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Suitable sets for topological groups

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Abstract

A subset S of a topological group G is said to be a suitable set if (a) it has the discrete topology, (b) it is a closed subset of $G \setminus \{1\}$, and (c) the subgroup generated by S is dense in G. K.H. Hoffmann and S.A. Morris proved that every locally compact group has a suitable set. In this paper it is proved that every metrizable topological group and every countable Hausdorff topological group has a suitable set. Examples of Hausdorff topological groups without suitable sets are produced. The free abelian topological group on the Stone–Čech compactification of any uncountable discrete space is one such example. Under the assumption of the Continuum Hypothesis or Martin's Axiom it is shown that examples exist of separable Hausdorff topological groups with no suitable set. It is not known if such examples exist in ZFC alone. An example is produced here of a compact connected abelian group with a one-element suitable set which has a dense σ-compact connected subgroup with no suitable set. © 1998 Elsevier Science B.V.

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0. Introduction

It is well known (see, for example, Hewitt and Ross [11, 25.14]) that a compact connected Hausdorff abelian group G has weight $w(G)$ less than or equal to $\mathfrak{c}$ if and only if it is monothetic; that is, if and only if it can be topologically generated by one element. (We say that a subset S of a topological group G topologically generates G if G is the smallest closed subgroup containing S.) Hoffmann and Morris [12] extended this by showing that a compact connected Hausdorff group can be topologically generated by two elements if and only if $w(G) \leqslant \mathfrak{c}$. It is clear that certain topological groups, for example, nonseparable groups, cannot be topologically generated by a finite set. So Hoffmann and Morris [12] introduced the concept of topological generating sets which are in some sense "thin". A subset S of a topological group G is said to be a suitable set if it topologically generates G, is discrete and $S \cup \{1\}$ is closed in G. A significant result of [12] was that every locally compact Hausdorff group has a suitable set. (For early overtures in the direction of what are now called suitable sets, see Iwasawa [17] and Koch [18, Section 4]. That every compact totally disconnected group has a suitable set was apparently first known by Tate and reported by Douady [6]. For a different and more detailed proof of this result based in part on a structure theorem of Varopoulos [23], see Hoffmann and Morris [16, Chapter 12].)

If G is a topological group with a suitable set, then Hoffmann and Morris defined the function s on G by $s(G) = \min\{|S|\colon S \text{ is a suitable set for } G\}$. They showed that if G is a connected locally compact Hausdorff group with $w(G) > \mathfrak{c}$, then $(s(G))^\omega = (w(G))^\omega$. Further results on suitable sets of locally compact groups were obtained in [2,13–15].

In this paper we examine suitable sets in nonlocally compact groups.

1. Preliminaries

We begin with some notation and terminology. If Y is a subset of a topological space X, then we denote the closure of Y in X by $\overline{Y}^X$, or $\overline{Y}$ if confusion is impossible. If X is a topological group, then the subgroup generated algebraically by the set Y is denoted by $\langle Y \rangle$.

Definition 1.1. Let G be a topological group and S a subset of G. Then S is said to be a *suitable set* for G if $\overline{\langle S \rangle} = G$, S has the discrete topology and $S \cup \{1\}$ is closed in G, where 1 denotes the identity element of the group.

For locally compact groups we have the following significant theorem:

Theorem 1.2 [12, Theorem 1.12]. *Every locally compact Hausdorff group has a suitable subset.*

Recall that the *weight* of a topological space X is denoted by $w(X)$ and is defined by $w(X) = \min\{|\mathcal{B}|\colon \mathcal{B} \text{ is a basis for the topology of } X\}$.

Somewhat surprising is the next result which says that many compact groups have finite suitable sets, indeed 2-element suitable sets.

Theorem 1.3 [12, Theorem 4.13]. *Every infinite compact connected Hausdorff group of weight $\leqslant \mathfrak{c}$ has a 2-element suitable set.*

The next result shows that we can and should restrict our attention to Hausdorff topological groups.

Proposition 1.4. *Let G be any non-Hausdorff topological group with at least three elements. Then G does not have a suitable set.*

Proof. As G is not Hausdorff, for each point $g \in G$ the set $\overline{\{g\}}$ contains a point $h \neq g$. Suppose that S is a suitable set for G and let s be any point in S. As S is discrete, $S \cap \overline{\{s\}} = \{s\}$. Since $S \cup \{1\}$ is closed in G, $\overline{\{s\}} = \{s, 1\}$, where $s \neq 1$ as G is not Hausdorff. Thus $\overline{\{1\}} = \{s, 1\}$. This implies that S has at most two points, s and 1. Further, as $\overline{\{1\}}$ is a group, we have that $s^2 = 1$. Finally, noting that $\langle S \rangle = G$, we see that G has at most two elements, which is a contradiction. Hence G has no suitable set. □

2. Countable topological groups

Recall that a topological space is said to be *0-dimensional* if it has a basis of clopen subsets.

Lemma 2.1. *Let G be a nondiscrete Hausdorff topological group and U a nonempty open subset which generates G. Then every point $x \in U$ has an open neighborhood $V_x \subseteq U$ such that $\langle U \setminus \overline{V_x} \rangle = G$. Further, if G is 0-dimensional, then V_x can be chosen to be clopen in G.*

Proof. Let x be any point in U, where $\langle U \rangle = G$. Since G is not discrete, $U \setminus \{x\}$ is dense in U, and hence $\langle U \setminus \{x\} \rangle$ is dense in $\langle U \rangle = G$. But $U \setminus \{x\}$ is open in G, so $\langle U \setminus \{x\} \rangle$ is open and closed in G. So $\langle U \setminus \{x\} \rangle = G$.

As $x \in \langle U \setminus \{x\} \rangle$, there exist $y_1, y_2, \dots, y_n \in U \setminus \{x\}$ and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n = \pm 1$, such that $x = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_n^{\varepsilon_n}$. Let O be an open neighborhood of x such that $y_i \notin \overline{O} \subseteq U$, for $i = 1, \dots, n$. Then for each $i \in \{1, \dots, n\}$ there exists an open neighborhood O_i of y_i such that $O_i \subseteq U$, $O \cap O_i = \emptyset$, and $O_1^{\varepsilon_1} O_2^{\varepsilon_2} \cdots O_n^{\varepsilon_n} \subseteq O$. Noting that $W = O_1^{\varepsilon_1} O_2^{\varepsilon_2} \cdots O_n^{\varepsilon_n}$ is an open neighborhood of x, we can find an open neighborhood V_x of x such that $\overline{V_x} \subseteq W \subseteq O$. So we have $U \setminus \overline{V_x} \supseteq U \setminus O \supseteq O_i$, for $i = 1, \dots, n$. This implies that $\langle U \setminus \overline{V_x} \rangle \supseteq O_1^{\varepsilon_1} \cdots O_n^{\varepsilon_n} \supseteq \overline{V_x}$. Therefore $\langle U \setminus \overline{V_x} \rangle = \langle U \rangle = G$, as required. The last statement of the lemma is obvious. □

Theorem 2.2. *Every countable Hausdorff topological group G has a closed discrete subset S such that $\langle S \rangle = G$. In particular, S is a suitable set for G.*

Proof. If G is discrete or finitely generated, then the claim is trivial. So we can assume G is neither discrete nor finitely generated. Let $G = \{g_n\colon n < \omega\}$. It suffices to find a subset S of G such that $\langle S\rangle = G$ and, for each $n < \omega$, an open neighborhood U_n of g_n such that $U_n \cap S$ is finite.

For this it will suffice to find for each $n < \omega$ a clopen set V_n in G and a finite set $S_n \subseteq G$ such that

(i) $g_n \in V_0 \cup V_1 \cup \cdots \cup V_n$;

(ii) $G = \langle G \setminus (V_0 \cup V_1 \cup \cdots \cup V_n)\rangle$;

(iii) for $n > 0$, $V_n \subseteq G \setminus (V_0 \cup V_1 \cup \cdots \cup V_{n-1})$;

(iv) $V_i \cap S_n = \emptyset$, for $i < n$; and

(v) $g_n \in \langle S_0 \cup S_1 \cup \cdots \cup S_n\rangle$.

That the above suffices is clear by putting $U_n = V_0 \cup V_1 \cup \cdots \cup V_n$ and $S = \bigcup_{n<\omega} S_n$.

We shall define the sets S_n and V_n inductively.

Put $S_0 = \{g_0\}$. As G is a countable topological group it is 0-dimensional (see [8, 6.2.8 and 6.2.6]), so applying Lemma 2.1 with G as the open set containing g_0, we find a clopen neighborhood V_0 of g_0 such that $G = \langle G \setminus V_0\rangle$. And these have the required properties.

