Embedding free amalgams of discrete groups in non-discrete topological groups

*Sidney A. Morris and Viatcheslav N. Obraztsov**

Abstract. The main result is the theorem: Assume that the set of nontrivial groups $\{G_\mu\}_{\mu \in I}$ contains either three groups or two groups of which one has order at least 3. Then the free amalgam Ω^1 of the groups G_μ, $\mu \in I$, can be embedded in a group $G = \mathrm{gp}\{\Omega^1\}$ in such a way that for each fixed cardinal $\beta < |G|$, G admits a non-discrete Hausdorff topology such that 1) G is a 0-dimensional topological group; 2) every neighbourhood is of cardinality $|G|$; 3) if a subgroup M of G is conjugate to a subgroup of G_μ for some $\mu \in I$ or M is a finite extension of a cyclic group, then M is discrete; 4) every subgroup of G of cardinality $\gamma \leq \beta$ is discrete; and 5) if β is finite, then G may be chosen to be metrizable. This result depends on the method of A.Yu. Ol'shanskii. A special case of the theorem, together with applications, is given in [4].

1. The main result

The structure of locally compact topological groups, especially compact groups, is reasonably well understood. Outside the class of locally compact groups, the standard techniques, for example using Lie groups, do not apply. Algebraic methods such as the method of graded diagrams developed by Ol'shanskii in [10] and central extensions of diagrammatically aspherical groups (see, for example, [10, Chapter 10]), seem to be very useful in understanding the structure of general topological groups. Some applications of these methods have already been used in topological group theory. It is sufficient to mention the following results: 1) Ol'shanskii, [8], settled the famous Markov problem about the existence of a countable non-topologizable group, that is, a countable group which admits the discrete topology only; 2) Zyabrev and Reznichenko, [11], constructed an example of a (path-) connected topological group in which all elements in some neighbourhood of the identity satisfy a law (namely, $x^n = 1$) which is not satisfied in the whole of the group; 3) the authors, together with Hofmann and Oates-Williams, obtained in [3] examples of non-abelian non-discrete locally compact Hausdorff groups in which every nontrivial closed subgroup is open.

The method of Ol'shanskii was extended by the second author in [5]–[7] to diagrams over the free products and applied to quotient groups of free products. As a result, embedding schemes of an arbitrary set of groups into a simple infinite group

*The second author was supported by a grant from the Australian Research Council.

with a "well-described" lattice of subgroups and a given outer automorphism group were established in [6] and [7]. The first step in introducing this technique to topological groups was made by tha authors in [4], where using tha Ol'shanskii method, any set of discrete groups without involutions was embedded in a non-discrete Hausdorff topological group with many discrete subgroups. One of the applications of this result is the existence, for each positive integer n, of a non-discrete Hausdorff topological group of cardinality $\aleph_n$ with no proper subgroups of tha same cardinality and with each proper suhgroup discrete.

The main result of [4] appeared there without the proof, because of its size, complexity and extensive use of algebraic methods. In this paper we not only provide the proof, but also give a generalization of the results in [4] to the class of all discrete groups.

For the statements, a few definithms are required.

Definition 1.1. The *free amalgam* Ω^1 of an arbitrary set of groups $\{G_\mu\}_{\mu\in I}$ is defined to be the set $\bigcup_{\mu\in I} G_\mu$ with $G_\mu \cap G_\nu = 1$ whenever $\mu \neq \nu$.

Definition 1.2. The mapping $g : \Omega^1 \to G$ is a *(topological) embedding* of the free amalgam Ω^1 of a set of (topological) groups G_μ, $\mu \in I$, into a (topological) group G if it is injective and G_μ is (topologically) isomorphic to $g(G_\mu)$ for each $\mu \in I$.

The main result of this paper is the following theorem.

Theorem 1.3. *Assume that the set of nontrivial groups $\{G_\mu\}_{\mu\in I}$ contains either three groups or two groups of which one has order at least 3. Then the free amalgam Ω^1 of the groups G_μ, $\mu \in I$, can be embedded in a group $G = gp\{\Omega^1\}$ in such a way that for each fixed cardinal $\beta < |G|$, G admits a non-discrete Hausdorff topology such that*

1) *G is a 0-dimensional topological group;*

2) *every neighbourhood is of cardinality $|G|$;*

3) *if a subgroup M of G is conjugate to a subgroup of G_μ for some $\mu \in I$ or M is a finite extension of a cyclic group, then M is discrete;*

4) *every subgroup of G of cardinality $\gamma \leq \beta$ is discrete;*

5) *if β is finite, then G may be chosen to be metrizable.*

Now we briefly describe the contents of the paper. In §2 we present a general embedding construction which fulfills the conditious of Theorem 1.3. Some applications of Theorem 1.3 will be discussed in §3. Topologization of the resulting group G in Theorem 1.3 involves the study of the products of "long" and "short" words over the alphabet Ω^1, where every "long" word is l-aperiodic for small values of l, which is heavily based on Ol'shanskii's technique developed in [10]. §4 contains the proof of Theorem 1.3, except for the proofs of the technical Lemma 4.2 and Lemma 4.9, which are relegated to §5 so that §4 will be accessible to more readers. The last section, §5, is devoted to these two lemmas.

2. Construction of the group G

As in [10], we introduce the positive parameters

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \iota,$$

where all the parameters are arranged according to "height", that is, the small positive value β is chosen after α, γ after β, and so on. Our proofs and some definitious are based on a system of inequalities involving these parameters. The values of the parameters can be chosen in such a way that all the inequalities hold. We then use the following notation:

$$\alpha\prime = 1/2 + \alpha,\ \beta\prime = 1 - \beta,\ \gamma\prime = 1 - \gamma,\ h = \delta^{-1},\ d = \eta^{-1},\ n = \iota^{-1}.$$

We assume that n is an integer.

Let Ω^1 be the free amalgam of the groups G_μ, $\mu \in I$, and $\Omega = \Omega^1 \setminus \{1\}$. We define $G(1)$ to be the free product of the groups G_μ, $\mu \in I$, and set $D_1 = \emptyset$. Assume, by induction, that we beve defined the set of relators D_{i-1}, $i \geq 2$, and define

$$G(i-1) = (G(1) || R = 1; R \in D_{i-1}).$$

A word X (over the alphabet Ω) is a *minimal* word in rank $i-1$ if it follows from $X = Y$ in $G(i-1)$ that $|X| \leq |Y|$, where $|Z|$ denotes the length of the word Z. A word Y is called *free* in rank $i-1$ if Y is not conjugate in rank $i-1$ to an element of Ω^1, that is, to an image in $G(i-1)$ of an element of one of the free factors G_μ. A non-empty word Z is said to be *simple* in rank $i-1$ if it is free in rank $i-1$, not conjugate in rank $i-1$ (that is, in $G(i-1)$) to a power of a shorter word and not conjugate in rank $i-1$ to a power of a period of rank $k < i$.

Now let P_i denote a set of words of length i which are simple in rank $i-1$ with the property that $A, B \in P_i$ and $A \not\equiv B$ ("$\equiv$" means letter-for-letter equality of words of the same length) implies that A is not conjugate in rank $i-1$ to B or B^{-1}. The words in P_i are called *periods* of rank i. A special role in the construction of the group G will be played by the sets P_i' of all periods of rank i which are not equal in rank $i-1$ to a product of two involutions (of $G(i-1)$). By the definition of the group $G(i-1)$, $P_i = P_i'$ if all groups G_μ, $\mu \in I$, are without involutions.

The set of relators S_i of rank i is constructed as follows. First, we include in S_i words of the form A^{n_A} (*relators of the first type*) for certain words $A \in P_i'$, where the odd number n_A, in general, depends on A and $n_A \geq hni$, and call any relation

$$A^{n_A} = 1 \tag{1}$$

a *defining relation of the first type of rank* i.

Secondly, given $A \in P_i'$, we also include in S_i a set of words of the form $T_1 A^{n_1} \ldots T_h A^{n_h}$ (*relators of the second type*) and call any relation

$$T_1 A^{n_1} \cdot T_2 A^{n_2} \ldots T_h A^{n_h} = 1 \tag{2}$$

a *defining relation of the second type* ($n_1, \dots, n_h$ depend on A and on (2)). Every relator of the second type must satisfy conditions R1–R7 of [10, pp. 271–272], and the following additional condition holds:

R8. $n_k \geq ni$ for each $k = 1, \dots, h$.

For each $i \geq 2$, we set $D_i = D_{i-1} \cup S_i$, and the group $G(i)$ is defined by its presentation:

$$G(i) = \langle G(1) || R = 1; R \in D_i \rangle. \tag{3}$$

Finally, we define

$$G = \langle G(1) || R = 1; R \in D = \bigcup_{i \geq 1} D_i \rangle.$$

Many embedding constructions based on the scheme given above are particular cases of the schemes in [6] (for groups without involutions) and [7] (in the general case). Here we give a consequence of these embedding schemes which will be used in the next section.

