The Variety of Topological Groups Generated by the Class of All Banach Spaces

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Abstract. A variety of topological groups is a class of (not necessarily Hausdorff) topological groups closed under the operations of forming subgroups, quotient groups and arbitrary products. The variety of topological groups generated by a class of topological groups is the smallest variety containing the class. The class of all topological groups underlying Banach spaces is considered. It is shown that the variety generated by this class equals the variety of all abelian topological groups.

1. Preliminaries

A non-empty class $\mathfrak{V}$ of (not necessarily Hausdorff) topological groups is said to be a *variety of topological groups* [**8**] if it is closed under the operations of forming subgroups, quotient topological groups and arbitrary products (with the Tychonoff product topology). (For a survey of varieties of topological groups see [**9**].) If Ω is a class of topological groups, then the smallest variety containing Ω is said to be the *variety generated by* Ω and is denoted by $\mathfrak{V}(\Omega)$ (cf. [**8**] and [**3**]).

For a topological group G, let $|G|$ denote the group obtained from G by "forgetting" the topology. We call $|G|$ the *group underlying* G. Further, let $|G|_I$ denote

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the topological group formed by taking $|G|$ with the indiscrete topology. We note that for a topological group G in a variety $\mathfrak{V}$, the topological group $|G|_I$ is also found in $\mathfrak{V}$ ([**8**], Lemma 2.7).

For topological groups G_1 and G_2, we say G_1 *is topologically isomorphic to* G_2 if there exists a map $f : G_1 \to G_2$ such that f is both an isomorphism of groups and a homeomorphism.

Finally, in the context of topological groups, by a Banach space we mean a topological group that underlies a Banach space.

2. Free Abelian Topological Groups and Free Locally Convex Topological Vector Spaces

The proof of our main result revolves around the relationship between the free abelian topological group on a completely regular space X and the free locally convex topological vector space on the same space X. First, we recall the definitions of these.

Definition 2.1 ([**7**]). Let X be a completely regular space and e a distinguished point in X. The topological group $FA(X)$ is said to be a *free abelian topological group on the space* X if it has the following properties:

1. X is a subspace of $FA(X)$;
2. X generates $FA(X)$ algebraically;
3. for any continuous mapping ϕ of X into any abelian topological group G which maps the point e onto the identity element of G, there exists a continuous homomorphism Φ of $FA(X)$ into G such that $\Phi(x) = \phi(x)$ on X. □

Definition 2.2 ([**4**]). Let X be a completely regular space. The topological group $FLCS(X)$ is said to be a *free locally convex topological vector space on the space* X if it has the following properties:

1. X is a subspace of $FLCS(X)$;
2. X is a (vector space) basis for $FLCS(X)$;
3. for any continuous mapping ϕ of X into any locally convex topological vector space V, there exists a continuous linear transformation Φ of $FLCS(X)$ into V such that $\Phi(x) = \phi(x)$ on X. □

The following theorem is not an obvious result. It shows that for a completely regular space X, $FA(X)$ is a subgroup of $FLCS(X)$.

Theorem 2.3 ([**14**], Theorem 3; [**15**]). *Let X be a completely regular Hausdorff space and let $FLCS(X)$ be the free locally convex topological vector space on X. Then the subgroup of $FLCS(X)$ that is algebraically generated by X is (with the induced topology) topologically isomorphic to the free abelian topological group on X.* □

3. The Variety Generated by Banach Spaces

We now turn our attention to the variety of topological groups generated by the class of all Banach spaces. However, before we present the main result, we recall the following two propositions.

Proposition 3.1 ([**2**], Section 7). *Every Hausdorff topological group is a quotient space of a topological space which admits a continuous metric.* □

Proposition 3.2 ([**1**], Lemma 1). *Let X be a topological space and G an abelian topological group such that there exists a quotient map from X onto G. Then there exists a quotient homomorphism from the free abelian topological group on X to G.*

Proof. Let the quotient map from X to G be ϕ and denote the free abelian topological group on X by F. Then, by the definition of free abelian topological group, there exists a continuous homomorphism Φ from F to G such that $\Phi(x) = \phi(x)$ for $x \in X$. Clearly, Φ is onto, so we need to show only that Φ is a quotient map.