Now assume that finite sets $S_0, S_1, \ldots, S_k$ and clopen sets $V_0, V_1, \ldots, V_k$ are defined and have the above properties (i)–(v). If $g_{k+1} \in \langle S_0 \cup S_1 \cup \cdots \cup S_k\rangle$, put $S_{k+1} = \emptyset$. If $g_{k+1} \notin \langle S_0 \cup S_1 \cup \cdots \cup S_k\rangle$, then by (ii) there exist $y_1, y_2, \ldots, y_m \in G \setminus (V_0 \cup V_1 \cup \cdots \cup V_k)$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m = \pm 1$ such that $g_{k+1} = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_m^{\varepsilon_m}$. Put $S_{k+1} = \{y_1, y_2, \ldots, y_m\}$. So in both cases (iv) and (v) are true.

Now if $g_{k+1} \in V_0 \cup V_1 \cup \cdots \cup V_k$, put $V_{k+1} = \emptyset$. If $g_{k+1} \notin V_0 \cup V_1 \cup \cdots \cup V_k$, then Lemma 2.1 shows that there exists a clopen neighborhood V_{k+1} of g_{k+1} such that $V_{k+1} \subseteq G \setminus (V_0 \cup V_1 \cup \cdots \cup V_k)$ and $G = \langle G \setminus (V_0 \cup V_1 \cup \cdots \cup V_{k+1})\rangle$. It is easily seen that conditions (i)–(iii) are also satisfied in both cases.

So by mathematical induction the sets S_n and V_n can be defined for all n with the required properties, which completes the proof. $\square$

Remark 2.3. Note that the above theorem says more than every countable Hausdorff group has a suitable set. Firstly the suitable set is closed. Secondly, the suitable set generates the group algebraically—it is not necessary to take the closure of the group it generates.

Open Question 1. Can there be found (without the assumption of axioms beyond ZFC) an example of a separable Hausdorff topological group which does not have a suitable set?

In Section 3 we produce such an example (indeed one which is countably compact) under the additional assumption of the Continuum Hypothesis (CH) or Martin's Axiom (MA). By contrast, in Section 5, we show that every separable metrizable group has a suitable set.

A question more general than Open Question 1 is the following one.

Open Question 2. Can there be found (again without the assumption of axioms beyond ZFC) an example of a Hausdorff topological group G which has a dense subgroup H with a suitable subset, but G itself does not have a suitable set?

Remark 2.4 [Added April, 1997]. We are indebted to Artur Tomita for the observation that Open Question 2 can be answered without difficulty on the basis of the results of Section 3 below. Indeed, let X be an uncountable discrete set, let $G = F(\beta X)$ be the free abelian topological group on βX (cf. Definition 3.2 below for the relevant definition), and let $H = \langle X \rangle$ be the subgroup of G algebraically generated by X. Then H is dense in G, the discrete set X generates H algebraically and hence topologically, and $X = H \cap \beta(X)$ is closed in H. Hence X is a suitable set for H, while according to Corollary 3.10 the group G itself has no suitable set. Independently after this manuscript had been completed, one of the present authors (Tkačenko) had constructed a similar example.

We are grateful to Professor Tomita for permission to cite this argument here.

3. Groups without suitable sets

Remark 3.1. If X is any Tychonoff space, then βX denotes the Stone–Čech compactification of X. In the terminology of Gillman and Jerison [9], a topological space X is said to be an *F-space* if every finitely generated ideal in the ring $C(X)$ is principal. Every countable (discrete) subspace of an F-space is C^*-embedded and in a compact F-space every infinite closed set K contains a homeomorph of the space $\beta\mathbb{N}$ (and hence satisfies $|K| \geqslant 2^{\mathfrak{c}}$ [24, 1.64]). Among the spaces known to be F-spaces [9, pp. 210 and 215] are: every discrete space; every space Y with $X \subseteq Y \subseteq \beta X$ and X an F-space; every space $\beta X \setminus X$ with X a locally compact F-space; every space $\beta X \setminus X$ with X locally compact and σ-compact. In particular we note that if D is any infinite discrete space the spaces βD and $\beta D \setminus D$ are compact F-spaces. Further, writing $\mathbb{R} = \mathbb{R}_+ \cup \mathbb{R}_-$ as usual and taking $X = \mathbb{R} \cup \overline{\mathbb{R}_-}^{\beta\mathbb{R}} = \beta\mathbb{R} \setminus [\overline{\mathbb{R}_+}^{\beta\mathbb{R}} \setminus \mathbb{R}]$, the space $\beta X \setminus X$ (which is $\overline{\mathbb{R}_+}^{\beta\mathbb{R}} \setminus \mathbb{R}_+$, that is, one of the connected components of $\beta\mathbb{R} \setminus \mathbb{R}$) is a compact connected F-space.

Definition 3.2. Let X be a Tychonoff space. Then the topological group $F(X)$ is said to be the *(Markov) free abelian topological group on X* [20,21] if X is a subspace of $F(X)$ and for every abelian topological group G and every continuous map of X to G extends uniquely to a continuous homomorphism of $F(X)$ into G.

It is known that for every Tychonoff space, $F(X)$ exists and is unique.

We now present some preliminary results needed to show that certain free abelian topological groups have no suitable sets.

Lemma 3.3. *Let $F_0, F_1, \ldots$ be a sequence of closed subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ such that $\beta\mathbb{N}\setminus\mathbb{N} \neq \bigcup_{n\in\mathbb{N}} F_n$. There exists an infinite subset P of $\mathbb{N}$ such that $\overline{P}^{\beta\mathbb{N}} \cap (\bigcup_{n\in\mathbb{N}} F_n) = \emptyset$.*

Proof. It is well known that each nonempty G_δ-set in $\beta\mathbb{N} \setminus \mathbb{N}$ has nonempty interior [9, 6S.8]. Therefore, $(\beta\mathbb{N} \setminus \mathbb{N}) \setminus \bigcup_{n\in\mathbb{N}} F_n$ contains a nonempty set V which is open in $\beta\mathbb{N} \setminus \mathbb{N}$. By 6S.4 of [9] the sets of the form $\overline{A}^{\beta\mathbb{N}} \setminus \mathbb{N}$, where A is an infinite subset of $\mathbb{N}$, form a basis for the topology on $\beta\mathbb{N} \setminus \mathbb{N}$. Therefore there exists an infinite subset P of $\mathbb{N}$ with $\overline{P}^{\beta\mathbb{N}} \setminus \mathbb{N} \subseteq V$. Clearly this P has the required property. $\square$

Notation 3.4. Let X be a set and $n \in \mathbb{N}$, $n \geqslant 1$. We say that a point $x \in X^n$ is *in general position* if all of the coordinates of x are different.

Lemma 3.5. *Let X be a compact F-space. Then for each integer $n \geqslant 1$ the subspace S_n of X^n consisting of points in general position is countably compact.*

Proof. For X finite we have $S_n = \emptyset$ (hence, countably compact) when $n > |X|$, and S_n is closed in X^n (hence is countably compact) when $n \leqslant |X|$; we assume henceforth that X is infinite. It is clear that $S_1 = X$ is compact. Let A be an infinite subset of $S_n \subset X^n$, $n \geqslant 2$. It suffices to show that A has a cluster point in S_n. Denote by p_i the projection of X^n onto the ith factor, $1 \leqslant i \leqslant n$. If $p_i(A)$ is finite for some $i \leqslant n$, there exist a point $x \in X$ and an infinite subset B of A such that $B \subset p_i^{-1}(x)$. Otherwise we can choose a countably infinite subset C of A satisfying $p_i(a) \neq p_i(b)$ for distinct $a, b \in C$. Thus, we can find a countably infinite subset $D \subset A$ such that the following condition is fulfilled for each $i \leqslant n$:

$$\text{either } |p_i(D)| = 1 \quad \text{or} \quad p_i(a) \neq p_i(b) \text{ for distinct } a, b \in D. \tag{$*$}$$

Without loss of generality one can assume that $|p_i(D)| = \aleph_0$ for each i with $1 \leqslant i \leqslant k$ and $|p_i(D)| = 1$ for each $i > k$, where $k \leqslant n$. Let a point $a_0 \in D$ be arbitrary. Suppose that the points $a_0, \ldots, a_r \in D$ have been defined for some $r < \omega$. Put $X_r = p_1(E_r) \cup \cdots \cup p_n(E_r)$, where $E_r = \{a_0, \ldots, a_r\}$. Since $|X_r| \leqslant nr < \omega$, the condition $(*)$ and the definition of k imply that for each $i \leqslant k$ there exist only finitely many points $y \in D$ with $p_i(y) \in X_r$. Thus, we can find $a_{r+1} \in D$ with $p_i(a_{r+1}) \notin X_r$ for each $i \leqslant k$.