Let $\{G_\mu\}_{\mu \in I}$ be a set of nontrivial groups, Ω^1 the free amalgam of the groups G_μ, $\mu \in I$, and also let $\Omega = \Omega^1 \setminus \{1\}$.

Definition 2.1. A mapping $f : 2^\Omega \setminus \{\emptyset\} \to 2^\Omega$ is called *generating* on the set Ω if the following conditions hold:

1) if $C \subseteq G_\mu$ for some $\mu \in I$, then $f(C) = \text{gp}\{C\} \setminus \{1\}$;

2) if $C \not\subseteq G_\mu$ for each $\mu \in I$ and $C = \{a, b\} \subseteq \Omega$, where a and b are involutions (such a subset of Ω will be called *dihedral*), then $f(C) = C$;

3) if C is a finite non-dihedral subset of Ω and $C \not\subseteq G_\mu$ for each $\mu \in I$, then $f(C) = B$, where B is an arbitrary countable subset of Ω such that $C \subseteq B$ and if D is a finite subset of B, then $f(D) \subseteq B$;

4) if C is an infinite subset of Ω, then $f(C) = \bigcup_{A \in T} f(A)$, where T is the set of all finite subsets of C.

For example, a generating mapping f on Ω can be defined in the following way: if $C \in 2^\Omega \setminus \{\emptyset\}$ and $C = \bigcup_{\mu \in I} C_\mu$, where $C_\mu = C \cap G_\mu$, $\mu \in I$, then $f(C) = \left(\bigcup_{\mu \in I} \text{gp}\{C_\mu\}\right) \setminus \{1\}$ (we assume that $\text{gp}\{C_\mu\} = \{1\}$ if $C_\mu = \emptyset$). It is obvious that in order to define a generating mapping f on Ω, it is sufficient to do it only on finite non-dihedral subsets C of Ω such that $C \not\subseteq G_\mu$ for each $\mu \in I$.

The following result follows immediately from Theorem B in [6] and Theorem B in [7].

Theorem 2.2. *Let $\{G_\mu\}_{\mu \in I}$ be an arbitrary set of nontrivial groups containing either three groups or two groups of which one has order at least 3. H an arbitrary (for example, trivial) group, Ω^1 the free amalgam of the groups H and G_μ, $\mu \in I$, and let*

f be an arbitrary generating mapping on $\Omega = \Omega^1 \setminus \{1\}$. Then the free amalgam Ω^1 can be embedded in a group $G = gp\{\Omega\}$ such that

1) *the free amalgam of the groups G_μ is embedded in a simple normal infinite subgroup L of G and G is the semidirect product of H and L;*

2) *every nontrivial subgroup of L is cyclic or infinite dihedral (if one of the groups G_μ, $\mu \in I$, or H has involutions), or conjugate in G to a subgroup $L_C = R_C \cap L$, where $R_C = gp\{C\}$ for some $C \in 2^\Omega \setminus 2^H$, $a \in \Omega \cap R_C$ if and only if $a \in f(C)$, and if $C \not\subseteq G_\mu$ for each $\mu \in I$, then L_C is simple and $L_C = gp\{cbab^{-1}c^{-1}, b, c \in f(C)\}$ for each $a \in f(C) \setminus H$;*

3) *if $C \not\subseteq G_\mu$ for each $\mu \in I$, then $\operatorname{Aut} L_C \cong R_C$ and $\operatorname{Out} L_C \cong R_C/L_C$ (in particular, $\operatorname{Aut} L \cong G$ and $\operatorname{Out} L \cong H$), and for each $g \in H \cap f(C)$, g is a regular automorphism of L_C (that is, $gag^{-1} = a$ if and only if $a = 1$);*

4) *if $X \in G$ and X is not conjugate in G to an element of one of the groups G_μ, $\mu \in I$, then X is not an involution;*

5) *if all groups G_μ, $\mu \in I$, are torsion-free (respectively, periodic and without involutions and H has no involutions), then G may be chosen so that the subgroup L is torsion-free (respectively, periodic and without involutions) too.*

3. Some applications of Theorem 1.3

First of all, we would like to emphasize that Theorem 1.3 provides an opportunity to reformulate any embedding result based on the schemes of §§34, 36 [10] (with the only restriction that the resulting group G is not of bounded exponent, it is caused by the additional condition R8 in the definition of G (see §2) and helps to avoid examples of groups like an infinite non-topologizable group in [8]) as a topological embedding of a set of discrete groups into a non-discrete Hausdorff topological group.

The detailed discussion of some applications of Theorem A [4] (and therefore of its generalization Theorem 1.3 of the present paper) may be foand in [4]. In this section we generalize some results in [4] to the class of all discrete groups.

Theorems F and G in [4] are devoted to construction of uncountable strongly minimal topological groups. (A non-discrete Hausdorff topologinal group is called *strongly minimal* if every proper subgroup of G is discrete.) Now we have

Theorem 3.1. *Let $\{G_\mu\}_{\mu\in I}$, $|I| > 1$, be an arbitrary set of nontrivial discrete groups such that $\sum_{\mu\in I} |G_\mu| = \aleph_n$ for some positive integer $n \geq 1$. Then the free amalgam Ω^1 of the groups G_μ can be topologically embedded in a simple strongly minimal topological group $G = gp\{\Omega^1\}$ such that if M is a proper subgroup of G and M is not contained in a subgroup conjugate in G to some G_μ, $\mu \in I$, then $|M| < \aleph_n$.*

Theorem 3.2. *Assume, for an infinite cardinal α, there exists a discrete Jonsson group M (that is, M has no proper subgroups of the same cardinality) of cardinality α. Then M can be topologically embedded in a simple Jonsson strongly minimal topological group G of cardinality α^+.*

Proof of Theorems 3.1 *and* 3.1. Repeat the proofs of Theorem F and Theorem G in [4] with references to Theorems A and B in [4] formally replaced by references to Theorem 1.3 and Theorem 2.2 of this paper. □

The following result is a generalization of Theorem H [4] about the groups of outer topological automorphisms of non-discrete groups.

Theorem 3.3. *Let $\{G_\mu\}_{\mu \in I}$ be an arbitrary set of nontrivial discrete groups and H an arbitrary discrete group. Then there is a non-discrete Hausdorff topological group G such that*

1) *the free amalgam of the groups G_μ is topologically embedded in a simple normal open subgroup L of G and G is the semidirect product of H and L;*

2) *the group* Aut L *of topological automorphisms of the group L is algebraically isomorphic to G (and the group* Out L *of outer topological automorphisms of L is algebraically isomorphic to H), and for each $g \in H \setminus \{1\}$, g is a regular automorphism of L;*

3) *G may be chosen to be metrizable.*

Proof. We may assume that $|I| > 1$. Theorem 2.2 applies to the free amalgam Ω^1 of the groups H and G_μ, $\mu \in I$, (and an arbitrary generating mapping f on $\Omega = \Omega^1 \setminus \{1\}$) and yields a group G with a simple normal infinite subgroup L such that 1) the free amalgam of the groups G_μ is embedded in L and G is the semidirect product of H and L, and 2) Aut $L \cong G$ and Out $L \cong H$, and g is a regular automorphim of L for each $g \in H \setminus \{1\}$.

Now Theorem 1.3 can be applied to the resulting group G. It follows from the proof of Lemma 4.1 (see §4) than an open basis at the identity of G can be chosen in such a way that it consists of neighbourhoods contained in the subgroup L. Hence L is an open subgroup of G. It remains to note that every inner algebraic automorphism of a topological group is a homeomorphism. □

By a famous embedding theorem of Higman, Neumann and Neumann, [2], every countable group can be embedded in a 2-generator group. But this embedding construction contains many subgroups other than the embedding group and its conjugates, and there is little information about the automorphism group of the resulting group. In [5] the second author proved a theorem on embeddability of every countable set of countable groups without involutions in a simple 2-generator infinite group in which every propar subgroup is either a cyclic group or contained in a subgroup conjugate to one of the embedding groups, and the generalizations of this theorem to the case of a countable set of arbitrary countable groups were given in [9] and [7]. For countable discrete groups we obtain

Theorem 3.4. *Let $\{G_\mu\}_{\mu \in I}$ be a countable set of nontrivial countable discrete groups containing either three groups or two groups of which one has order at least 3 and H an arbitrary countable discrete group. Then the free amalgam Ω^1 of the groups H and G_μ, $\mu \in I$, can be topologically embedded in a non-discrete Hausdorff topological group $G = gp\{\Omega^1\}$ with the following properties:*

1) *tha free amalgam of the groups G_μ is topologically embedded in a simple normal open strongly minimal topological subgroup L of G and G is the semidirect product of H and L;*

2) *the group* Aut L *of topological automorphisms of L is algebraically isomorphic to G (and the group* Out L *of outer topological automorphisms of L is algebraically isomorphic to H), and g is a regular automorphism of L for each $g \in H \setminus \{1\}$;*

3) *if $X, Y \in L$ with $X \in G_\mu \setminus \{1\}$, $Y \notin G_\mu$ for some $\mu \in I$, then either L is algebraically generated by the pair (X, Y) or X and Y are involutions, or X and XY are involutions;*

