It seems appropriate to record the following result at this point; though it does not require Theorem 2.6 for its proof, it adds significantly to our knowledge of the structure of $T(\kappa)$.

Theorem 2.8. *For any infinite cardinal κ, there is set of at most 2^{2^κ} wide subvarieties of $T(\kappa)$.*

Proof. Proposition 3.6 of [7], referred to above, shows not merely that each wide subvariety of $T(\kappa)$ requires only a set of limit laws in addition to the laws defining $T(\kappa)$, but that, independent of the subvariety, there is an upper bound on the cardinalities of the directed sets involved in the additional laws, and that the variables in those laws may be assumed to belong to a fixed set. It follows that the class of wide subvarieties of $T(\kappa)$ is a set, and the specific bound above follows from calculation of the relevant cardinalities. □

We do not know whether the above bound is best possible, nor whether the class of (not necessarily wide) subvarieties of $T(\kappa)$ is a set.

3. Outside the $T(\kappa)$

A major open question on wide varieties is whether every wide variety is definable by a set of limit laws (together with a set of algebraic laws). By the results of the previous section, any wide variety which requires a proper class of limit laws cannot be contained in any $T(\kappa)$. We have no example of a wide variety requiring a proper class, but we explore, in this section, the class of wide varieties outside the $T(\kappa)$.

Clearly any abstract variety of groups $\mathcal{G}$ defines a *full* wide variety of topological groups [14]—the class of all topological groups whose underlying abstract groups are in $\mathcal{G}$. The $2^{\aleph_0}$ distinct abstract varieties [12] therefore give rise to $2^{\aleph_0}$ distinct full wide varieties of topological groups, and it is easy to see that none of these lies in any $T(\kappa)$. It is of interest to seek other wide varieties not lying in the $T(\kappa)$, and in particular to determine whether they form a proper class.

We say that a limit law $[w_d]$, with index set D, is *eventually trivial* if there exists $d_0 \in D$ such that $w_d = e$ for all $d \geq d_0$.

Proposition 3.1. *If Λ is a class of limit laws such that the wide variety of all topological groups is* $\sin(\Lambda)$*, then each law in Λ is eventually trivial.*

PROOF. Let λ be a law in Λ, and suppose that λ uses τ variables, for some cardinal τ. Let F be a discrete free group of rank τ. Then the distinct words of λ may be mapped injectively into F under a suitable variable valuation, and the fact that the resultant net in F converges to e implies that λ is eventually trivial. □

More generally, let $\mathcal{G}$ be any abstract variety of groups. Then we say that a limit law $\lambda = [w_d]$, with index set D, is *eventually $\mathcal{G}$-trivial* if there exists $d_0 \in D$ such that the algebraic law $w_d = e$ is satisfied in $\mathcal{G}$ for all $d \geq d_0$. A similar argument to that above then shows that if Λ is a class of limit laws such that the full wide variety of topological groups over $\mathcal{G}$ is $\sin(\Lambda)$, then each law in Λ is eventually $\mathcal{G}$-trivial.

In the proof of the next theorem, we require the existence of a strictly increasing sequence $\{\mathcal{V}_n\}$ of varieties of abstract groups. Examples of such sequences are given by taking $\mathcal{V}_n$ to be (i) the class of nilpotent groups of class n, or (ii) the class of groups satisfying the algebraic law $x^{n!} = e$.

Theorem 3.2. *There exists a wide variety $\mathcal{V}$ of topological groups such that*:

(i) *$\mathcal{V}$ is defined by a single limit law*;
(ii) *$\mathcal{V}$ contains a strictly increasing countable sequence of full wide varieties*;
(iii) *$\mathcal{V}$ is not the wide variety of all topological groups*;
(iv) *the variety of abstract groups underlying $\mathcal{V}$ is the variety of all groups*; *and*
(v) *$\mathcal{V}$ does not lie in $T(\kappa)$, for any cardinal κ.*

Proof. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of varieties of abstract groups, such that $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ and $\mathcal{V}_n \neq \mathcal{V}_{n+1}$, for each $n \in \mathbb{N}$. Choose an algebraic law $w_n = e$ which is satisfied in $\mathcal{V}_n$ but not in $\mathcal{V}_{n+1}$, for each n. Consider the limit law $\lambda = [w_n]$ (indexed by the natural numbers), and let $\mathcal{V}$ be the wide variety of topological groups defined by this single law. Then (i) holds by definition. For any n, the algebraic law $w_m = e$ holds in $\mathcal{V}_n$, for all $m \geq n$, showing that λ holds in any topological group whose underlying abstract group is in $\mathcal{V}_n$, and therefore proving (ii). We have (iii), by Proposition 3.1, since w_n must clearly be non-trivial for each n. Also, all indiscrete topological groups are in $\mathcal{V}$, so (iv) follows. Lastly, (v) follows by our earlier remarks about full wide varieties. □

Theorem 3.3 below constructs a proper class of wide varieties none of which lies in any $T(\kappa)$, and the particular varieties constructed form a strictly increasing transfinite sequence, indexed by the cardinal numbers. In view of this argument, and the proof of the previous theorem, it seems worth noting the following general observation on increasing chains of wide varieties.