From the definition of the set $E = \{a_r\colon r < \omega\}$ it follows that $p_i(E)$ is infinite for each $i \leqslant k$ and $p_i(E) \cap p_j(E) = \emptyset$ for all $i, j \leqslant k$, $i \neq j$. Let E_0 be an infinite subset E such that $p_i(E_0)$ is discrete for each $i \leqslant k$. Our aim is to define an infinite subset E^* of E_0 so that $\overline{p_i(E^*)} \cap \overline{p_j(E^*)} = \emptyset$ for all distinct $i, j \leqslant k$.

For every $x \in E_0$, put $F_x = \{p_2(x), \ldots, p_n(x)\} \cap \overline{p_1(E_0)}$. Obviously, F_x is a finite subset of $\overline{p_1(E_0)} \setminus p_1(E_0)$, and hence is closed. Since $p_1(E_0)$ is countably infinite and discrete in the F-space X, the set $\overline{p_1(E_0)} \setminus p_1(E_0)$ is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$. Apply Lemma 3.5 to choose an infinite subset E_1 of E_0 so that

$$\left(\bigcup_{x\in E_1} F_x\right) \cap \overline{p_1(E_1)} = \emptyset.$$

This gives us $p_j(E_1) \cap \overline{p_1(E_1)} = \emptyset$ for each $j \neq 1$.

Apply the same procedure for $i = 2, \ldots, k$ and define a decreasing sequence $E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_k$ of infinite subsets of E_0 satisfying

$$p_j(E_i) \cap \overline{p_i(E_i)} = \emptyset \quad \text{for each } j \neq i.$$

Clearly, the set $E^* = E_k$ satisfies $p_j(E^*) \cap \overline{p_i(E^*)} = \emptyset$ for all distinct $i, j \leqslant k$. The latter means that $Y = \bigcup_{i \leqslant k} p_i(E^*)$ is a countable discrete subspace of the F-space X, whence $\overline{p_i(E^*)} \cap \overline{p_j(E^*)} = \emptyset$ whenever $i, j \leqslant k,\ i \neq j$.

If y is a cluster point of E^* in X^n, then $p_i(y) \in \overline{p_i(E^*)}$ for each $i \leqslant k$ so that $p_i(y) \neq p_j(y)$ for distinct $i, j \leqslant k$. This gives us a cluster point $y \in S_n$ for E^* if $k = n$. The case $k < n$ requires more work. Let $p_i(D) = \{x_i\},\ k+1 \leqslant i \leqslant n$. Choose disjoint infinite subsets E', E'' of E^*. Then $p_1(E')$ and $p_1(E'')$ are disjoint subsets of the discrete set Y, whence $\overline{p_1(E')} \cap \overline{p_1(E'')} = \emptyset$. Therefore, either $x_{k+1} \notin \overline{p_1(E')}$ or $x_{k+1} \notin \overline{p_1(E'')}$. We put $Q_1^1 = E'$ in the first case and $Q_1^1 = E''$ otherwise. Continuing this way we define infinite sets $Q_1^1 \supset Q_2^1 \supset \cdots \supset Q_{n-k}^1$ satisfying $x_{k+i} \notin \overline{p_1(Q_i^1)}$ for $i = 1, \ldots, n-k$. Put $Q^1 = Q_{n-k}^1$. Clearly, $\{x_{k+1}, \ldots, x_n\} \cap \overline{p_1(Q^1)} = \emptyset$.

Repeat this procedure considering the projection p_2 and define an infinite set $Q^2 \subset Q^1$ satisfying $\{x_{k+1}, \ldots, x_n\} \cap \overline{p_2(Q^2)} = \emptyset$. At the step k we shall have an infinite subset Q^k of Q^{k-1} with

$$\{x_{k+1}, \ldots, x_n\} \cap \overline{p_i(Q^k)} = \emptyset \quad \text{for each } i \leqslant k. \tag{$**$}$$

Put $E_* = Q^k$. Then $(**)$ and the choice of the set E^* imply that every cluster point of E_* belongs to S_n. This completes the proof. □

Lemma 3.6. *Let X be as in Lemma 3.5, and let $X_i = X$, for $i = 1, \ldots, n+1$, $n \in \mathbb{N}$. Let B be an infinite subset of the product $X^{n+1} = X_1 \times \cdots \times X_n \times X_{n+1}$, $n \geqslant 1$, and $p_n(b) = p_{n+1}(b)$ for each $b \in B$, where $p_i : X^{n+1} \to X_i$ is the projection mapping. If $\pi_{n+1}(B)$ has a cluster point $a = (a_1, a_2, \ldots, a_n)$, then $(a_1, a_2, \ldots, a_n, a_n)$ is a cluster point of B, where $\pi_{n+1} : X^{n+1} \to X_1 \times X_2 \times \cdots \times X_n$ is the projection omitting the $(n+1)$st coordinate.*

Proof. Substituting $X_1 \times \cdots \times X_{n-1}$ by a single factor, it suffices to consider the case $n = 2$. Let (a_1, a_2) be a cluster point of $\pi_3(B) \subseteq X_1 \times X_2$ and $O = O_1 \times O_2 \times O_3$ be an open neighborhood of the point $a^* = (a_1, a_2, a_2)$. Put $O_2' = O_2 \cap O_3$. By assumption there exists a point $b \in B$ with $\pi_3(b) \in O_1 \times O_2'$, whence $b \in O_1 \times O_2' \times O_2' \subseteq O$. This proves that $O \cap B \neq \emptyset$ for any neighborhood O of a^*. □

Lemma 3.7. *Let X be a compact F-space, B an infinite subset of X^n $(n \geqslant 1)$ and $\delta \in \{+1, -1\}^n$. Let $F(X)$ be the free abelian topological group on X and j_δ the multiplication map of X^n into $F(X)$ given by $j_\delta(x_1, \ldots, x_n) = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $\delta = (\varepsilon_1, \ldots, \varepsilon_n)$. If $j_\delta(b)$ has length n for each $b \in B$, then there exists a cluster point a of B such that $j_\delta(a)$ also has length n.*

Proof. We apply mathematical induction on n. If $n = 1$, the mapping $j_\delta : X \to F$ is a homeomorphic embedding and the claim is obvious. Now assume that the lemma is proved for $n = m$, and let $n = m + 1$. Let S_n be a subset of X^n consisting of points in general position. If the intersection $B \cap S_n$ is infinite, it suffices to use Lemma 3.5 and the continuity of the mapping j_δ.

Otherwise one can assume that B and S_n are disjoint. Every point $b \in B$ has at least two equal coordinates and since B is infinite, there exist indices $k, l \leqslant n$, $k < l$, and an infinite subset $C \subseteq B$ such that $p_k(x) = p_l(x)$ for each $x \in C$, where p_k and p_l are respectively the projections of X^n onto the kth and lth factors. Let $\delta = (\varepsilon_1, \ldots, \varepsilon_n)$. Then $\varepsilon_k = \varepsilon_l$, for the length of $j_\delta(b)$ is n for each $b \in C$ (and C is not empty). Denote by π_l the projection of X^n onto X^m omitting the lth coordinate: $\pi_l(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, \hat{x}_l, \ldots, x_n)$. By the inductive hypothesis there exists a cluster point $a = (a_1, \ldots, a_k, \ldots, \hat{a}_l, \ldots, a_n)$ of $\pi_l(C)$ in X^m such that the length of $j_{\delta'}(a)$ is equal to m, where $\delta' = (\varepsilon_1, \ldots, \varepsilon_k, \ldots, \hat{\varepsilon}_l, \ldots, \varepsilon_n)$. By Lemma 3.6, $a^* = (a_1, \ldots, a_k, \ldots, a_l, \ldots, a_n)$ with $a_l = a_k$ is a cluster point of C and one easily sees that the length of $j_\delta(a^*)$ is n, for $\varepsilon_k = \varepsilon_l$. □

Theorem 3.8. *If X is a nonseparable compact F-space, then the free abelian topological group on X, $F(X)$, does not have a suitable set.*

Proof. *Case* 1. $|S| \leqslant \omega$. Each $s \in S$ is a word using finitely many symbols of X. All told, the (minimal) set C of symbols from X needed to give S is then countable. Since X is not separable there is continuous $f : X \to T$ such that $f|C = 1$, and some $p \in X$ satisfies $f(p) \neq 1$. The continuous homomorphism h from $F(X)$ extending f then satisfies $h = 1$ on $F(X)$ (since C generates that group topologically) but $h(p) = f(p) \neq 1$, contradiction.