4) *every proper subgroup of L is either a cyclic group or infinite dihedral (if one of the groups G_μ, $\mu \in I$, or H has involutions), or contained in a subgroup conjugate in G to some G_μ;*

5) *if $X \in G$ and X is not conjugate in G to an element of one of the groups G_μ, $\mu \in I$, then X is not an involution;*

6) *if all groups G_μ, $\mu \in I$, are torsion-free (respectively, periodic and without involutions and H has no involutions), then G may be chosen such that the subgroup L is torsion-free (respectively, periodic and without involutions) too;*

7) *G may be chasen to be metrizable.*

Proof. We define a generating mapping f on $\Omega = \Omega^1 \setminus \{1\}$ in the following way: if $C \subseteq \Omega$ such that $C \nsubseteq G_\mu$ for each $\mu \in I$, $C \nsubseteq H$ and C is not dihedral (it follows from the statement of the theorem that such a subset C exists), then $f(C) = \Omega$. Then Theorem B applies to Ω^1 and this mapping f and yields a group $G = \text{gp}\{\Omega^1\}$ with a simple normal infinite subgroup L such that 1) the free amalgam of the groups G_μ, $\mu \in I$, is embedded in L and G is the semidirect product of H and L, 2) Aut $L \cong G$ and Out $L \cong H$, and for each $g \in H \setminus \{1\}$, g is a regular antomorphism of L, and 3) G has properties 4-6 in the statement of the theorem. Assertion 3 of the Theorem 3.4 can be proved in the same way as in Theorem 2 [9].

Now Theorem 1.3 supplies the group G with the required non-discrete Hausdorff topology. That the subgroup L is open can be explained in a way as in the proof of Theorem F in [4]. □

4. Proof of Theorem 1.3

For each positive integer $d \geq 1$, we denote by U_d a set of 7-aperiodic cyclically reduced words (over the alphabet Ω) of length $l_d = 2^{d(d+1)}n^d d!$ written in the normal form (that is, every element X in U_d is written in the form $X_1, \ldots, X_t$, where each X_l, $1 \leq l \leq t$, is a nontrivial element of $G_{\mu(l)}$, $\mu(l) \in I$, and $\mu(l) \neq \mu(l-1)$ for $l = 1, \ldots, t-1$) with the properties that $U_d = U_d^{-1}$, where $U_d^{-1} = \{a^{-1},\ a \in U_d\}$, and if $A, B \in U_d$ and $A \not\equiv B$, then every common subword of A and B has length less then $l_d/3$. The words in U_d are called *distinguished words of depth d.*

Lemma 4.1. *For each $d \geq 1$, there is a non-empty set U_d of distinguished words of depth d and moreover, it can be assumed that $|U_d| = |\Omega|$ if the set Ω is infinite.*

Proof. There exists $\mu \in I$ such that $|G_\mu \setminus \{1\}| \leq |\Omega \setminus G_\mu|$. Let a and b be arbitrary fixed elements of $G_\mu \setminus \{1\}$ and $\Omega \setminus G_\mu$, respectively. By Theorem 4.6, [10], there exists a 6-aperiodic word W of length $l_d/2$ in a 2-letter alphabet $\{x, y\}$. If c is an arbitrary element of $\Omega \setminus G_\mu$ such that $b \neq c$ (such a c always exists, since $|\Omega| \geq 3$), then putting $x \to (ab)ac(ab)^2ac(ab)^3ac$, $y \to ac$ yields a cyclically reduced word W_c of length $> l_d$ which is 7-aperiodic relative to Ω. But the word W_c does not contain the subword $(ab)^{-3}(ac)^{-1}(ab)^{-2}$. Hence every common subword X of W_c and W_c^{-1} is of length $|X| < 20 < l_d/3$. Thus the assertion of the lemma is proved if the set Ω is finite.

If the set Ω is infinite, then repeating the previous considerations, we obtain a 7-aperiodic cyclically reduced word W_c of length $> l_d$ for each $c \in \Omega \setminus (G_\mu \cup \{b\})$ and include a cyclically reduced subword V_c of W_c of length l_d together with V_c^{-1} in U_d. It is easy to see, as above, that $A, B \in U_d$ and $A \not\equiv B$ implies that every common subword X of A and B is of length $|X| < 20 < l_d/3$. Now the assertion of the lemma follows from the equation

$$|U_d| = |\Omega \setminus (G_\mu \cup \{b\})| = |\Omega|. \qquad \square$$

The following important reselt about the words in U_d, $d \geq 1$, will be proved in §5.

Lemma 4.2. *Let $\{T_i\}_{1 \leq i \leq f}$, $\{L_j\}_{1 \leq j \leq r}$ be arbitrary sets of distinguished words of depths t and s, respectively, with $f < 2^t$ and $r < 2^s$, and $A_1 T_1^{\varepsilon_1} \ldots A_f T_f^{\varepsilon_f} A_{f+1} = B_1 L_1^{\delta_1} \ldots B_r L_r^{\delta_r} B_{r+1}$ in G, where $|A_i| < 2\beta l_t/5$, $|B_j| < 2\beta l_s/5$ $(1 \leq i \leq f+1$, $1 \leq j \leq r+1)$, $|\varepsilon_i| = |\delta_j| = 1$ $(1 \leq i \leq f, 1 \leq j \leq r)$ and $A_i, B_j \neq 1$ in G if $T_{i-1}^{\varepsilon_{i-1}} \equiv T_i^{-\varepsilon_i}$ and $L_{j-1}^{\delta_{j-1}} \equiv L_j^{-\delta_j}$, respectively, for $i \in \{2, \ldots, f\}$, $j \in \{2, \ldots, r\}$. Then $t = s$, $f = r$, $T_i^{\varepsilon_i} \equiv L_i^{\delta_i}$ and $A_j = B_j$ in G for $i \in \{1, \ldots, f\}$, $j \in \{1, \ldots, f+1\}$.*

For each $d \geq 1$, we assume that $U_d \neq \emptyset$ (it is possible by Lemma 4.1). Let $\Lambda_m = \bigcup_{d=m}^{\infty} U_d$, and also let Π be an arbitrary family of subsets of Λ_1 with the following properties:

i) if $A \in \Pi$, then $A = A^{-1}$ and $A \cap U_d \neq \emptyset$ for an infinite set of positive integers d;

ii) if $A, B \in \Pi$, then $A \cap B \in \Pi$.

Of course, such a set Π exists, for example, $\Pi = \{A_1\}$. For each $A \in \Pi$, we define a family of sets $\Theta_k(A)$, where $k \geq 1$, in the following way.

a) If $T \in A \cap U_d$ for some positive integer d and Z is a minimal word in G (that is, Z is minimal in each rank $i \geq 1$) with $s\gamma l_d/d \leq |Z| < (s+1)\gamma l_d/d$ for some $s \in \{0, 1, \dots, d-1\}$, then we include $Z^{-1}T^{\pm 1}Z$ in $\Theta_{d-s}(A)$. It follows from Lemma 4.2 that if $Z^{-1}T^{\pm 1}Z$ are included in $\Theta_m(A)$ for some $m \geq 1$ in the above-mentioned way, then $Z^{-1}T^{\pm 1}Z \notin \Theta_t(A)$ whenever $t \neq m$.

b) Let $\Theta_k(A)$, $k \geq 1$, be defined as in a) and $L = L_1 \dots L_t$, where for each i, $1 \leq i \leq t$, $L_i = Z_i^{-1}T_i^{\varepsilon_i}Z_i$, $|\varepsilon_i| = 1$, T_i is a distinguished word of some depth $d(i)$ such that $T_i \in A$, and $|Z_i| < \gamma l_{d(i)}$. Now for each $k \geq 1$, we define $s(k) = |\{L_1, \dots, L_t\} \cap \Theta_k(A)|$. If $0 \leq s(k) < 2^k$ for each positive integer k, then we set $L \in \Theta_s(A)$ for $s = \min_{s(k)>0}(k - [\log_2 s(k)])$, where $[l]$ denotes the integer part of l.

It is easy to see that $\Theta_k(A) \supseteq \Theta_{k+1}(A)$ for each $A \in \Pi$ and $k \geq 1$. Now we prove some properties of the sets $\Theta_k(A)$.

Lemma 4.3. *If $T \in \Theta_k(A)$ for some $A \in \Pi$ and $k \geq 1$ and $T \neq 1$ in G, then T may be represented in the unique form $T = C_1T_1^{\varepsilon_1} \dots C_mT_m^{\varepsilon_m}C_{m+1}$, where $|\varepsilon_i| = 1$ and T_i is a distinguished word of some depth d such thet $T_i \in A$, $1 \leq i \leq m$, $d \geq k$, $|C_i| < 2\beta l_d/5$ for each $i \in \{1, \dots, m+1\}$, $m < 2^d$ and $C_i \neq 1$ in G if $T_{i-1}^{\varepsilon_{i-1}} \equiv T_i^{-\varepsilon_i}$, $2 \leq i \leq m$.*

(This form will be called *canonical*.)