Let $U \subseteq G$ such that $\Phi^{-1}(U)$ is open in F. We know that $\Phi^{-1}(U) \cap X = \phi^{-1}(U)$ is open in X. Therefore, U is open in G. □

The following theorem is our main result asserting that each Hausdorff abelian topological group is contained in the variety of topological groups generated by the class of Banach spaces. Not only this, but each Hausdorff abelian topological group is a quotient group of a *closed* subgroup of a product of Banach spaces.

Theorem 3.3. *Every Hausdorff abelian topological group G is topologically isomorphic to a quotient group of a closed subgroup of a product of Banach spaces.*

Proof. Let G be a Hausdorff abelian topological group. Then by Proposition 3.1, G is a quotient space of a topological space X which admits a continuous metric. Further, by Proposition 3.2, G is a quotient topological group of $FA(X)$, the free abelian topological group on X. By Theorem 2.3, $FA(X)$ is topologically isomorphic to a subgroup of $FLCS(X)$, the free locally convex topological vector space on X. Recall that every locally convex Hausdorff topological vector space, in particular $FLCS(X)$, is isomorphic as a topological vector space to a subspace of a product of Banach spaces. Now, we know that the free abelian topological group on X is Dieudonné complete ([**14**], Theorem 4) and hence is closed in any Hausdorff group containing it as a subgroup. Therefore, $FA(X)$ is topologically isomorphic to a closed subgroup of a product of Banach spaces. Thus, G is topologically isomorphic to a quotient topological group of a closed subgroup of a product of Banach spaces. □

This theorem leads us to the fact that the variety of topological groups generated by the class of all Banach spaces is exactly the variety of all abelian topological groups.

Corollary 3.4. *The variety of topological groups generated by the class of all topological groups that underlie Banach spaces is exactly the variety of all abelian topological groups.*

Proof. Let $\mathcal{B}$ be the class of all Banach spaces and let $\mathcal{A}$ be the variety of all abelian topological groups. Clearly, $\mathfrak{V}(\mathcal{B}) \subseteq \mathcal{A}$.

By Theorem 3.3, $\mathfrak{V}(\mathcal{B})$ contains all Hausdorff abelian topological groups. We note that $\mathfrak{V}(\mathcal{B})$ contains the topological groups $\mathbb{R}$ (the additive group of real numbers with the Euclidean topology) and $\mathbb{Z}$ (the discrete group of integers taken as a subgroup of $\mathbb{R}$) and so the circle group $\mathbb{R}/\mathbb{Z}$ is in $\mathfrak{V}(\mathcal{B})$. But it is well-known that every abelian group is algebraically isomorphic to a subgroup of a power of the divisible group $\mathbb{R}/\mathbb{Z}$. Hence, every abelian group appears in $\mathfrak{V}(\mathcal{B})$ with some topology. Therefore, $\mathfrak{V}(\mathcal{B})$ contains all abelian groups with the indiscrete topology ([**8**], Lemma 2.7).

Finally, we note that for any abelian topological group H with identity e, H is topologically isomorphic to a subgroup of the product group $H/\overline{\{e\}} \times |H|_I$ where $|H|_I$ is the group $|H|$ with the indiscrete topology and $\overline{\{e\}}$ is the closure of $\{e\}$. Clearly, $H/\overline{\{e\}}$ is a Hausdorff abelian topological group and is thus contained in $\mathfrak{V}(\mathcal{B})$. Also, $|H|_I \in \mathfrak{V}(\mathcal{B})$. Therefore, $H \in \mathfrak{V}(\mathcal{B})$.

Therefore, $\mathfrak{V}(\mathcal{B})$ contains all abelian topological groups, and so $\mathfrak{V}(\mathcal{B}) = \mathcal{A}$, as required. □

Remark 3.5. An alternative proof of Corollary 3.4 uses the extension of a metric as described in Graev's paper [**7**] in the following manner.