Case 2. $|S| > \omega$. Algebraically $F(X)$ is the free abelian group on X. For each integer $k \geqslant 1$ and each sequence $\delta = (\varepsilon_1, \ldots, \varepsilon_k) \in \{-1, +1\}^k$, denote by j_δ the mapping of X^k to $F(X)$ defined by $j_\delta(x_1, \ldots, x_k) = x^{\varepsilon_1} \cdots x^{\varepsilon_k}$. Considered as a mapping to $F(X)$, the mapping j_δ is continuous in each case. Clearly, we have

$$F(X) = \bigcup \{j_\delta(X^k) \colon k \in \mathbb{N},\ \delta \in \{-1, +1\}^k\}.$$

Therefore, there exist $k \in \mathbb{N}$ and $\delta \in \{-1, +1\}^k$ such that the intersection $S \cap j_\delta(X^k)$ is infinite. Let n be the minimal integer with this property, and choose $\delta \in \{-1, +1\}^n$ corresponding to this n. By the choice, there exists a countably infinite subset $A \subset S \cap j_\delta(X^n)$ all elements of which have length exactly n. Put $B = j_\delta^{-1}(A)$. Note that the mapping $j_\delta : B \to F(X)$ is finite-to-one (in fact, $|j_\delta^{-1}(g)| \leqslant n!$ for each $g \in A$). Since A is discrete, we conclude that B is countably infinite and discrete. By Lemma 3.7, there exists a cluster point y of B in X^n such that the length of $j_\delta(y)$ is equal to n. Thus, $g = j_\delta(y)$ is a cluster point of A and $g \neq e_F(X)$. This contradicts the fact that S (and a subset A of S) has no cluster points in $F(X) \backslash \{e_{F(X)}\}$. The same argument shows that S cannot be suitable for $F(X)$. □

Remark 3.9. In Theorem 3.8 we do not use all the power of the assumption on X that it is an F-space; rather we used only that every countable discrete subspace of X is C^*-embedded in X.

Corollary 3.10. *If X is any uncountable discrete space, then $F(\beta X)$ has no suitable set.*

Later we shall show that if X is a countable discrete space, then $F(\beta X)$ does have a suitable set.

Corollary 3.11. *If X is any infinite discrete space, then $F(\beta X \setminus X)$ has no suitable set.*

Corollary 3.12. *If X is a connected component in $\beta\mathbb{R} \setminus \mathbb{R}$, then $F(X)$ has no suitable set.*

Recall that a *variety of topological groups* [22] is defined to be a class of topological groups closed under the formation of subgroups, quotient groups, and arbitrary Cartesian products.

Remark 3.13. We now observe that a dense subgroup of a topological group with a suitable set may itself fail to have a suitable set.

Corollaries 3.10, 3.11 and 3.12 remain true if we replace free abelian topological group by free topological group in the variety, $\mathfrak{V}(\mathbb{T})$, of topological groups generated by the circle group, $\mathbb{T}$. In particular if X is as in Corollary 3.12, then $F(X, \mathfrak{V}(\mathbb{T}))$ has no suitable set; but it is a σ-compact connected dense subgroup of its closure in $\mathbb{T}^{\mathfrak{c}}$, and that closure, like every compact connected abelian group of weight not exceeding $\mathfrak{c}$, is monothetic (cf. [11, 25.14]) and hence has a one-element suitable set.

Remark 3.14. In the above corollaries we produced examples of topological groups without suitable sets which had cardinality at least $2^{\mathfrak{c}}$. However, it is possible to modify Corollary 3.11 to produce in ZFC a Hausdorff topological group of cardinality $\mathfrak{c}$ with no suitable set.

We will define a nonseparable subspace Y of $\beta\mathbb{N} \setminus \mathbb{N}$ with $|Y| = \mathfrak{c}$ such that the set $Y^n \cap S_n$ is countably compact for each integer $n \geqslant 1$, where S_n is the subset of $(\beta\mathbb{N}\setminus\mathbb{N})^n$ consisting of points in general position. Then the free abelian topological group $F(Y)$ also is not separable and the argument in Case 2 of the proof of Theorem 3.8 shows that $F(Y)$ has no suitable set.

Now let us define Y. For every integer $n \geqslant 1$ and every countably infinite subset A of S_n, take a point $x(A, n) \in \overline{A} \cap S_n$, by Lemma 3.5. Let γ be an uncountable family of nonempty disjoint open subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ (see Example 3.6.18 of [8]). By transfinite recursion one defines an increasing chain $Y_0, Y_1, \ldots, Y_\alpha, \ldots$, $\alpha < \omega_1$, of subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ satisfying the following conditions for each $\alpha < \omega_1$:

(0) Y_0 intersects every member of the family γ;

(1) $|Y_\alpha| \leqslant 2^{\aleph_0}$;

(2) if $n \geqslant 1$ and A is a countably infinite subset of $Y_\alpha^n \cap S_n$, then every coordinate of the point $x(A, n)$ belongs to $Y_{\alpha+1}$.

Let Y be the union of the sets Y_α, $\alpha < \omega_1$. Since $Y_0 \subseteq Y$, from (0) it follows that Y is not separable. It remains to show that $Y^n \cap S_n$ is countably compact for each $n \in \mathbb{N}^+$.

Let A be a countably infinite subset of $Y^n \cap S_n$ for some $n \geqslant 1$. Since the sequence $Y_0, Y_1, \dots, Y_\alpha, \dots$ is increasing, there exists $\alpha < \omega_1$ such that $p_1(A) \cup \dots \cup p_n(A) \subset Y_\alpha$, where p_i is the projection of $(\beta\mathbb{N} \setminus \mathbb{N})^n$ onto the ith factor, $1 \leqslant i \leqslant n$. Then $A \subset Y_\alpha^n \cap S_n$, whence $x(A, n) \in \overline{A} \cap Y_{\alpha+1} \cap S_n \subseteq Y^n \cap S_n$. This proves our claim. □

Now before giving a partial response to Open Question 1 of Section 2, we recall Martin's Axiom (MA) which is independent of the usual axioms of set theory (ZFC). We state it in a topological form (see [24]).

Martin's Axiom. *If X is a compact space in which every collection of disjoint open sets is countable, then X is not the union of fewer than $\mathfrak{c}$ nowhere dense subsets.*

Theorem 3.15. *In the axiom system* ZFC + MA *there is a separable group with no suitable subset.*

Proof. It is proved by van Douwen [7, 8.1] in that axiom system that the Boolean group $\{-1, +1\}^{\mathfrak{c}}$ contains a (dense) separable countably compact subgroup G with no convergent sequences. As remarked by van Douwen himself (loc. cit., 6.1), in such a group G every infinite subset A satisfies $|\overline{A}^G| \geqslant \mathfrak{c}$. It is therefore clear that G has no infinite suitable set. That G has no finite suitable set is also clear: G itself is infinite, but every finite subset of G generates a finite (hence closed) subgroup. □

Open Question 3. Does there exist in ZFC a pseudocompact Hausdorff topological group with no suitable set?

Remark 3.16 [Added April, 1997]. In joint work in progress, one of the present co-authors (Tkačenko) and Dikran Dikranjan and Vladimir Tkačuk have answered Question 3 affirmatively. Indeed, the example (in ZFC) may be chosen ω-bounded in the sense that each of its countable subsets has compact closure. A manuscript is in preparation.

4. Groups with suitable sets

Let $G = \prod_{i \in I} G_i$ be a Cartesian product of topological groups G_i. For every element $x \in G$, denote $\operatorname{supp}(x) = \{i \in I\colon x_i \neq 1_i\}$ where 1_i is the identity of G_i. It is easy to verify that the sets

$$\bigoplus_{i \in I} G_i = \{x \in G\colon |\operatorname{supp}(x)| < \aleph_0\} \quad \text{and} \quad \sum_{i \in I} G_i = \{x \in G\colon |\operatorname{supp}(x)| \leqslant \aleph_0\}$$

are dense subgroups of G; these are called, respectively the *σ-product* and *Σ-product* of the groups G_i.

Theorem 4.1. *Let $H = \bigoplus_{i \in I} G_i$, $A = \sum_{i \in I} G_i$ and $G = \prod_{i \in I} G_i$, for any index set I. If each G_i has a suitable set then H, A and G each have a suitable set.*

Proof. Let S_i be a suitable set for G_i, for $i \in I$. We assume without loss of generality that $1_i \notin S_i$. Define $S = \bigcup_{i \in I} (S_i \times \{1_{I \setminus \{i\}}\})$.