Proof. If $T \in \Theta_k(A)$, then by the definition of the set $\Theta_k(A)$, we have that $T = L_1 \dots L_t$, where for each $i \in \{1, \dots, t\}$, $L_i = Z_i^{-1}T_i^{\delta_i}Z_i$, $|\delta_i| = 1$, T_i is a distinguished word of some depth $d(i) \geq k$, $T_i \in A$ and $|Z_i| < \gamma l_{d(i)}$. Let $T_{i_1}^{\varepsilon_1}, \dots, T_{i_m}^{\varepsilon_m}$ ($i_1 < \dots < i_m$ and $|\varepsilon_i| = 1$ for each $i \in \{1, \dots, m\}$) are all distinguished words of the maximal depth d occurring in the expression of T. We may assume that $T \neq 1$ in G and the value of d is minimal for all such expressions of T. It follows from the definition of the set $\Theta_k(A)$ that $m < 2^d$ and $d \geq k$.

By the definition of the set $\Theta_k(A)$, the total length of all L_i, where $i \in \{1, \dots, t\} \setminus \{i_1, \dots, i_m\}$ does not exceed

$$(d-1)2^{d-1}(1+2\gamma)l_{d-1} < \gamma l_d.$$

So T can be represented as a product $C_1T_{i_1}^{\varepsilon_1} \dots C_mT_{i_m}^{\varepsilon_m}C_{m+1}$, where

$$|C_s| < 3\gamma l_d < 2\beta l_d/5$$

for each $s \in \{1, \dots, m+1\}$. After corresponding reductions we may assume that $C_s \neq 1$ in G if $T_{i_{s-1}}^{\varepsilon_{s-1}} \equiv T_{i_s}^{-\varepsilon_s}$, $2 \leq s \leq m$. □

The uniqueness of such a presentation follows from Lemma 4.2.

Lemma 4.4. *For each $A \in \Pi$, $\bigcap_{k=1}^{\infty} \Theta_k(A) = \{1\}$.*

Proof. Let $k \geq 2$ and T be an arbitrary distinguished word of depth $d \geq k$ such that $T \in A$. (The existence of such a word T follows from tha definition of Π.) Then $TT^{-1} = 1 \in \Theta_{k-1}(A)$ and $\bigcap_{k=1}^{\infty} \Theta_k(A) \supseteq \{1\}$.

Let C be an arbitrary nontrivial element of $\Theta_1(A)$ and $C_1 T_1^{\varepsilon_1} \dots C_m T_m^{\varepsilon_m} C_{m+1}$ the canonical form of C. If d is tha depth of tha distinguished words $T_1, \dots, T_m$, then it follows from the definition of the sets $\Theta_k(A)$ and Lemma 4.3 that $C \notin \Theta_k(A)$ for each $k > d$. Thus $\bigcap_{k=1}^{\infty} \Theta_k(A) = \{1\}$ as required. □

Lemma 4.5. $\Theta_{m+1}(A)\Theta_{m+1}^{-1}(A) \subseteq \Theta_m(A)$ *for each $A \in \Pi$ and $m \geq 1$.*

Proof. It follows from the definition of the set $\Theta_{m+1}(A)$ that $\Theta_{m+1}(A) = \Theta_{m+1}^{-1}(A)$. Let a and b be arbitrary elements of $\Theta_{m+1}(A)$, $s_1(k)$, $s_2(k)$ and $s(k)$ the numbers $s(k)$ from tha definition of tha set $\Theta_{m+1}(A)$ for some presentations of the elements a, b and the corresponding presentation of ab, respectively. Then by the definition of $\Theta_{m+1}(A)$, we have that $s_1(k) = s_2(k) = 0$ for $k \in \{1, \dots, m\}$ and $0 \leq s_1(k), s_2(k) < 2^{k-m}$ for each $k \geq m+1$. Hence

$$s(k) = s_1(k) + s_2(k) < 2^{k-m+1}$$

for each $k \geq m+1$ and $s(k) = 0$ for $k \in \{1, \dots, m\}$. Then it follows from the definition of $\Theta_m(A)$ that $ab \in \Theta_m(A)$ as required. □

Lemma 4.6. *If $X \in \Theta_m(A) \setminus \Theta_{m+1}(A)$ for some $A \in \Pi$ and $m \geq 1$, and tbe distinguished words $T_1, \dots, T_k$ in the canonical form $C_1 T_1^{\varepsilon_1} \dots C_k T_k^{\varepsilon_k} C_{k+1}$ of X are of depth d, then $X\Theta_{d+1}(A) \subseteq \Theta_m(A) \setminus \Theta_{m+1}(A)$.*

Proof. Let Y be an arbitrary element of $\Theta_{d+1}(A)$, $s_1(k)$, $s_2(k)$ and $s(k)$ the numbers $s(k)$ for the canonical forms of Y, X and the corresponding presentation of XY, respectivaly (see the definition of the sets $\Theta_{d+1}(A)$ and $\Theta_m(A)$). Then it follows from the definition of tha set $\Theta_{d+1}(A)$ that $s_1(k) = 0$ for each $k \in \{1, \dots, d\}$ and $0 \leq s_1(k) < 2^{k-d}$ in the case $k \geq d+1$, and by Lemma 4.3, the statement of the lemma and the definition of the set $\Theta_m(A)$, we have that $s_2(l) = 0$ for $1 \leq l < m$ or $l \geq d+1$, and $0 \leq s_2(l) < 2^{l-m+1}$ for each $l \in \{m, \dots, d\}$. Thus $s(k) = s_2(k)$ for $k \in \{1, \dots, d\}$ and $s(k) = s_1(k)$ whenever $k \geq d+1$. Hence it follows from the statement of the lemma and the definition of tbe sets $\Theta_m(A)$ and $\Theta_{d+1}(A)$ that

$$m = \min_{s_2(k)>0} (k - [\log_2 s_2(k)]) < d + 1 = \min_{s_1(k)} (k - [\log_2 s_1(k)]),$$

hence $XY \in \Theta_m(A)$. Now by Lemma 4.2 and the choice of the integer d, we have that $XY \notin \Theta_{m+1}(A)$, which completes the proof of the lemma. □

Lemma 4.7. *For each $A \in \Pi$, $m \geq 1$ and $X \in G$, there exists a positive integer q such that $X^{-1}\Theta_q(A)X \subset \Theta_m(A)$.*

Proof. Let q be an arbitrary positive integer such that $q > m + 2$ and $|X| < \gamma l_q/q$, and let b be an arbitrary element of $\Theta_q(A)$. Then it follows from the definition of the set $\Theta_q(A)$ that $b = L_1 \ldots L_t$ with $L_i = Z_i^{-1} T_i^{\varepsilon_i} Z_i$ for each i, $1 \leq i \leq t$, where $|\varepsilon_i| = 1$, T_i is a distinguished word of some depth $d(i) \geq q$ such that $T_i \in A$,

$$0 \leq |Z_i| < (d(i) - q + 1)\gamma l_{d(i)}/d(i), \tag{4}$$

and $|\{L_1, \ldots, L_t\} \cap \Theta_k(A)| = s(k)$, where $s(k) = 0$ for each $k < q$ and $0 \leq s(k) < 2^{k-q+1}$ in the case $k \geq q$.

Now we consider the element $X^{-1}bX = (X^{-1}L_1X)\ldots(X^{-1}L_tX)$. By (4),

$$|Z_iX| < (d(i) - q + 2)\gamma l_{d(i)}/d(i)$$

for each $i \in \{1, \ldots, t\}$, since $l_p/p \leq l_r/r$ whenever $p \leq r$. Then by the definition of the sets $\Theta_k(A)$, $k \geq 1$, if $L_i \in \Theta_k(A)$ for some $k \geq 1$, then either $X^{-1}L_iX \in \Theta_{k-1}(A)$ or $X^{-1}L_iX \in \Theta_k(A)$. Hence if $s_1(k) = |\{X^{-1}L_1X, \ldots, X^{-1}L_tX\} \cap \Theta_k(A)|$ for each $k \geq 1$, then $s_1(k) = 0$ for each $k < q - 1$, $s_1(q-1) \leq s(q)$ and

$$0 \leq s_1(k) \leq s(k) + s(k+1) < 2^{k-q+1} + 2^{k-q+2} < 2^{k-q+3}$$

for each $k \geq q$. Thus it follows from the definition of the set $\Theta_{q-2}(A)$ that $X^{-1}bX \in \Theta_{q-2}(A) \subset \Theta_m(A)$ as required. □

Lemma 4.8. *For each $A, B \in \Pi$ and arbitrary positive integers m and q, $\Theta_m(A) \cap \Theta_q(B) \supseteq \Theta_k(A \cap B)$, where $k = \max(m, q)$.*

Proof. The assertion of the lemma follows immediately from the definition of the sets $\Theta_s(C)$, where $s \geq 1$ and $C \in \Pi$. □

The last lemma of this section will also be proved in §5.

