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A topological generalization of the Higman–Neumann–Neumann theorem

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Abstract. We generalize in a substantial way the celebrated result by Graham Higman, Bernhard Neumann and Hanna Neumann on embedding countable groups into 2-generator groups as follows: every countable topological group is isomorphic to a topological subgroup of a topological group algebraically generated by two elements. A number of corollaries are derived. In particular, we characterize those topological groups embeddable into groups with two topological generators: they are the groups covered by countably many translations of each neighbourhood of the identity and having weight at most $\mathfrak{c}$. In particular, they include all separable topological groups.

0 Introduction

A famous result of Graham Higman, Bernhard Neumann and Hanna Neumann [9] asserts that every countable group is isomorphic to a subgroup of a 2-generator group. Their proof uses free products with amalgamations. Later Bernhard and Hanna Neumann gave a proof using wreath products [14]. Subsequently the latter proof was converted into a very transparent form, and there is an excellent exposition by Fred Galvin in the American Mathematical Monthly [5]. (See [6] for even more refined results using the same construction.)

Here we show that the proof as presented by Galvin can be reconstructed at the level of topological dynamics so as to lead to the following general result for countable topological groups.

Theorem A. *Let G be any countable topological group. Then there exists a topological group H which has G as a topological subgroup and which is algebraically generated by some $a, b \in H$.*

The original Higman–Neumann–Neumann theorem is recovered in the special case where the original group G has the discrete topology.

As a direct corollary of Theorem A, one deduces the following new result: every separable topological group is isomorphic to a topological subgroup of a group having two topological generators (that is, having an everywhere dense 2-generator subgroup). Bearing in mind that not every topological subgroup of a group with two topological generators is separable, we go further to obtain a complete description of those topological groups embeddabe into groups with two topological generators or, equivalently, into separable topological groups (Theorem B). A number of other corollaries are derived, including a new description of the important class of $\aleph_1$-precompact topological groups.

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1 Conventions and preliminaries

All topological spaces (including those of topological groups) in this note are assumed to be Tychonoff (completely regular Hausdorff), and all uniform spaces $(X, \mathscr{U})$ separated ($\bigcap \mathscr{U} = \Delta_X$). A good reference on uniform structures and completions of topological groups is [16].

Given a family $\{(X_\alpha, \mathscr{U}_\alpha) : \alpha \in A\}$ of uniform spaces, we denote by $\bigoplus_{\alpha \in A} X_\alpha$ their coproduct, that is, the disjoint union of all X_α equipped with the finest uniformity inducing the given uniformity on each X_α. A basis of the uniformity on $\bigoplus_{\alpha \in A} X_\alpha$ is formed by all entourages of the form $\bigcup_{\alpha \in A} U_\alpha$ as $U_\alpha \in \mathscr{U}_\alpha$, where each $X_\alpha \times X_\alpha$ is canonically identified with a subset of the Cartesian square of $\bigoplus_{\alpha \in A} X_\alpha$. A mapping $f : \bigoplus_{\alpha \in A} X_\alpha \to Y$, where Y is any uniform space, is uniformly continuous if and only if is every restriction $f \,|\, _{X_\alpha} : X_\alpha \to Y$, $\alpha \in A$ is uniformly continuous.

The full symmetric group of a set X is denoted by $\mathrm{Sym}(X)$. Any action $\tau : G \to \mathrm{Sym}(X)$ of a group G on a set X is interpreted as an action *on the right,* that is, we associate with it a map $X \times G \to X, (g, x) \mapsto \tau_g x \equiv xg$. The disjoint union X of a family $\{X_\alpha : \alpha \in A\}$ of sets equipped with an action of a group G itself admits a natural action of G. For the basic concepts of dynamics, see e.g. [19].