S is discrete. Given $p = (x_i, 1_{I \setminus \{i\}}) \in S$ there is a neighborhood U_i of x_i in G_i such that $1_i \notin U_i$ and $U_i \cap S_i = \{x_i\}$. Let π_i be the projection mapping of G onto G_i. Then $\pi_i^{-1}(U_i) \cap S = \{p\}$.

$S \cup \{1\}$ is closed in G. Suppose that x is an accumulation point of $S \cup \{1\}$, with $x \notin S \cup \{1\}$. If some two distinct elements i and j of I satisfy $\pi_i(x) = x_i \neq 1_i$ and $\pi_j(x) = x_j \neq 1_j$ then we readily find a neighborhood U of x (of the form $\pi_i^{-1}(U_i) \cap \pi_j^{-1}(U_j)$) such that $U \cap (S \cup \{1\}) = \emptyset$, a contradiction. Thus $x_i \neq 1_i$ for exactly one $i \in I$. Then $x_i \in S_i$ is impossible since S_i is discrete, and $x_i \notin S_i$ is impossible since $S_i \cup \{1_i\}$ is closed in G_i.

Being a subset of H which is closed in G, $S \cup \{1\}$ is closed in H. It is clear that S generates a dense subgroup of H. The proof that S is suitable in H is complete.

Since H is dense in A and in G we have: S generates A topologically and S generates G topologically. □

Theorem 4.2. *Let H be an open subgroup of a topological group G. If H has a suitable set, then G has a suitable set. If H has a closed suitable set, then G has a closed suitable set.*

Proof. Let S be a suitable set for H and let A select one point from each coset of H in G; that is, $x \in G$ implies $|A \cap xH| = 1$. We claim that $S \cup A$ is suitable for G. Surely $S \cup A$ is discrete, and 1 is its only (possible) accumulation point. Further, S generates a dense subgroup of H; so $S \cup A$ generates a dense subgroup of G. If S is closed in H, then $S \cup A$ is clearly closed in G. □

The next theorem is a special case of one which we shall prove later. However, its proof yields information we use in the next result.

Theorem 4.3. *Let $G = (G, \rho)$ be a metric group with a suitable set X, and let H be a dense subgroup of G. Then H has a suitable set. Further, if X is a closed suitable set for G, then H has a closed suitable set.*

Proof. Let $X = \{x_i\colon i \in I\}$ be suitable in G. Since X is discrete and G is metric, X is strongly discrete in the sense that its elements admit pairwise disjoint neighborhoods $B_{\varepsilon_i}(x_i)$, where $B_\varepsilon(x)$ denotes the open ball with respect to the metric ρ with center x and radius ε. For $i \in I$ we shall define a set $Y_i \subseteq H$ and then $Y = \bigcup_{i \in I} Y_i$; and then show that Y is suitable for H.

If $x_i \in H$ we take $Y_i = \{x_i\}$. If $x_i \notin H$ we choose a faithfully indexed sequence $y_{i,n}$ in H with x_i as limit; indeed we choose $y_{i,n} \in B_{\varepsilon_i/3}(x_i)$; then, we take $Y_i = \{y_{i,n}\colon n < \omega\}$. Put $Y = \bigcup_{i \in I} Y_i$.

Y is discrete. If $y = x_i \in Y$ then $B_{\varepsilon_i}(x_i) \cap Y = \{y\}$. If $y = y_{i,n} \in Y_i \subseteq Y$ then since the sequence $\{y_{i,k}\colon k < \omega\}$ is discrete there is a neighborhood U of y meeting no point $y_{i,k}$ (same i) when $k \neq n$. Then $(U \cap B_{\varepsilon_i/3}(y)) \cap Y = \{y\}$.

Y generates H topologically. The closed subgroup of G generated by Y contains X, hence is G itself. Thus the closed subgroup of H generated by Y is H.

$Y \cup \{1\}$ is closed in H. Let $p \in H$ be a limit point of Y. We will show that $p = 1$. Let $\{z_k\colon k < \omega\}$ be a sequence in Y converging to p.

Case 1. z_k has a subsequence of points of the form x_i, say $z_k = x_{i_k} \in X$. Then $x_{i_k} \to p \in G$ so $p = 1$.

Case 2. Case 1 fails. Passing to a subsequence if necessary, we assume each z_k has the form $z_k = y_{i_k, n_k}$. Note that no fixed $i \in I$ arises infinitely often. For if $z_k = y_{i,n_k}$ for fixed i and for infinitely many $k < \omega$ then $z_k \to x_i \notin H$, a contradiction since $p \in H$. We assume therefore, passing to a subsequence if necessary, that $z_k = y_{i_k,n_k}$ with the indices i_k pairwise distinct.

We claim in this case that $\varepsilon_{i_k} \to 0$. If not, passing again to a subsequence, we have (for some $\varepsilon > 0$) that $\varepsilon_{i_k} > \varepsilon$ for all k. Now let l and m be distinct values of k; without loss of generality we assume $\varepsilon_l \leqslant \varepsilon_m$. Since $x_l \notin B_{\varepsilon_m}(x_m)$, $\rho(x_l, x_m) \geqslant \varepsilon_m$. Also $\rho(x_m, z_m) \leqslant \varepsilon_m/3$ and $\rho(x_l, z_l) \leqslant \varepsilon_l/3 \leqslant \varepsilon_m/3$. Hence $\rho(z_l, z_m) \geqslant \varepsilon_m/3 \geqslant \varepsilon/3$. This is a contradiction as the sequence z_k converges to p. So the claim is proved.

Now from $z_k \to p$ and $z_k \in B_{\varepsilon_{i_k}}(x_{i_k})$ and $\varepsilon_{i_k} \to 0$ follows $x_{i_k} \to p \in G$ with $x_{i_k} \in X$. Thus $p = 1$, as desired.

Finally, it is clear that if X is closed in G, then Y is closed in H. □

The following is a corollary of the proof of Theorem 4.3. For the statement recall that if the topological group G has a suitable set, then we define $s(G)$ to be $\min\{|S|\colon S \text{ is a suitable set for } G\}$.

Corollary 4.4. *Let G be a metrizable topological group with a suitable set and let H a dense subgroup. Then $s(H) \leqslant \max\{s(G), \omega\}$.*

Remark 4.5. Firstly we note that $\mathbb{Q}$ is a dense subgroup of $\mathbb{R}$ and $s(\mathbb{Q}) = \omega$ while $s(\mathbb{R}) = 2$. So, in the notation of Corollary 4.4, we can have $s(H) > s(G)$.

Remark 4.6. At first sight one might think that if H is a dense subgroup of a Hausdorff topological group G and both have suitable sets then $s(H)$ would be no larger than $s(G)$. This is certainly false. For example, if G_i, $i \in I$, is such that each G_i is topologically isomorphic to $\mathbb{T}$ and $|I| = \mathfrak{c}$, then $H = \sum_{i \in I} G_i$ is dense in $G = \prod_{i \in I} G_i$, while $s(H) = \mathfrak{c}$, and $s(G) = s(\mathbb{T}^{\mathfrak{c}}) = 1$. Note that this demonstrates that if the condition of metrizability were deleted from the statement of Corollary 4.4 then it would be false. Indeed we know of no example where $s(H) < s(G)$, for H dense in G. [Note added April, 1997. Recently Dmitri Shakhmatov has provided such examples.]

Theorem 4.7. *Let G be any Hausdorff topological group. Then there exists a Hausdorff topological group F such that G is topologically isomorphic to an open subgroup of F and F has a closed suitable set; indeed F is generated algebraically by a closed discrete subset.*

Proof. Given G, let H be the underlying group of G with the discrete topology. Define $F = G \times H$. Then G is an open subgroup of the topological group F.

Let $G = \{x_\xi: \xi < \kappa\}$ and $H = \{w_\xi: \xi < \kappa\}$. For each ξ, choose $y_\xi \in G \times \{w_\xi\}$—for example, $y_\xi = (x_\xi, w_\xi)$—and define $z_\xi = y_\xi \cdot (x_\xi, 1)^{-1}$. Since $(x_\xi, 1) = y_\xi \cdot z_\xi^{-1}$, the point $(x_\xi, 1)$ (which we have identified with x_ξ) lies in any subgroup of F which contains both y_ξ and z_ξ. Thus the set $A := \{y_\xi: \xi < \kappa\} \cup \{z_\xi: \xi < \kappa\}$ generates F algebraically, hence topologically. Clearly A is closed and discrete in F, since (a) the open sets $G \times \{w_\xi\}$ are disjoint and cover F and (b) the intersection of A with $G \times \{w_\xi\}$ is the two-element set $\{y_\xi, z_\xi\}$. □

Remark 4.8. Since there are Hausdorff topological groups with no suitable subset, the result above provides a strong negative answer to the question: does every open subgroup of a topological group with a suitable set itself have a suitable set?