Lemma 4.9. *If T is a period of some rank or a simple word in G (that is, a simple word in each rank $i \geq 1$), then there exists a positive integer k such that $T^p \notin \Theta_k(A)$ for each $A \in \Pi$ and $p \geq 1$.*

We turn now to the proof of Theorem 1.3.

Proof. Let Π be an arbitrary set from the beginning of this section. It follows from Lemmas 4.4–4.8 that G admits a unique non-discrete Hausdorff topology such that a family $\Theta = \{\Theta_k(A),\ A \in \Pi, k \geq 2\}$ is an open basis at 1 (see, for example, Theorem 4.5 [1]).

Let X be an arbitrary element of G, $A \in \Pi$ and $k \geq 2$. Then by the definition of the set $\Theta_k(A)$, there exists a distinguished word T of some depth $d \geq k$ such that $T \in \Theta_k(A)$ and $|X| < \gamma l_d/d$. Hence $X^{-1}TX \in \Theta_k(A)$, and it follows from Lemma 4.3 that tha set $\Theta_k(A)$ is of cardinality $|G|$. Therefore, every neighbourhood of G is of cardinality $|G|$.

Let $X \in G \setminus \Theta_k(A)$ for some $A \in \Pi$ and $k \geq 2$. If $X \notin \Theta_{k-1}(A)$, then it follows from Lemma 4.5 that $X\Theta_k(A) \cap \Theta_k(A) = \emptyset$, otherwise $X \in \Theta_{k-1}(A) \setminus \Theta_k(A)$ and by Lemma 4.6, there is a positive integer d such that $X\Theta_{d+1}(A) \cap \Theta_k(A) = \emptyset$. Hence the set $\Theta_k(A)$ is closed and a group G is 0-dimensional.

Let X be an arbitrary element of G_μ for some $\mu \in I$. If $X \in \Theta_k(A)$ for some $A \in \Pi$ and $k \geq 2$, then we arrive at a contradiction to Lemma 4.3 and Lemma 4.2, since $|X| = 1$. Thus G_μ is a discrete subgroup of G. Moreover, every inner algebraic automorphism of G is a homeomorphism, then if M is conjugate to a subgroup of G_μ for some $\mu \in I$, then M is discrete.

Let $M = \text{gp}\{A\}$ be a cyclic subgroup of G not conjugate to a subgroup of G_μ for some $\mu \in I$. Then by Theorem 34.7 [10], A is conjugate in G to a power of a period T of some rank or to a power of a simple word T in G. It follows from Lemma 4.9 that $\text{gp}\{T\}$, and therefore also M, is a discrete subgroup of G. Since every cyclic subgroup of G is discrete, it is obvious that every finite extension of a cyclic subgroup is discrete as well.

Let β be an arbitrary cardinal number such that $\beta < |G|$. If β is finite, then, of course, every subgroup of G of cardinality $\leq \beta$ is discrete, and in order to obtain the group G to be metrizable, it is sufficient to take a family $\Theta = \{\Theta_k(\Lambda_2), k \geq 2\}$ as an open basis at 1.

Hence we may consider the case when β is infinite, and therefore, the set Ω is infinite too. Then by Lemma 4.1, we may assume that $|U_d| = |\Omega| = |G|$ for each $d \geq 1$.

A set Π is defined in the following way:

$$\Pi = \{A : A \subseteq \Lambda_1, A = A^{-1}, |\Lambda_1 \setminus A| \leq \beta\}.$$

Now for each $A \in \Pi$ and $d \geq 1$, we beve that $A \cap U_d \neq \emptyset$, since otherwise $\Lambda_1 \setminus A \supseteq U_d$ and $|\Lambda_1 \setminus A| = |G| > \beta$. Let $A, B \in \Pi$. Then $A \cap B \subseteq \Lambda_1$ and

$$|\Lambda_1 \setminus (A \cap B)| \leq |\Lambda_1 \setminus A| + |\Lambda_1 \setminus B| \leq \beta.$$

Thus the set Π satisfies both conditions from the beginning of this section, and G admits a non-discrete Hausdorff topology such that a family $\{\Theta_k(A),\ A \in \Pi, k \geq 2\}$ is an open basis at 1.

Let M be an arbitrary subgroup of G of cardinality $\gamma \leq \beta$, $K = \Theta_2(\Lambda_1) \cap M$. If $K = \{1\}$, then $M \cap \Theta_2(\Lambda_1) = \{1\}$ and M is discrete. Otherwise let S be the set of all distinguished words occurring in the expressions of the canonical forms of all nontrivial elements of K. Then it follows from Lemma 4.3 that $|S| \leq \max(\aleph_0, K)$, hence $|S| \leq \beta$. Now $\Lambda_1 \setminus (S \cup S^{-1}) \in \Pi$ and by Lemma 4.3, and Lemma 4.2, $M \cap \Theta_2(\Lambda_1 \setminus (S \cup S^{-1})) = \{1\}$. Hence M is a discrete subgroup of G, which completes the proof of the theorem. □

5. Proofs of Lemma 4.2 and Lemma 4.9

Before embarking on the proofs, we need some auxiliary results. All definitions and notation which are not introduced in this paper may be found in [10].

By a *diagram of rank i*, where $i \geq 2$, we mean a diagram over the presentation (3). Relators of the first type correspond, in the diagrams under consideration, to *cells of the first type* whose contour is taken as one long cyclic section. But if a cell Π corresponds to a word of the form (2), then it is called a *cell of the second type*. Those sections of Π with labels $A^{\pm n_i}$ are called *long sections* while the others (with labels $T_k^{\pm 1}$) are called *short sections* of the contour.

By an *H-diagram* we understand a circular B-diagram Δ whose contour has the form $p_1 s_1 \ldots p_k s_k p_{k+1} q$, where $s_1, \ldots s_k$ are called *long sections of the first kind*, $p_1, \ldots p_{k+1}$ *short sections*, and q a *long section of the second kind*. All sections are assumed reduced and, for some positive integer j, the following conditions hold.

H1. All long sections of the first kind and short sections are geodesic.

H2. For each $i \in \{1, \ldots, k\}$, $|s_i| = 0$ or the label of s_i is a 7-aperiodic reduced word.

H3. $k > 3$ and $|s_2|, \ldots, |s_{k-1}| > j/10$.

H4. $|p_1| + \ldots + |p_{k+1}| < \beta(k+1)j$.

H5. If p is a section of a cell, then there are contiguity submaps of p to at most 10 distinct sections of $\partial\Delta$ (by meaning the standard partition of the contour).

H6. The contiguity degree of any contiguity submap of a long section of the first kind to q is less than $9/10$.

H7. The long section q of the second kind is either smooth or geodesic.

A contiguity submap Γ of zero rank of t to s in any diagram Δ, where t and s are sections of cells or of $\partial\Delta$, is called *maximal* if Δ has no distinct contiguity submaps of t to s of zero rank containing Γ.

Lemma 5.1. *In any H-diagram Δ, there is a maximal contiguity submap Γ of zero rank of s_i to s_j, where $1 \leq i < j \leq k$.*

Proof. We define tha distinguished contiguity submaps in an H-diagram Δ in the same way as for C-maps (see §23 [10]). The Ω-edges of the contigeity arcs of s_i to s_j, where $1 \leq i, j \leq k$, for the distinguished submaps are called *outer* edges while all the other edges are called *inner*. The weight of the edges belonging to the contours of the cells is left unchanged. Moreover, we define the weight of an edge e of a long section s of the first kind in $\partial\Delta$ by

$$\nu(e) = |s|^{-1/3}.$$

The weights of other edges are set equal to zero. The weights of paths, cells and submaps are defined as in §21 [10].

1) Let Γ be a contiguity submap of a cell π to a long section s of the first kind. If $(\pi, \Gamma, s) \geq \varepsilon$, then by the proof of Theorem 22.2 [10], there are a long section p of a cell π_1 and a contiguity submap Γ_1 of p to s such that $r(\Gamma_1) = 0$ and $(p, \Gamma_1, s) \geq \varepsilon$. We arrive at a contradiction to H2 and the choice of defining relators in G. Thes $(\pi, \Gamma, s) < \varepsilon$.

2) Let Γ be a contiguity submap of a long section s of the first kind to t, where t is a section of a cell or of the contour of Δ. Then it follows from the definition of contiguity submaps and item 1) that Γ is a 0-contiguity submap and $\partial\Gamma = l_1 l_2$, where l_1, l_2 are subpaths of s and t, respectively. If $r(\Gamma) > 0$, then by Corollary 22.1 [10], Γ has a D-cell π and disjoint contiguity submaps Γ_1, Γ_2 of π to l_1, l_2 such that $(\pi, \Gamma_1, l_1) + (\pi, \Gamma_2, l_2) > \gamma'$. It follows from Lemma 21.7 [10] that $(\pi, \Gamma_1, l_1) > \gamma' - \alpha' > \varepsilon$, which contradicts item 1). Thus $r(\Gamma) = 0$.