The group $\mathrm{Aut}(X)$ of all automorphisms of a uniform space $X = (X, \mathscr{U})$ is always equipped with the topology of uniform convergence, of which a neighbourhood basis at the identity is formed by the sets

$$\tilde{U} = \{g \in \mathrm{Aut}(X) : (x, xg) \in U \text{ for all } x \in X\},$$

where $U \in \mathscr{U}$. The topology of uniform convergence is a Hausdorff group topology. It is well known that every topological group G is isomorphic to a topological subgroup of the group of the form $\mathrm{Aut}(X)$ for a suitable uniform space X; in fact, one can always choose as X a compact space with its unique compatible uniformity (see [17], Theorem 2). In our argument to follow the particular form of X plays no role whatsoever.

2 Proof of Theorem A

Our proof generally follows the argument of Galvin [5], with topological and uniform structures incorporated at every stage.

Index elements of G with odd positive integers: $G = \{g_1, g_3, \ldots, g_{2k+1}, \ldots\}$. Choose a uniform space X and an effective (i.e. faithful) action τ of G upon X such that the topology of uniform convergence induced on G via the monomorphism $\tau : G \hookrightarrow \mathrm{Aut}(X)$ coincides with the original group topology.

We make the product $\mathbb{Z} \times \mathbb{Z} \times X$ into a uniform space by identifying it with the coproduct of countably many copies of the uniform space X indexed with elements of $\mathbb{Z} \times \mathbb{Z}$:

$$\mathbb{Z} \times \mathbb{Z} \times X \cong \bigoplus_{(m,n) \in \mathbb{Z}\times\mathbb{Z}} \{(m,n)\} \times X\}$$

Now we define permutations a and b of $\mathbb{Z} \times \mathbb{Z} \times X$ in exactly the same fashion as Galvin does: $(m,n,x)a = (m+1,n,x)$, and

$$(m,n,x)b = \begin{cases} (m, n+1, x) & \text{if } m = 0 \\ (m, n, xg_m) & \text{if } m \text{ is odd, } m > 0,\ n \geq 0, \\ (m, n, x) & \text{otherwise.} \end{cases}$$

Both a and b are uniform automorphisms of $\mathbb{Z} \times \mathbb{Z} \times X$, rather than just permutations of it. This follows from the identification of the latter space with the coproduct of a family of copies of X and the fact that the restriction of both a or b and their inverses to any copy of the uniform space X of the form $\{(m,n)\} \times X$ is either an automorphism of this copy of the form $(m,n,x) \mapsto (m,n,xg_m^{\varepsilon})$, with $\varepsilon = \pm 1$, or else a canonical isomorphism of it with a piece of the form $\{(m \pm 1, n)\} \times X$ or $\{(m, n \pm 1)\} \times X$.

Set $b_i = a^i b a^{-i}$ and $\hat{g}_i = b_i b^{-1} b_i^{-1} b$ for all $i = 1, 3, 5, \ldots$. All b_i, and therefore all $\hat{g}_i$, are uniform automorphisms of $\mathbb{Z} \times \mathbb{Z} \times X$, being finite compositions of such maps.

By Galvin's calculations,

$$(0,0,x)\hat{g}_i = (0,0,xg_i) \tag{1}$$

and

$$(m,n,x)\hat{g}_i = (m,n,x) \quad \text{whenever either } m \neq 0 \text{ or } n \neq 0. \tag{2}$$

Algebraically, this implies that the correspondence

$$i : G \ni g_m \mapsto \hat{g}_m \in \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z} \times X)$$

establishes an isomorphism of G with a subgroup H of $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z} \times X)$, algebraically generated by a and b.

A closer look at (1) and (2) reveals that the action i of the group G upon the uniform space $\mathbb{Z} \times \mathbb{Z} \times X$ is the direct sum of the original action τ of G upon the uniform subspace $X \equiv \{(0,0)\} \times X$ and the trivial action of G upon the rest, $[(\mathbb{Z} \times \mathbb{Z}) \setminus \{(0,0)\}] \times X$. In other words, for each $g \in G$ the restriction of the g-motion $i_g : \mathbb{Z} \times \mathbb{Z} \times X \to \mathbb{Z} \times \mathbb{Z} \times X$ to $\{(0,0)\} \times X$ coincides with τ_g, while $i_g|_{\{(m,n)\}\times X} = \mathrm{Id}_{\{(m,n)\}\times X}$ whenever $(m,n) \neq (0,0)$. Given any element U of the uniform structure of $\mathbb{Z} \times \mathbb{Z} \times X$, any $g \in G$ and any $x \in X$, one has therefore $(x, i_g x) \in U$ if and only if $(x, \tau_g x) \in U \cap (X \times X)$.