Remark 4.9. The proof of Theorem 4.7 also shows that a Hausdorff quotient group of a topological group with a suitable set need not have a suitable set.

5. Separable topological groups

Recall that a Tychonoff space X is called a *k_ω-space* [19] if it has compact subspaces X_n, $n \in \mathbb{N}$, such that $X = \bigcup_{n\in\mathbb{N}} X_n$, and a subset A of X is closed in X if and only if $A \cap X_n$ is compact for each $n \in \mathbb{N}$. Our approach to the proof of Theorem 5.1 uses Stone–Čech compactifications in the manner introduced in [10].

Theorem 5.1. *Let X be a separable Tychonoff space and let $F(X)$ be the free abelian topological group on X. Then $F(X)$ has a closed suitable set.*

Proof. If X is finite, then $F(X)$ is discrete, and clearly $F(X)$ has a closed suitable set. So without loss of generality, assume that X is infinite.

Let $F(\beta X)$ be the free abelian topological group on the Stone–Čech compactification, βX, of X. The natural map ϕ of X into βX extends to a continuous one-to-one homomorphism Φ of $F(X)$ into $F(\beta X)$. Then $F(\beta X)$ is a k_ω-space with k_ω-decomposition $F(\beta X) = \bigcup_{n\in\mathbb{N}} F_n(\beta X)$, where $F_n(\beta X)$ is the set of all words in $F(\beta X)$ of length with respect to βX of length $\leqslant n$.

Let $Y = \{y_1, y_2, \ldots, y_n, \ldots\}$ be a dense subset of X. Put

$$S = \{y_1, y_1 y_2, \ldots, y_1 y_2 \cdots y_n, \ldots\}.$$

Clearly each of the sets

$$\Phi(S) \cap F_n(\beta X) = \{\Phi(y_1), \Phi(y_1 y_2), \ldots, \Phi(y_1 y_2 \cdots y_n)\}$$

is finite, so $\Phi(S)$ is closed in $F(\beta X)$. Indeed for each subset T of S, $\Phi(T) \cap F_n(\beta X)$ is finite. Thus $\Phi(T)$ is closed. Hence $\Phi(S)$ is discrete. Therefore S is a closed discrete subspace of $F(X)$.

As $\langle S\rangle$ contains Y, $\overline{\langle S\rangle}$ contains X and hence equals $F(X)$. Thus S is a closed suitable set for $F(X)$. $\square$

Remark 5.2. Theorem 5.1 remains true if free abelian topological group is replaced by free topological group.

Definition 5.3. A topological group G is said to be *totally bounded* if for every nonempty open subset U of G there is finite $F \subseteq G$ such that $G = FU$. For κ an infinite cardinal, the topological group G is said to be κ*-totally bounded* if for every nonempty open subset U of G there is $F \subseteq G$ such that $|F| < \kappa$, $G = FU$. (So, totally bounded is ω-totally bounded.)

Every group G is $|G|^+$-totally bounded, where $|G|^+$ denotes the first cardinal greater than $|G|$.

Notation 3.4. Given a topological group G, let $b(G)$ (the boundedness number of G) be the least cardinal κ such that G is κ-totally bounded.

Remark 5.4. The condition $\kappa < b(G)$ means that for some nonempty open subset U of G no $F \subseteq G$ with $|F| < \kappa$ satisfies $G = FU$. It is then easy using induction to find a set $\{x_\xi\colon\ \xi < \kappa\} \subseteq G$ such that each x_ξ satisfies $x_\xi \notin \bigcup_{\eta<\xi} x_\eta U$. It then follows for some nonempty open neighborhood V of 1 that some κ-many translates of V form a pairwise disjoint family which is uniformly V-discrete in the sense that for each $p \in G$ the neighborhood pV of p meets only finitely many members of that family (in fact, at most one). Indeed given U as above, with U a neighborhood of 1, let V be a neighborhood of 1 such that $V = V^{-1}$ and $V^4 \subseteq U$. Then $\{x_\xi V\colon\ \xi < \kappa\}$ is a disjoint, uniformly V-discrete family: given $p \in G$, the neighborhood pV of p meets at most one of the sets $x_\xi V$, since if $v_i \in V$ (with $1 \leqslant i \leqslant 4$) satisfy $pv_1 = x_\xi v_2$ and $pv_3 = x_\eta v_4$ with $\eta < \xi$ then $x_\xi = x_\eta v_4(v_3)^{-1}v_1(v_2)^{-1} \in x_\eta V^4 \subseteq x_\eta U$, a contradiction.

Notation 3.4. We write $d(G)$ for the *density character* of G, that is the least cardinal of a dense subset of G.

Theorem 5.5. *Every topological group G satisfies $b(G) \leqslant (d(G))^+$.*

Proof. Suppose instead that $b(G) > (d(G))^+$, so there is κ such that $d(G) < \kappa < b(G)$. Since $\kappa < b(G)$ there is (according to Remark 5.4) an open neighborhood V of 1 and a subset X of G with $|X| = \kappa$ such that the sets xV with $x \in X$ are pairwise disjoint. This is incompatible with the condition $d(G) < \kappa$. $\square$

Remark 5.6.

(a) The "gap" between $b(G)$ and $d(G)$ may be arbitrarily large. For examples to this effect let $\alpha \geqslant \omega$ and define $\kappa = (2^\alpha)^+$. Then with $G = \{-1, +1\}^\kappa$ or $G = \mathbb{T}^\kappa$ we have $b(G) = \omega$ since G is compact, but $d(G) = \log(w(G)) = \log\kappa > \alpha$ (cf. [8, 2.3.25] or [3, 3.9(v)]).

(b) For metrizable groups G the inequality $b(G) \leqslant (d(G))^+$ can be sharpened to read $d(G) \leqslant b(G) \leqslant (d(G))^+$, with $b(G) = (d(G))^+$ in the case $\mathrm{cf}(d(G)) > \omega$. To see this let ρ be a left-translation-invariant compatible metric for G and for $0 < n < \omega$ let D_n be a maximal $1/n$-dispersed subset of G (in the sense that D_n is maximal with respect to the property $x, y \in D_n$, $x \neq y \Rightarrow \rho(x, y) \geqslant 1/n$) and set $|D_n| = \kappa_n$. Each set of the form $pB_{1/(2n)}(1)$ ($p \in G$) contains at most one element of D_n, so no $F \subseteq G$ with $|F| < \kappa_n$ satisfies $G = FB_{1/2n}(1)$. Thus $b(G) \geqslant \kappa_n^+$ for each $n < \omega$, so $b(G) \geqslant |D| \geqslant d(G)$. If $\mathrm{cf}(d(G)) > \omega$ then some κ_n satisfies $\kappa_n = d(G)$ and we have $d(G)^+ = \kappa_n^+ \leqslant b(G) \leqslant (d(G))^+$, as asserted.

(c) The relation $d(G) = b(G)$ can occur for metrizable groups (with $\mathrm{cf}(d(G)) = \omega$). Given a strictly increasing sequence κ_n of infinite cardinal numbers, set $\kappa = \sup\{\kappa_n\colon n < \omega\}$, choose (discrete) groups G_n with $|G_n| = \kappa_n$ and let G be the product group $\prod_{n<\omega} G_n$ with the usual product topology. Every basic neighborhood U of 1_G has the form

$$U = \bigcap_{n \leqslant N} \pi_n^{-1}\{1_n\} = \{1_H\} \times \prod_{n>N} G_n.$$

for some $N < \omega$ (and with $H := \prod_{n \leqslant N} G_n$); since $|H| = \prod_{n \leqslant N} \kappa_n = \kappa_N < \kappa$, fewer than κ-many translates of U suffice to cover G. It follows that $b(G) \leqslant \kappa = w(G) = d(G)$, as asserted.

(d) In view of Theorem 5.5, the hypothesis "$d(G) < b(G)$" in the next theorem is equivalent to the condition $b(G) = (d(G))^+$.