Let τ_0 be a cardinal, and suppose that for all cardinals $\tau < \tau_0$, wide varieties $\mathcal{V}_\tau$ are chosen in such a way that $\tau \leq \tau'$ implies $\mathcal{V}_\tau \subseteq \mathcal{V}_{\tau'}$. Then of course $\bigcup_{\tau<\tau_0} \mathcal{V}_\tau$ is not in general a wide variety, being closed under the formation of continuous homomorphic images and subgroups, but not in general under formation of products. However, if we have a transfinite increasing sequence of wide varieties $\mathcal{V}_\tau$ for every cardinal τ, then $\bigcup_\tau \mathcal{V}_\tau$ is again a wide variety: closure under taking of products follows because any set of groups in $\bigcup_\tau \mathcal{V}_\tau$ lies in some single variety $\mathcal{V}_\tau$ in the sequence.

Examples of such sequences are of course provided by the varieties $T(\kappa)$, $S(\kappa)$ and $B(\kappa)$ discussed earlier, and another by the varieties $\mathcal{V}_\kappa$ defined in the proof of the following theorem. (In each of these cases, the union is simply the variety of all topological groups.)

Theorem 3.3. *There is a proper class of wide varieties which do not lie in any of the varieties $T(\kappa)$.*

Proof. For any infinite cardinal κ, let $\mathcal{V}_\kappa$ be the wide variety generated by all abelian topological groups and a single discrete free group $F(\kappa)$ of rank κ. It is easy to see that (i) $\mathcal{V}_\kappa$ does not lie in $T(\kappa')$, for any cardinal κ', (ii) the union of the $\mathcal{V}_\kappa$, for all cardinals κ, is the wide variety of all topological groups, and (iii) the variety of abstract groups underlying $\mathcal{V}_\kappa$ is the variety of all groups (by Theorem 15.4 of [11], for example). We shall show that $\mathcal{V}_\kappa$ contains no discrete free group of rank greater than κ, so that the $\mathcal{V}_\kappa$ form a proper class of distinct wide varieties.

Let F be a discrete free group in $\mathcal{V}_\kappa$. Then for some cardinal τ and some abelian topological group A, and for some subgroup H of the product $G = A \times F(\kappa)^\tau$, there exists a continuous homomorphism ϕ from H onto F. Without loss of generality, we may take τ to be infinite. Suppose that F is generated freely (algebraically) by a subset $X = \{x_\alpha\}$. It is easy to see that the choice of a set of arbitrary preimages $\{g_\alpha\}$ in G of the $\{x_\alpha\}$ is a set of free generators for the subgroup of G that it generates, and that this subgroup maps bijectively to F under ϕ. Since ϕ is continuous and F is discrete, it follows that H, and hence G, contains a discrete free subgroup of the same rank as F. Thus, without loss of generality, we may take F to be a discrete free subgroup of G.

Let N be a basic neighbourhood of the identity in G which intersects F only in $\{e\}$. Then we may write $G = A \times B \times C$, where B is a finite product $F(\kappa)^n$, for some integer n, and $C = F(\kappa)^\tau$, and where for some neighbourhood U of the identity in A, we have $N = U \times \{e\} \times C$ and $N \cap F = \{(e, e, e)\}$. Let π be the natural projection from $A \times B \times C$ onto $A \times B$. If (a_1, b_1, c_1) and (a_2, b_2, c_2) are in F and $\pi(a_1, b_1, c_1) = \pi(a_2, b_2, c_2)$, then $(a_1 a_2^{-1}, b_1 b_2^{-1}, c_1 c_2^{-1}) = (e, e, c_1 c_2^{-1}) \in N \cap F$, and so $c_1 = c_2$. Therefore π is bijective on F, and so $A \times B$ contains (at least algebraically) a free subgroup F' of the same rank as F.

Consider the projection π' from $A \times B$ onto B. Clearly $K = \ker \pi'$ is algebraically isomorphic to the abelian group A, and so $\ker \pi' \cap F'$ is an abelian subgroup of F', and is therefore either trivial or infinite cyclic. Hence $\pi'(F')$ has the same (infinite) cardinality, and therefore rank, as F', and this is bounded by the cardinality κ of B. This completes the proof. $\square$

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