It follows immediately that the topology induced on G via the monomorphism $i : G \hookrightarrow \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z} \times X)$ coincides with the topology induced on G via the original action $\tau : G \hookrightarrow \mathrm{Aut}(X)$ and therefore with the original topology on G. From here it follows that G is a topological subgroup of the topological subgroup H of $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z} \times X)$ generated by a and b, as required.

3 Applications

A subset X of a topological group G *topologically generates* G, or forms a set of *topological generators*, if the subgroup $\langle X \rangle$ algebraically generated by X is everywhere dense in G. The following is a new result.

Corollary 1. *Every separable topological group embeds, as a topological subgroup, into a topological group with two topological generators.*

Proof. Let G be a separable topological group. Then G contains a countable everywhere dense subgroup F. Embed F into a 2-generator topological group H using Theorem A. Since F lies inside H as a topological subgroup, the completion $\hat{F}$ of F with respect to the two-sided uniformity is canonically isomorphic with a topological subgroup of the completion of H, and clearly $\hat{H}$ is topologically generated by two elements. Because of the well-known uniqueness of the completion of a topological group with respect to the two-sided uniformity (see [16]), whenever a topological subgroup F is everywhere dense in a topological group G, the completions $\hat{F}$ and $\hat{G}$ are canonically isomorphic. Therefore G embeds into $\hat{F} \cong \hat{G}$ as a topological subgroup. As a consequence, G embeds into the 2-generator topological group $\hat{H}$.

Remark 1. Corollary 1 shows that the topological group $\mathbb{R}^{\aleph_0}$ can be embedded in a topological group with two topological generators. At the same time, $\mathbb{R}^{\aleph_0}$ itself requires $\aleph_0$ topological generators, as $\mathbb{R}^n$, which requires $n+1$ generators, is a topological quotient group for each n (see [2]).

In general, a topological subgroup of a separable topological group need not be separable ([7]; [4], Theorem 3.3), apart from the locally compact case ([4]; [3], Theorem 3.14), and therefore Corollary 1 is not a final result. We now proceed to obtain a complete description of topological subgroups of groups with two topological generators.

A topological group G is called *κ-precompact* (see [13]), where κ is an infinite cardinal, if G can be covered by fewer than κ translates of any non-empty open subset.

Equivalently, G is κ-precompact if and only if it is isomorphic with a topological subgroup of the direct product of a family of topological groups of weight less than κ. The class of all κ-precompact topological groups is closed under continuous homomorphisms, topological subgroups and infinite direct products. In particular, $\aleph_0$-precompact groups are the usual precompact topological groups, while $(\kappa+)$-precompact groups were studied in [8] under the name of κ-bounded groups; see also [1]. Every Lindelöf (in particular, every compact) and every separable topological group is $\aleph_1$-precompact (*ibid.*). (Our terminology agrees with that in [12].)

The following result describes subgroups of separable (and of topologically 2-generated) topological groups in easily verifiable terms.

Theorem B. *For every topological group G the following are equivalent:*

(i) *G embeds as a topological subgroup into a topological group with* 2 *topological generators;*

(ii) *G embeds as a topological subgroup into a separable topological group;*

(iii) *G is an* $\aleph_1$*-precompact group of weight at most* $\mathfrak{c}$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, (ii) $\Rightarrow$ (i) is just Corollary 1, and (ii) $\Rightarrow$ (iii) follows from the well-known facts that every separable topological group is $\aleph_1$-precompact [1, 8] and has weight at most $\mathfrak{c}$.