Theorem 5.7. *Let G be a Hausdorff topological group such that $d(G) < b(G)$. Then G has a closed suitable subset.*

Proof. Let $\kappa = d(G)$ and (using Remark 5.4) choose $\{p_\xi\colon \xi < \kappa\} \subseteq G$ and an open neighborhood V of 1 such that each set of the form pV (with $p \in G$) meets at most one of the sets $p_\xi V$. Since V is open we have $d(V) \leqslant \kappa$ so there is a (not necessarily faithfully indexed) dense subset $\{x_\xi\colon \xi < \kappa\}$ of V. For each $\xi < \kappa$, let $y_\xi = p_\xi x_\xi$. Then define

$$S = \{p_\xi\colon \xi < \kappa\} \cup \{y_\xi\colon \xi < \kappa\},$$

and let H be the (open) subgroup of G generated by $V \cup \{p_\xi V\colon \xi < \kappa\}$. Then $S \subseteq H$. The subgroup of G generated by S contains for each $\xi < \kappa$ the point $x_\xi = (p_\xi)^{-1} y_\xi$, so the closure of that group contains V and hence H. Each point p in G has a neighborhood (namely pV) which meets at most one of the sets $p_\xi V$; this neighborhood contains at most two elements of S. Thus S is closed and discrete in G. The upshot is that S is a closed suitable subset of H, with H an open subgroup of G. Then G itself has a closed suitable subset, by Theorem 4.2. □

Corollary 5.8. *Every separable Hausdorff topological group which is not totally bounded admits a closed suitable subset.*

Proof. Such a group G has $d(G) = \omega$ and $b(G) > \omega$ so Theorem 5.7 applies. □

Corollary 5.9. *Let G be a Hausdorff topological group with a nonempty open subset U such that $d(U) < b(G)$. Then G has a closed suitable set. Thus every locally separable Hausdorff topological group which is not totally bounded has a closed suitable subset.*

Proof. We take $1 \in U$. With $\kappa = d(U) < b(G)$, again from Remark 5.4 there is an open neighborhood V of 1 (with $V \subseteq U$) and $\{p_\xi\colon \xi < \kappa\} \subseteq G$ such that the family $\{p_\xi V\colon \xi < \kappa\}$ is uniformly V-discrete. Now with $\{x_\xi\colon \xi < \kappa\}$ chosen dense in V, the proof proceeds and concludes *verbatim* as in Theorem 5.7. □

The next corollary will be subsumed in a more general result later, but the proof is quite different.

Corollary 5.10. *Every locally separable metrizable topological group has a suitable subset.*

Proof. If such a topological group G is not totally bounded then Corollary 5.9 applies.

If G is totally bounded then its Weil completion [25] $\overline{G}$ is compact (hence, complete) and metrizable. By Theorem 1.2, $\overline{G}$ admits a suitable subset. Theorem 4.3 implies that a dense subgroup (in this case, G) of a metrizable group with a suitable subset (in this case, $\overline{G}$) itself has a suitable subset. □

It should be noted that neither Corollary 5.10 nor Theorem 1.12 can be strengthened to demanding the existence of a closed suitable set. Theorem 5.11 shows that, for example, the compact group $\{-1, 1\}^\kappa$, where κ is any cardinal number $\geqslant \omega$, has no closed suitable set. In particular, this is the case when $\kappa = \omega$ and the group is compact metrizable.

Theorem 5.11. *Let G be a countably compact Hausdorff group. If $|G| > 2^{\mathfrak{c}}$ or G is an infinite abelian torsion group, then G does not have a closed suitable set.*

Proof. Let S be a closed suitable subset of G. Then S is a closed discrete subspace of the countably compact group G. Hence S is finite. Then $|\overline{\langle S\rangle}| \leqslant 2^{\mathfrak{c}}$. So if $|G| > 2^{\mathfrak{c}}$, this is a contradiction and G has no suitable set.

On the other hand, if G is an abelian torsion group, then $\langle S\rangle$ is a finite group and so is closed in G. Hence $\overline{\langle S\rangle}$ is also finite and so does not equal G. In this case also G has no closed suitable set. □

Definition 5.12. A topological space is said to be of *countable pseudocharacter* if every singleton set is the intersection of a countable number of open sets.

Theorem 5.13. *Let H be a separable totally bounded Hausdorff topological group of countable pseudocharacter. If S is a countable dense subgroup of H, then there exists a discrete subset L of S such that L is closed in $H \setminus \{1\}$ and $S = \langle L\rangle$. So L is a suitable set for both S and H.*

Proof. Denote by G the completion of H. Then G is a compact topological group which contains H as a dense subgroup. Let $S = \{x_n\colon n \in \omega\}$ be a enumeration of S. Choose a decreasing sequence $\{U_n\colon n \in \omega\}$ of open neighborhoods of the identity e_G in G satisfying the following conditions:

(1) $U_{n+1}^2 \subset U_n$ for each $n \in \omega$;

(2) $\{1_H\} = H \cap (\bigcap_{n\in\omega} U_n)$.

We shall construct by induction an increasing sequence $\{L_k\colon k \in \omega\}$ of finite subsets of S to satisfy the following conditions for each $k \in \omega$:

(i) $x_k \in \langle L_k \rangle$;

(ii) $L_{k+1} \setminus L_k \subset U_k$;

(iii) $G = \langle L_k \rangle \cdot U_k$.

Being dense in H, the group S is dense in G. We have, therefore, the equality $G = S \cdot U_0$. Since G is compact, there exists a finite subset K_0 of S such that $K_0 \cdot U_0 = G$. In particular, there are $a_0 \in K_0$ and $u_0 \in U_0$ with $x_0 = a_0 \cdot u_0$. Then $u_0 = a_0^{-1} \cdot x_0 \in S$ and we put $L_0 = K_0 \cup \{u_0\}$. It is clear that $L_0 \subset S$.

Let $n \in \omega$ and suppose that we have defined an increasing sequence $L_0, \ldots, L_n$ of finite subsets of S satisfying (i)–(iii) for each $k \leqslant n$. Since S is dense in G, the set $\langle U_n \cap S \rangle$ is dense in the group $G_n = \langle U_n \rangle$. Obviously, G_n is open in G, and hence is closed compact. Therefore, one can find a finite subset F_{n+1} of $\langle U_n \cap S \rangle$ so that $F_{n+1} \cdot U_{n+1} = G_n$. In particular, there exists a finite subset K_{n+1} of $U_n \cap S$ with $F_{n+1} \subset \langle K_{n+1} \rangle$, whence $\langle K_{n+1} \rangle \cdot U_{n+1} = G_n$. Put $L'_{n+1} = L_n \cup K_{n+1}$. By (iii), we have

$$\langle L'_{n+1} \rangle \cdot U_{n+1} = \langle L'_{n+1} \rangle \cdot \langle K_{n+1} \rangle \cdot U_{n+1} = \langle L'_{n+1} \rangle \cdot G_n \supset \langle L_n \rangle \cdot U_n = G. \quad (***)$$

By $(***)$, there are $a_{n+1} \in \langle L'_{n+1} \rangle$ and $u_{n+1} \in U_{n+1}$ such that $x_{n+1} = a_{n+1} \cdot u_{n+1}$. Since $a_{n+1} \in \langle L'_{n+1} \rangle \subset S$ and $x_{n+1} \in S$, we conclude that $u_{n+1} \in S$ and put $L_{n+1} = L'_{n+1} \cup \{u_{n+1}\}$. Clearly, L_{n+1} is a finite subset of S and $L_n \subset L_{n+1}$. From $(***)$ it follows that $\langle L_{n+1} \rangle \cdot U_{n+1} = G$. Thus, the conditions (i)–(iii) hold at the step $n + 1$.

Put $L = \bigcup_{n\in\omega} L_n$. From (ii) it follows that $L \setminus U_k \subset L_k$ is a finite set for each $k \in \omega$. The latter, along with (1) and (2), implies that L is closed discrete subset of $H \setminus \{1_H\}$. It remains to apply (i) in order to conclude that $S = \langle L \rangle$. The result is proved. □

Theorem 5.14. *A locally separable Hausdorff topological group of countable pseudocharacter has a suitable subset. In particular a locally separable metrizable topological group has a suitable set.*

Proof. Let H be a locally separable topological group of countable pseudocharacter. Then it has an open separable neighborhood U of the identity. Let G be the separable open subgroup of H generated by U.

If G is not totally bounded, then G has a suitable set by Corollary 5.8. Otherwise let D be a countable dense subset of G and $S = \langle D \rangle$. Then S is a dense countable subgroup of G and an application of Theorem 5.13 shows G has a suitable set. Finally, as G has a suitable set Theorem 4.2 implies that H has a suitable set too. □

Note that we now have two proofs that every (locally) separable metrizable group has a suitable set, see Theorem 5.14 and Corollary 5.10. In the next section we will prove a more general result.