1. If F is the free abelian group (with identity e) on a completely regular space X that admits a metric d, then the extension of d to d' on F as described in Graev's paper [**7**] has the property that for each $a \in F$, $d'(a^2, e) = 2d'(a, e)$. (This is not entirely obvious.)
2. By Enflo ([**6**], Theorem 2.4.1), every abelian topological group with the property mentioned in part 1, in particular (F, d'), is topologically isomorphic to a subgroup of a Banach space.
3. If X is a completely regular space that admits a continuous metric then its topology is clearly defined by a family of continuous metrics $\{d_i : i \in I\}$. Extending each of these metrics d_i to a metric d_i' on the free abelian group F on X using Graev's method, then forming the sum of these topologies on F yields the free abelian topological group on X (see [**10**], pp. 378–379; [**11**], Proposition 1). Further, this sum is clearly topologically isomorphic to a subgroup of the product $\prod_{i \in I}(F, d_i')$.
4. Using parts 2 and 3, the free abelian topological group on X is topologically isomorphic to a subgroup of a product of Banach spaces.
5. The proof is completed by applying Proposition 3.1 and Proposition 3.2 to G, a Hausdorff abelian topological group, then proceeding as in the proof of Corollary 3.4.

This approach is used later to show that the free abelian topological group on a metrizable space X contains no small subgroups (cf. [**13**]).

4. Further Results

We recall the following concept, introduced by Enflo [**6**]. A topological group G is said to be *uniformly free from small subgroups* if it contains a neighbourhood of the identity, U, such that for every neighbourhood of the identity, V, there exists a positive integer n_V with the property that $x \notin V \Rightarrow x^n \notin U$ for some $n \leq n_V$.

Proposition 4.1 ([**12**], Proposition 2.2). *A topological group G is uniformly free from small subgroups if and only if for some neighbourhood of the identity, U, the sets*

$$(1/n)U = \{x \in G : \forall k = 1, 2, \ldots, n, x^k \in U\}$$

form a neighbourhood basis at the identity. □

Remark 4.2. Using Proposition 4.1, it is routine to show that a subgroup of a topological group uniformly free from small subgroups is also uniformly free from small subgroups. Further, every topological group uniformly free from small subgroups clearly has no small subgroups [**12**]. Morris and Pestov noted that every Banach space is uniformly free from small subgroups ([**12**], Theorem 2.6). This allows us to prove Corollary 4.3.

The following result is a corollary to the proof in Remark 3.5.

Corollary 4.3. *Let X be a completely regular space that admits a continuous metric. Then the free abelian topological group on X has no small subgroups.*

Proof. Let X be defined by the family of continuous metrics $\{d_i : i \in I\}$. If one of the metrics d_i is extended to a metric d'_i on the free abelian group F on X using Graev's method, then the topological group (F, d'_i) is topologically isomorphic to a subgroup of a Banach space (see Remark 3.5, parts 1, 2 and 3). Therefore, (F, d'_i) is a subgroup of a topological group uniformly free from small subgroups and so is uniformly free from small subgroups. Hence, (F, d'_i) has no small subgroups (see Remark 4.2). Since the free abelian topological group on X, $FA(X)$, is the group F with a finer topology than that induced by d'_i, $FA(X)$ has no small subgroups. □

Before presenting our final result, we present the following lemma from the work by Morris and Pestov.

Lemma 4.4 ([**12**], Lemma 3.2). *Let G be a topological group uniformly free from small subgroups. Then, whenever G is topologically isomorphic to a quotient group of a topological subgroup of the direct product of a family $\mathcal{G}$ of topological groups, G is topologically isomorphic to a quotient group of a subgroup of the product of a finite subfamily of $\mathcal{G}$.* □

We can now prove an interesting result concerning Hausdorff abelian topological groups that are uniformly free from small subgroups.