3) Let Γ_1, Γ_2 be contiguity submaps of a long section s of the first kind to t, where t is a section of a cell or of the contour of Δ. By item 2), $r(\Gamma_1), r(\Gamma_2) = 0$. Assume first that these submaps are not disjoint. Then it is obvious that there exists a contiguity submap of zero rank of s to t containing Γ_1 and Γ_2. If Γ_1 and Γ_2 are disjoint contiguity submaps of s to t, then they are the bonds of a 0-contiguity submap Γ_3 of s to t. It follows from item 2) that $r(\Gamma_3) = 0$, and Γ_1, Γ_2 are submaps of Γ_3. Therefore, a maximal contiguity submap of s to t is unique.

4) Let Γ be a distinguished contiguity submap of a long section p of a cell to a long section s of the first kind. Then by item 2), $r(\Gamma) = 0$ and it follows from H2 and the choice of defining relators in G that $(p, \Gamma, s) < \varepsilon$.

5) If Γ is a contiguity submap of a long section s of the first kind to t, where t is a section of a cell or of the contour of Δ, then by item 2), $r(\Gamma) = 0$, and so Γ has no special cells. Then repeating the proof of Lemma 21.9 [10], we obtain that the sum H_0 of the weights of all the special cells of an H-diagram Δ is at most $\alpha^{-1}\varepsilon\nu(\Delta)$.

6) Let Γ be a distinguished contiguity submap of a long section q_1' of a cell Π to a section q_2' of $\partial\Delta$ with $(q_1', \Gamma, q_2') < \varepsilon$, K_Γ the sum of the weights of the edges in q_1 and q_2, where $p_1 q_1 p_2 q_2 = \partial(q_1', \Gamma, q_2')$. Then $\nu(q_1) < \varepsilon\nu(q_1')$. In order to evaluate $\nu(q_2)$, we may assume that q_2' is a long section of the first kind (otherwise $\nu(q_2) = 0$). Hence by item 2), $|q_2| = |q_1| < \varepsilon|q_1'|$, and we have that

$$\nu(q_2) = |q_2||q_2'|^{-1/3} \leq |q_2|^{2/3} < (\varepsilon|q_1'|)^{2/3} = \varepsilon^{2/3}\nu(q_1').$$

Thus $K_\Gamma = \nu(q_1) + \nu(q_2) < 2\varepsilon^{2/3}\nu(q_1')$.

Let K_0 be the sum of the numbers K_Γ as Γ runs through all such submaps Γ in Δ. By definition, all Ω-edges of cells are inner, and it follows from H5 that $K_0 < 20\varepsilon^{2/3}M$, where M is the sum of the weights of all the inner edges in Δ.

7) Let C_0 and G_0 be defined for an H-diagram in the same way as in Lemmas 23.9 and 23.11 [10] for a C-map, respectively. Then, as in Lemmas 23.9 and 23.11 [10], we obtain that $C_0 \le 3\alpha^{-1}\delta^{2/3}\nu(\Delta)$ and $G_0 \le \alpha' M$.

8) Let Γ be a distinguished contiguity submap of a short section p of a cell Π of rank k to a long section s of the first kind of $\partial\Delta$. Set $\partial(p,\Gamma,s) = p_1q_1p_2q_2$, and let L be the sum of the $\nu(q_2) = \nu(q_2^\Gamma)$ over all such submaps Γ. By item 2) and the choice of defining relators in G, $|q_2| = |q_1| < dk$. Hence

$$\nu(q_2) < |q_2|^{2/3} < (dk)^{2/3} = (d\iota)^{2/3}(nk)^{2/3} \le (d\iota)^{2/3}\nu(\Pi).$$

Taking into account H5 and the fact that the number of short sections of Π is not greater then h, we obtain $L \le 20h(d\iota)^{2/3}M \le \alpha M$.

9) Let Γ be a distinguished contiguity submap of a short section p of a contour to a long section s of the first kind with $\partial(p,\Gamma,s) = p_1q_1p_2q_2$, F the sum of the weights of all such submaps Γ. By item 2), $|q_1| = |q_2|$ and $F = \sum \nu(q_2)$. It follows from H3, H4 and the definition of the weight function that

$$F < 2(\beta(k+1)j)^{2/3} + (\beta(k+1)j)(j/10)^{-1/3}$$

and

$$F/\nu(\Delta) < (2(\beta(k+1))^{2/3} + 10^{1/3}\beta(k+1))((k-2)10^{-2/3})^{-1}$$

$$= 2(k+1)^{2/3}(k-2)^{-1}10^{2/3}\beta^{2/3} + 10(k+1)(k-2)^{-1}\beta < 30\beta^{2/3}.$$

10) Let Γ be a contiguity submap of a long section s of the first kind to s. Then by item 2), $r(\Gamma) = 0$, and we arrive at a contradiction to H1.

11) Repeating the proof of Lemma 23.14 [10] (and using H6 and the estimates from items 5)–9)), we obtain that the sum of the weights of all inner edges in an H diagram Δ is less than $(\delta^{1/2} + 30\beta^{2/3} + 9/10)\nu(\Delta)$. Hence the sum of the weights of all outer edges in Δ is greater than

$$(1 - \delta^{1/2} - 30\beta^{2/3} - 9/10)\nu(\Delta) > 0,$$

and the assertion of the lemma follows immediately from items 3) and 10). □

Lemma 5.2. *Any H-diagram Δ contains a contiguity submap Γ of zero rank of s_i to s_{i+1} for some $i \in \{1, \dots, k-1\}$.*

Proof. We proceed by induction on k. For $k = 3$, it follows from Lemma 5.1 that there is a maximal contiguity submap Γ of zero rank of s_i to s_j, where $1 \le i < j \le 3$. If i or j is equal to 2, then the assertion of the lemma is proved in this case, otherwise we have that $i = 1$ and $j = 3$. Hence Δ contains a subdiagram Δ_1 with contour $s_1' p_2 s_2 p_3 s_3'$, where s_1' and s_3' are subpaths of s_1 and s_3, respectively. Of course, Δ_1 is an H-diagram and by the maximality of Γ, there is no contiguity submaps of s_1' to s_3'. Now we derive the assertion of the lemma for Δ_1, and therefore also for Δ, from Lemma 5.1.

Now let $k > 3$. By Lemma 5.1, there is a maximal contiguity submap Γ of zero rank of s_i to s_j, where $1 \le i < j \le k$. If $j = i + 1$, then the assertion is proved, otherwise Δ consists of two subdiagrams Δ_1 and Δ_2 which are joint in Δ by subpaths of s_i and s_j. It is easy to see that at laast one of the subdiagrams Δ_1 and Δ_2, say Δ_1, is an H-diagram with the standard partition of the contour $p_1' s_1' \dots p_t' s_t' p_{t+1}' q'$, where $|p_1'| = |p_{t+1}'| = |q'| = 0$, and either $t < k$ or $i = 1$ and $j = k$. In the second case, it follows from the maximality of Γ and Lemma 5.1 for Δ_1 that the assertion of Lemma 5.2 is true for Δ_1 or Δ_1 contains an H-subdiagram Δ_3 in which the number of long sections of the first kind is less than k. By the induction hypothesis we may assume that the lemma is true for Δ_1 (or Δ_3)and therefore also for Δ.

By an *L-diagram* we mean a circular B-diagram Δ with contour $p_1 s_1 \dots p_k s_k p_{k+1} q$, where $s_1, \dots, s_k$ and q are celled *long sections of the first* and *second kind*, respectivaly, $p_1, \dots, p_{k+1}$ are called *short sections*, all sections are reduced, Δ satisfies H1, H2, H6 and H7, and, for some positive integer j, the following conditions hold:

L1. $k \le j$;

L2. $|s_i| \le j$ for each $i \in \{1, \dots, k\}$;

L3. $k \ge 3$ and $|s_2|, \dots, |s_{k-1}| > j/4$;

L4. $|p_i| < \beta j$ for each $i \in \{1, \dots, k+1\}$. □

Lemma 5.3. *In any L-diagram Δ, there exists a contiguity submap Γ of zero rank of s_i to s_{i+1} for some $i \in \{1, \dots, k-1\}$.*

Proof. To obtain the assertion of the lemma we need induction on k.

1) If Δ satisfies H5 (in particular, if $k = 3, 4$), then Δ is an H-diagram, and the assertion of the lemma follows from Lemma 5.2.

2) Let Γ be a contiguity submap of a long section p of a cell Π of rank i to a long section s of the first kind, $t = p_1 s_1 \dots p_k s_k p_{k+1}$. Repeating the proof of Lemma 23.16 [10], we can define a contiguity submap Γ_1 of Π to $\partial\Delta = tq$ such that $(\Pi, \Gamma_1, tq) > 1 - 3\beta$. By Lemma 21.7 [10] and H7, the contiguity degree of Π to q is lass then α', hence there exists a contiguity submap Γ_2 of Π to t with $(\Pi, \Gamma_2, t) > 1 - \alpha' - 3\beta$, and it follows from Lemma 21.3 [10] that

$$|t| > (1 - \alpha' - 5\beta)|\partial\Pi| > |\partial\Pi|/3. \tag{5}$$

On the other hand, it follows from the choice of defining relations (1) and (2) of G that $|\partial\Pi| \geq hni^2$, and by L1, L2 and L4,

$$|t| < (k + \beta(k+1))j < (1+2\beta)j^2. \tag{6}$$

Hence

$$i < (3(1+2\beta)\delta\iota)^{1/2}j < j/1000.$$

By item 2) from the proof of Lemma 5.1, $r(\Gamma) = 0$ and it follows from H2 that the length of $\Gamma \wedge p$ is less then $10i < j/100$.