Now assume (iii). Then G embeds, as a topological subgroup, into the direct product of a family of topological groups with countable base; without loss of generality the cardinality of such a family can be assumed at most $\mathfrak{c}$. The Tychonoff product of such a family is separable by the famous Hewitt–Marczewski–Pondiczery theorem ([10], Theorem 11.2), which establishes (ii).

Remark 2. It is instructive to compare Theorem B to the following result [11]: every compact connected group of weight at most $\mathfrak{c}$ has 2 topological generators. Notice that not every topological group of weight at most $\mathfrak{c}$ embeds into a group with two topological generators: Theorem B tells us that no discrete group of cardinality $\mathfrak{c}$ is a topological subgroup of a separable group because it is not $\aleph_1$-precompact.

In general, the 2-generator topological group H constructed in the proof of Theorem A need not be metrizable even if G is metrizable because the coproduct of infinitely many non-discrete uniform spaces is never metrizable. Nevertheless, weakening the topology on H by means of a well-established technique [1], one can prove the following.

Corollary 2. *Every countable metrizable group is isomorphic to a topological subgroup of a metrizable group topologically generated by two elements.*

Proof. Since G is metrizable, it is possible to choose a countable prefilter γ of neighbourhoods of the identity in H whose restrictions to G form a base for the neighbourhood filter. Without loss of generality one can assume first that γ contains all cofinite subsets of H, and then that for every $V \in \gamma$ there is a $W \in \gamma$ with $W^{-1}W \subseteq V$. Since H is countable, the minimal prefilter $\tilde{\gamma}$ on H containing γ and invariant under inner automorphisms is countable as well. Now it is easy to see that $\tilde{\gamma}$ satisfies all the usual properties (see e.g. [3], 1.11) of a neighbourhood basis at the identity for a Hausdorff group topology $\mathfrak{T}$ on H, which is metrizable (since $\tilde{\gamma}$ is countable), clearly contained in the original topology of H, and induces the original topology on G.

Corollary 3. *Every topological group of countable weight is isomorphic to a topological subgroup of a group of countable weight topologically generated by two elements.*

Proof. It is enough to choose a countable everywhere dense subgroup F of the group G of countable weight, then to embed F into a 2-generator metrizable group L using Corollary 2 and to observe first that G is isomorphic to a subgroup of the completion $\hat{L}$ (as in the proof of Theorem B), and then that every separable metrizable group (in particular $\hat{L}$) has countable weight (see e.g. [3], Theorem 3.5.i).

In fact, Corollary 3 can be further strengthened.

Corollary 4. *There exists a topological group G of countable weight topologically generated by two elements such that every topological group of countable weight is topologically isomorphic with a subgroup of G.*

Proof. The group of all self-homeomorphisms of the Hilbert cube $Q = I^{\aleph_0}$ forms a universal topological group with countable base; see [18]. Applying to Homeo Q our Corollary 3 yields the desired result.

The following strengthens an earlier result from [15], originally established for zero-dimensional compacta instead of Cantor cubes.

Corollary 5. *For a topological group G the following are equivalent:*

(i) *G is isomorphic to a topological subgroup of a topological group topologically generated by a Cantor cube D^τ;*

(ii) *G is $\aleph_1$-precompact.*

Proof. It is well known that a topological group topologically generated by a compact subset is $\aleph_1$-precompact; see [1], [8]. Going in the opposite direction, if G is $\aleph_1$-precompact, then it embeds as a topological subgroup into the direct product $\prod_{\alpha<\tau} G_\alpha$ of a suitable family of groups with countable base; see again [1], [8]. For each $\alpha < \tau$, fix a topological group H_α topologically generated by a 2-element subset $\{a_\alpha, b_\alpha\}$ and containing G_α as a topological subgroup. The topological group $\prod_{\alpha<\tau} H_\alpha$ contains G as a topological subgroup and is topologically generated by a subset $\prod_{\alpha<\tau}\{a_\alpha, b_\alpha\}$ homeomorphic to D^τ.

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