Proposition 4.5. *If a Hausdorff abelian topological group G is uniformly free from small subgroups then it is topologically isomorphic to a quotient group of a subgroup of a Banach space.*

Proof. By Theorem 3.3, G is topologically isomorphic to a quotient group of a closed subgroup of a product of Banach spaces. Therefore, by Lemma 4.4, G is topologically isomorphic to a quotient group of a subgroup of a finite product of Banach spaces. Since a finite product of Banach spaces is a Banach space, G is topologically isomorphic to a quotient group of a subgroup of a Banach space. □

This Proposition is extended in Remark 4.6 (3) below.

Remark 4.6. To conclude, we note the following points.

1. The class of all topological groups which are topologically isomorphic to a subgroup of a Banach space is the class of all abelian topological groups G (with identity e) that admit a metric d which has the property that for all $a \in G$, $d(a^2, e) = 2d(a, e)$ (Remark 3.5, parts 1 and 2).
2. The class of all topological groups which are topologically isomorphic to a quotient group of a subgroup of a product of Banach spaces is the class of all abelian topological groups (Corollary 3.4).
3. The class of all topological groups (resp., Hausdorff topological groups) topologically isomorphic to a quotient group of a subgroup of a Banach space is the class of all pseudo-metrizable abelian topological groups (respectively, metrizable abelian topological groups). To see this observe that in the Hausdorff case, for example, if G is a metrizable abelian topological group with invariant metric d, then we can extend d to a metric d' on the free abelian group on the space G as in Remark 3.5 and by the same Remark, the free abelian group with this metric can be embedded as a subgroup of a Banach space. Further, the natural map of the free abelian group $(F(G), d')$ with the metric d' onto the topological group G is continuous as (d', d) is non-expanding and is clearly, then, a quotient mapping, which completes the proof.

References

[1] A. V. Arhangel'skii, *Mappings related to topological groups*, Soviet Math. Dokl. **9** (1968), 1011–1015.

[2] A. V. Arhangel'skii, *Classes of topological groups*, Russian Math. Surveys **36** (3) (1981), 151–174.

[3] M. S. Brooks, Sidney A. Morris, and Stephen A. Saxon, *Generating varieties of topological groups*, Proc. Edinburgh Math. Soc. (2) **18** (1973), 191–197.

[4] J. Diestel, Sidney A. Morris, and Stephen A. Saxon, *Varieties of linear topological spaces*, Trans. Amer. Math. Soc. **172** (1972), 207–230.

[5] J. Dugundji, *Topology*, Allyn and Bacon Inc., Boston, 1966, ISBN 0205002714.

[6] P. Enflo, *Uniform structures and square roots in topological groups*, Israel J. Math. **8** (1970), 230–272.

[7] M. I. Graev, *Free topological groups*, Izvestiya Akademii Nauk SSSR Seriya Matematiceskaya **12** (1948), 279–324 (Russian). English translation, Amer. Math. Soc. Transl. **35** (1951). Reprint Amer. Math. Soc. Transl. (1) **8** (1962), 305–364.

[8] Sidney A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. **1** (1969), 145–160.
[9] Sidney A. Morris, *Varieties of topological groups — A survey*, Colloq. Math. **46** (1982), 147–165.
[10] Sidney A. Morris, *Free abelian topological groups*, Proc. Internat. Conf. Categorical Topology, Toledo, Ohio (1983), 375–391, Heldermann Verlag, Berlin, 1984.
[11] Sidney A. Morris and Peter Nickolas, '*Locally invariant topologies on free groups*, Pacific J. Math. (2) **103** (1982), 523–537.
[12] Sidney A. Morris and Vladimir Pestov, *On Lie groups in varieties of topological groups*, to appear.
[13] O. V. Sipacheva and V. V. Uspenskii, *Free topological groups with no small subgroups, and Graev metrics*, Moscow Univ. Math. Bull. **42** (1987), No. 4, 24–29.
[14] M. Tkachenko, *On completeness of free abelian topological groups*, Soviet Math. Dokl. (2) **27** (1983), 341–345.
[15] V. Uspenskii, *Free topological groups of metrizable spaces*, Math. USSR Izvestiya (3) **37** (1991), 657–680.