3) Let Γ be a contiguity submap of a short section p of a cell Π of rank i to a long section s of the first kind of $\partial\Delta$. Then it follows from the choice of defining relators in G that $|p| < \delta d\iota i^{-1}|\partial\Pi|$, and by (5) and (6),

$$|p| < 3(1+2\beta)\delta d\iota i^{-1} j^2. \tag{7}$$

First we consider the case $j > 100di$. Then it follows from the choice of defining relators in G that

$$|p| < di < j/100.$$

We now assume that $j \leq 100di$. Then by (7),

$$|p| < 300(1+2\beta)\delta d^2\iota j < j/100.$$

Hence, in either case, $|p| < j/100$.

By item 2) from tha proof of Lemma 5.1, $r(\Gamma) = 0$, and moreover, we have that the length of $\Gamma \wedge p$ is not greater than $|p| < j/100$.

4) Let $k \geq 5$ and Δ does not satisfy H5, that is, there exists a cell Π of rank i with a section p such that there are contiguity submaps of p to at least 11 distinct sections of $\partial\Delta$. Then it follows from items 2) and 3) that at least one of the following cases is possibla.

a) There exists a subdiagram Δ_1 of Δ with contour $l_1t_1l_2p_1$, where l_1, l_2 are subpaths of long neighbouring sections of the first kind (it is possihle that $|l_i| = 0$ for some $i \in \{1, 2\}$), l_1 and l_2 are not simultaneously subpaths of s_1 and s_k (or s_k and s_1), respectively, p_1 and t_1 are subpaths of p and a short section of $\partial\Delta$, respectively, there is no contiguity submaps of l_1, l_2 to p_1 and

$$\max(|l_1|, |l_2|) > (1/4 - 1/100)j/2 = 3j/25.$$

Hence Δ_1 is an H diagram, and by Lemma 5.2, $|l_1|, |l_2| > 0$ and there is a contiguity submap of zero rank of l_1 to l_2, which completes the proof of the lemma in this case.

b) There exists a subdiagram Δ_1 of Δ with contour $t_1l_1 \ldots t_fl_ft_{f+1}p_1$, where $3 \leq f < k$, sections $t_1, \ldots t_{f+1}$ and p_1 are subpaths of short sections and p, respectively, l_1, l_f are subpaths of long sections of the first kind, $l_2, \ldots l_{f-1}$ are long sections of the first kind of lengths greater than $j/4$, such that there is no contiguity submaps of sections $l_1, \ldots, l_f$ to p_1. Then Δ_1 is a L-diagram with $f < k$, and we can assume that the lemma is true for Δ_1 and therefore also for Δ. □

By a *K-diagram* we understand a circular B-diagram Δ whose contour has the form $p_1s_1 \dots p_ks_kp_{k+1}q$, where $s_1, \dots, s_k$ and $p_1, \dots, p_{k+1}$ are called *long sections of the first kind* and *short sections*, respectively, q is called a *long section of the second kind*. All sections are assumed reduced, Δ satisfies H1, H7 and, for some positive integer j, the following conditions hold:

K1. the label of each long section of the first kind is a 7-aperiodic reduced word;

K2. $|s_i| = j$ for each $i \in \{1, \dots, k\}$;

K3. the contiguity degree of any contiguity submap of a long section of the first kind to q is less than 1/5;

K4. $k \leq j$;

K5. the length of each short section is less then βj.

The main auxiliary lemma is

Lemma 5.4. *In any* K*-diagram* Δ, $k > 1$ *and for some* $i \in \{1, \dots, k-1\}$, *there is a contiguity submap* Γ *of zero rank of* s_i *to* s_{i+1} *such that* $(s_i, \Gamma, s_{i+1}) > 1/3$ *and the lengths of the initial segments of paths* s_i^{-1} *and* s_{i+1} *to the initial paint of* $\Gamma \wedge s_{i+1}$ *are less than* βj.

Proof. 1) Let Γ be a contiguity submap of a cell π to a long section s of the first kind. Then, as in item 1) from the proof of Lemma 5.1, we have that $(\pi, \Gamma, s) < \varepsilon$.

2) If $k = 1$, then $\partial\Delta = p_1's_1'p_1s_1p_2s_2'p_2'q$ with $|p_1'| = |s_1'| = |s_2'| = |p_2'| = 0$, and Δ is an H-diagram. We arrive at a contradiction to Lemma 5.1.

3) Let Γ be a contiguity submap of a long section s of the first kind to t, where t is a section of a cell or of the contour of Δ. Then repeating the proofs of items 2) and 3) from the proof of Lemma 5.1, we obtain that Γ is a submap of the unique maximal contiguity submap (of zero rank) of s to t.

4) Let Γ be a maximal contiguity submap of zero rank of s_i to s_{i+1} for some $i \in \{1, \dots, k-1\}$. Hence there is a subdiagram Δ_1 of Δ with contour $t_1p_{i+1}t_2$, where t_1^{-1} and t_2 are the initial segments of paths s_i^{-1} and s_{i+1}, respectively, to the initial point of $\Gamma \wedge s_{i+1}$. If $r(\Delta_1) > 0$, then by Corollary 22.1 [10], Δ_1 has a D-cell π and disjoint contiguity submaps $\Gamma_1, \Gamma_2, \Gamma_3$ of π to t_1, t_2 and p_{i+1}, respectively, such that $(\pi, \Gamma_1, t_1) + (\pi, \Gamma_2, t_2) + (\pi, \Gamma_3, p_{i+1}) > \gamma'$. It follows from Lemma 21.7 [10] that $(\pi, \Gamma_3, p_{i+1}) < \alpha'$, then there is $s \in \{1, 2\}$ such that $(\pi, \Gamma_s, t_s) > (\gamma' - \alpha')/2 > \varepsilon$, which contradicts item 1). Hence $r(\Delta_1) = 0$ and it follows from the maximality of Γ that $|t_s| \leq |p_{i+1}| < \beta j$ for each $s \in \{1, 2\}$.

5) By items 2)–4), it remains to prove that for some $i \in \{1, \dots, k-1\}$, there exists a contiguity submap Γ of s_i to s_{i+1}, such that $(s_i, \Gamma, s_{i+1}) > 1/3$.

Assuming the contrary, we have that there is no $i \in \{1, \dots, k-1\}$ such that there exists a contiguity submap Γ of s_i to s_{i+1} with $(s_i, \Gamma, s_{i+1}) > 1/3$. If there is a contiguity submap Γ of s_1 to s_2, then by item 3), there exist a maximal contiguity submap Γ_1 (of zero rank) of s_1 to s_2 and a subdiagram Δ^1 of Δ with contour $t_1 p_2 t_2$, where t_1^{-1} and t_2 are the initial segments of paths s_1^{-1} and s_2, respectively, to the initial point of $\Gamma_1 \wedge s_2$. Excising Γ_1 and Δ^1 from Δ, we obtain a diagram Δ_1. Then by repeating the same trick for each $i \in \{2, \dots, k-1\}$, we obtain a B-diagram $\Delta_{k-1} = \Delta'$ with contour $p'_1 s'_1 \dots p'_k s'_k p_{k+1} q$, where p'_i and s'_i are subpaths of p_i and s_i, respectively, $1 \le i \le k$ (and it is possible thet $|p'_i| = 0$ for some $i \in \{1, \dots, k\}$), such that there is no contiguity submaps of s'_i to s'_{i+1}, $1 \le i \le k-1$, and it follows from our assumption and item 4) that

$$|s'_i| > (1 - 2(1/3 + \beta))j > j/4 \quad (8)$$

for each $i \in \{1, \dots, k\}$.

It is obvious thet Δ' satisfies H1, H2, H7, and L1–L4. Moreover, if Γ is a contiguity submap of s'_i to q for some $i \in \{1, \dots, k\}$, then by K3, $|\Gamma \wedge s'_i| \le |\Gamma \wedge s_i| < j/5$, and it follows from (8) that

$$(s'_i, \Gamma, q) = |\Gamma \wedge s'_i|/|s'_i| < (j/5)/(j/4) = 4/5.$$

Hence Δ' is a L-diagram, and we arrive at a contradiction to Lemma 5.3.

The proof of Lemma 5.4 is complete. □

Now all necessary machinery has been developed, and we pass to the proof of Lemma 4.2.

Proof. Assume first that $t \neq s$ and consider, for example, the case $t < s$. Then it follows from the statement of the lemma and the definition of distinguished words thet

$$|A_1 T_1^{\varepsilon_1} \dots A_f T_f^{\varepsilon_f} A_{f+1}| < 2^t (1 + 2\beta/5) l_t < \beta l_s/5 \, . \quad (9)$$

Let Δ be a reduced circular diagram for an equation

$$D L_1^{\delta_1} B_2 L_2^{\delta_2} \dots B_r L_r^{\delta_r} = 1,$$

where $D = B_{r+1}(A_1 T_1^{\varepsilon_1} \dots A_f T_f^{\varepsilon_f} A_{f+1})^{-1} B_1$, with contour $p_1 s_1 \dots p_r s_r$, where $\phi(s_i) = L_i^{\delta_i}$, $\phi(p_1) \equiv D$ and $\phi(p_j) \equiv B_j$ for each $i \in \{1, \dots, r\}$, $j \in \{2, \dots, r\}$. We may assume that $D, B_2, \dots, B_r$ are minimal words in G, hence Δ is a K-diagram with long sections s_i of the first kind ($1 \le i \le r$) and $|p_{r+1}| = |q| = 0$, since by the statement of the lemma and (9), $r < 2^s < l_s$ and

$$|D| < (4/5 + 1/5)\beta l_s = \beta l_s.$$

By Lemma 5.4, $r > 1$ and for some $i \in \{1, \dots, r-1\}$, there is a contiguity submap Γ of zero rank of s_i to s_{i+1} such that $(s_i, \Gamma, s_{i+1}) > 1/3$ and the lengths of the initial segments of paths s_i^{-1} and s_{i+1} to the initial point of $\Gamma \wedge s_{i+1}$ are less than βl_s. Then there are decompositions of the words $L_i^{-\delta_i}$ and $L_{i+1}^{\delta_{i+1}}$ such that $L_i^{-\delta_i} = XYZ$,

$L_{i+1}^{\delta_{i+1}} = X_1 Y Z_1$ in rank 1, where $|X|, |X_1| < \beta l_s$ and $|Y| > l_s/3$. By the definition of a set U_s, we have that $L_i^{-\delta_i} = L_{i+1}^{\delta_{i+1}}$ in rank 1.

Let $X \neq X_1$ and suppose, for example, that $|X| < |X_1|$. Then $X_1 = XX_2$, where $1 \leq |X_2| < \beta l_s$, and $YZ = X_2 Y Z_1$. It follows from $|X_2| < |Y|$ that $Y = X_2 Y_1$ and $Y_1 Z = X_2 Y_1 Z_1$, and so on. As a result, we obtain that Y contains a subword X_2^7, since

$$7|X_2| < 7\beta l_s < l_s/3 < |Y|.$$

We arrive at a contradiction to the definition of a set U_s. Thus $X = X_1$ in rank 1 and the paths s_i^{-1} and s_{i+1} are compatible in Δ. Then $L_i^{\delta_i} \equiv L_{i+1}^{-\delta_{i+1}}$ and $B_{i+1} = 1$ in G, which contradicts the statement of the lemma. Hence $t = s$.

Now let Λ be a reduced circular diagram for an equation

$$D_2 L_r^{-\delta_r} \dots B_2^{-1} L_1^{-\delta_1} D_1 T_1^{\varepsilon_1} \dots A_f T_f^{\varepsilon_f} = 1,$$

where $D_1 = B_1^{-1} A_1$ and $D_2 = A_{f+1} B_{r+1}^{-1}$, with contour $p_1 s_1 \dots p_{r+f} s_{r+f}$, where $\phi(s_i) = L_{r-i+1}^{\delta_{r-i+1}}$, $\phi(s_{r+j}) \equiv T_j^{\varepsilon_j}$, $\phi(p_1) \equiv D_2$, $\phi(p_{r+1}) \equiv D_1$, $\phi(p_k) = B_{r+2-k}^{-1}$ and $\phi(p_{r+q}) \equiv A_q$ for $i \in \{1, \dots, r\}$, $j \in \{1, \dots, f\}$, $k \in \{2, \dots, r\}$, $q \in \{2, \dots, f\}$. We can assume that $D_1, D_2, B_2, \dots, B_r, A_2, \dots, A_f$ are minimal words in G, hence Λ is a K-diagram with long sections s_i of the first kind ($1 \leq i \leq r + f$) and $|p_{r+f+1}| = |q| = 0$, since $|D_1|, |D_2| < 4\beta l_t/5$ and $r + f < 2^{t-1} < l_t$.

Repeating the previous considerations and using Lemma 5.4 and the statement of Lemma 4.2, we have that $L_1^{-\delta_1} \equiv T_1^{\varepsilon_1}$, $D_1 = B_1^{-1} A_1 = 1$ and $A_2 T_2^{\varepsilon_2} \dots T_f^{\varepsilon_f} A_{f+1} = B_2 L_2^{\delta_2} \dots L_r^{\delta_r} B_{r+1}$ in G. Assuming thet $f \leq r$ and using induction on f, we obtain the assertion of the lemma or (in the case $f < r$) an equation $A_{f+1} = B_{f+1} L_{f+1}^{\delta_{f+1}} \dots L_r^{\delta_r} B_{r+1}$, and, as in the consideration of the case $t < s$, we arrive at a contradiction to the statement of the lemma.

The proof of Lemma 4.2 is complete. □

It remains to prove Lemma 4.9.

Proof. Let k be an arbitrary positive integer such that $|T| < l_k/50$, and suppose thet $T^p \in \Theta_k(A)$ for some $A \in \Pi$ and $p \geq 1$. Let $C_1 T_1^{\varepsilon_1} \dots C_m T_m^{\varepsilon_m} C_{m+1}$ be the canonical form of T^p, where $T_1, \dots, T_m \in A$. By Lemma 4.3, $|T_1| = \dots = |T_m| = l_d$, where $d \geq k$.

Let Λ be a reduced circular diagram for an equation $C_1 T_1^{\varepsilon_1} \dots T_m^{\varepsilon_m} C_{m+1} T^{-p} = 1$ with contour $p_1 s_1 \dots p_m s_m p_{m+1} q$, where $\phi(s_i) \equiv T_i^{\varepsilon_i}$, $\phi(p_j) \equiv C_j$, $1 \leq i \leq m$, $1 \leq j < m + 1$, and $\phi(q) \equiv T^{-p}$. We may assume that $C_1, \dots, C_{m+1}$ are minimal words in G. If T is a period of some rank, then by Lemma 26.5 [10], the section q can be assumed smooth in Δ if we modify its label in accordance with Lemma 13.3 [10]. Now it is obvious that Δ satisfies H1, H7 and K1, K2, K4, K5 (since by Lemma 4.3, we have that $m < 2^d < l_d$).

Suppose that Γ is a contiguity submap of s_i to q for some $i \in \{1, \dots, m\}$. Then by item 2) from the proof of Lemma 5.1, $r(\Gamma) = 0$, and it follows from 7-aperiodicity of the word T_i that $|\Gamma \wedge s_i| < 10|T| < |s_i|/5$. Hence Δ is a K-diagram, and repeating the proof of Lemma 4.2, we obtain that there is $i \in \{2, \dots, m\}$ such that $C_i = 1$ in G and $T_{i-1}^{\varepsilon_{i-1}} \equiv T_i^{-\varepsilon_i}$, wlèch contradicts the definition of the canonical form.

The proof of Lemma 4.9 is complete. □

References

[1] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis I, Springer-Verlag, Berlin 1963.

[2] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, J. London Math. Soc. 24 (1949), 247–254.

[3] K. H. Hofmann, S. A. Morris, S. Oates-Williams and V. N. Obraztsov, Locally compact groups with closed subgroups open and p-adic, Math. Proc. Cambridge Philos. Soc. 118 (1995), 303–313.

[4] S. A. Morris eud V. N. Obraztsov, Non-discrete topological groups with many discrete subgroups, Topology Appl. 84 (1998), 105–120.

[5] V. N. Obraztsov, An embedding theorem for groups and its corollaries, Mat. Sb. 180 (4) (1989), 529–541; English translation: Math. USSR-Sb. 66 (2) (1990), 541–553.

[6] —, A new embedding scheme for groups and some applications, J. Austral. Math. Soc. Ser. A 61 (1996), 267–288.

[7] —, Embedding into groups with well-described lattices of subgroups, Bull. Austral. Math. Soc. 54 (1996), 221–240.

[8] A. Yu. Ol'shanskii, A remark on a countable non-topologized group, Vestnik Moskov. Univ. 3 (1980), 103.

[9] —, On thrifty embeddings of groups, Vestnik Moskov. Univ. 2 (1989), 28–34.

[10] A. Yu. Ol'shanskii, Geometry of Defining Relations in Groups, Nauka, Moscow 1989; English translation: North Holland, Dordrecht 1991.

[11] I. N. Zyabrev and E. A. Reznichenko, Identities in connected topological groups, 18th All-Union Algebra Conf., Res. Comm., Part 1, Kishinev 1985, 208.