6. Metrizable topological groups

Recall that a family $\{F_i\colon\ i \in I\}$ of subsets of a topological space X is said to be *locally finite* if for each $x \in X$ there is an open neighborhood U of x such that U intersects only a finite number of the F_i.

The following lemma is easily verified.

Lemma 6.1. *Let X be a topological space and $\mathcal{F}$ a locally finite family of subsets of X, each discrete and closed in X. Then $\bigcup_{F\in\mathcal{F}} F$ is discrete, and closed in X.*

Proposition 6.2. *Let G be a topological group of countable pseudocharacter which is not totally bounded. If G has a suitable set S, then it has a closed suitable set S'. Further, if $G = \langle S\rangle$ then the closed set S' can be chosen so that $G = \langle S'\rangle$.*

Proof. Let U be a neighborhood of 1 such that no finite $F \subseteq G$ satisfies $G = FU$. Let $x_0 \notin U$ and recursively choose $x_{n+1} \in G\backslash(U \cup \bigcup_{k\leqslant n} x_k U)$. Choose a symmetric neighborhood V of 1 such that $V^4 \subseteq U$ and note this:

For every $p \in G$ the set pV meets at most one of the sets $x_n V$. $\quad (*)$

[*Proof.* If $pv_1 = x_n v_2$ and $pv_3 = x_m v_4$, all $v_i \in V$, say with $n < m$, then $x_m = x_n v_2 v_1^{-1} v_3 v_4^{-1} \in x_n U$, contradiction.]

Now let $\{1\} = \bigcap_{n<\omega} U_n$ with each U_n open and with $\overline{U}_{n+1} \subseteq U_n$. We assume without loss of generality, replacing U_n by $U_n \cap V$ if necessary, that each $U_n \subseteq V$. Define the (possibly empty) annulus A_n by $A_n = U_n\backslash U_{n+1}$.

Now let S be suitable for G. If $1 \in S$ then S is closed and there is nothing to prove, so we assume $1 \notin S$.

For $n < \omega$ define $X_n = \{x_n\} \cup x_n \cdot (A_n \cap S) = x_n(\{1\} \cup (A_n \cap S))$, and define $S' = (S\backslash U_0) \cup \bigcup_{n<\omega} X_n$.

We want to see that S' is a closed suitable set.

(1) $\overline{\langle S'\rangle} = G$. It is enough to show that $\langle S'\rangle \supseteq S$. Surely $S' \supseteq S\backslash U_0$. If $p \in S \cap U_0$ then since $p \neq 1$ there is a largest $n < \omega$ such that $p \in U_n$. Then $p \in A_n \cap S$ so $p \in x_n^{-1}(x_n(A_n \cap S)) \subseteq \langle S'\rangle$.

(2) *S' is closed and discrete.* According to Lemma 6.1, it is enough to see that
   - (a) the set $S\backslash U_0$ is closed and discrete;
   - (b) each set X_n is closed and discrete; and
   - (c) the family $\{X_n\colon\ n < \omega\} \cup \{S\backslash U_0\}$ is locally finite.

(a) Surely $S\backslash U_0$ is discrete. An accumulation point of $S\backslash U_0$ must be an accumulation point of S, i.e., must be 1; but U_0 is a neighborhood of 1 missing $S\backslash U_0$. Thus $S\backslash U_0$ is closed.

(b) Conceptually it is easier to consider $\{1\} \cup (A_n \cap S)$, of which X_n is a translate. This set is closed because 1 adjoined to any subset of S yields a closed set. It is discrete at points of S, and it is discrete at 1 because U_{n+1} is a neighborhood of 1 whose intersection with this set is 1.

(c) It is clear from $(*)$ that the family $\{x_n V\colon n < \omega\}$ is locally finite. Since $X_n \subseteq x_n V$ (all n) the family $\{X_n\colon n < \omega\}$ is therefore locally finite. Adjoining the single set $\{S \backslash U_0\}$ cannot destroy local finiteness.

We noted in the proof of (1) that $\langle S' \rangle \supseteq S$. Thus S' generates G algebraically if S does. □

Theorem 6.3. *Let G be a noncompact topological group of countable pseudocharacter. If G has a suitable set, then it has a closed suitable set. Further, if $G = \langle S \rangle$ then the closed set S' can be chosen so that $G = \langle S' \rangle$.*

Proof. Consider two cases.

Case 1. G is not totally bounded. Then the claim follows directly from Proposition 6.2.

Case 2. G is totally bounded. Denote by H the group completion of G. Then H is compact and $H \neq G$ because G is not compact. If G intersects every nonempty G_δ-set in H, then G is pseudocompact (see [3, 6.4]; [4]). However, a pseudocompact group of countable pseudocharacter is metrizable, and hence compact [1, 3.1]; [5], which gives us the contradiction $H = G$.

We have thus proved that there exists a nonempty G_δ-set P in H disjoint from G. Pick a point $h \in P$. One easily defines a sequence $\{V_n\colon n \in \omega\}$ of open neighborhoods of h in H so that $\overline{V_{n+1}} \subseteq V_n$ for each $n \in \omega$ and $\bigcap_{n\in\omega} V_n \subset P$. Let $\{U_n\colon n \in \omega\}$ be a sequence of open neighborhoods of 1 in H with the following properties:

(1) $h \cdot U_n \subseteq V_n$ for each $n \in \omega$;

(2) $U_{n+1}^2 \subseteq U_n$ for each $n \in \omega$;

(3) $\{1\} = G \cap \bigcap_{n\in\omega} U_n$.

For every $n \in \omega$ pick a point $x_n \in G \cap h \cdot U_n$ and put $A_n = U_n \setminus U_{n+1}$ and $X_n = x_n \cdot (\{1\} \cup (S \cap A_n))$.

Let S be a suitable set for G. We claim that

$$S' = (S \setminus U_0) \cup \bigcup_{n\in\omega} X_n$$

is a closed suitable subset in G.

(1) $\overline{\langle S' \rangle}^G = G$. It is enough to show that $\langle S' \rangle \supseteq (S \setminus \{1\})$. Surely $S' \supseteq S \setminus U_0$. If $p \in S \cap U_0$ and $p \neq 1$, then there is the largest $n < \omega$ such that $p \in U_n$. Then $p \in A_n \cap S$ so $p \in x_n^{-1} \cdot (x_n \cdot (A_n \cap S)) \subseteq \langle S' \rangle$.

(2) *S' is closed and discrete.* According to Lemma 6.1, it is enough to see that

(a) the set $S \setminus U_0$ is closed and discrete;

(b) each set X_n is closed and discrete;

(c) the family $\{X_n\colon n \in \omega\} \cup \{S \setminus U_0\}$ is locally finite.

The property (a) follows from the fact that 1 can be the only cluster point for any subset of S, in particular, for $S \setminus U_0$; but U_0 is a neighborhood of 1 missing $S \setminus U_0$. Thus, $S \setminus U_0$ is closed in G as well as all its subsets. Hence $S \setminus U_0$ is discrete.

To see (b), it suffices to show that the set $\{1\} \cup (S \cap A_n)$, a translate of X_n, is closed and discrete in G. This set is closed because 1 adjoined to any subset of S yields a closed set. It is discrete at points of S, and it is discrete at 1 because U_{n+1} is a neighborhood of 1 whose intersection with this set is 1.

Let us verify (c). Since $\{1\} \cup (S \cap A_n) \subseteq U_n$ for each $n \in \omega$, we have

$$X_{n+1} \subseteq x_{n+1} \cdot U_{n+1} \subseteq h \cdot U_{n+1} \cdot U_{n+1} \subseteq h \cdot U_n \subseteq V_n. \tag{$*$}$$

From the definition of the sets V_n it follows that $V_{n+1} \subseteq \overline{V}_{n+1} \subseteq V_n$ for each $n \in \omega$ and the set $K = \bigcap_{n\in\omega} V_n$ is disjoint from G, for $K \subseteq P$. Now apply ($*$) to conclude that all cluster points of the family $\{X_n\colon n \in \omega\}$ lie in K, that is, outside of G. This means that this family is locally finite in G. Adjoining the single set $\{S \setminus U_0\}$ cannot destroy local finiteness.

We noted in the proof of (1) that $\langle S'\rangle \supseteq (S \setminus \{1\})$. Thus S' generates G algebraically if S does. □

For our principal result, Theorem 6.6, we use a couple of simple lemmas.

Lemma 6.4. *Let X be a subset of a topological group G and U an open neighborhood of the identity in G. Then there exist an ordinal γ and a subset $Y = \{x_\alpha\colon \alpha < \gamma\}$ of X such that $x_\beta \notin x_\alpha \cdot U$ whenever $\alpha < \beta < \gamma$ and $X \subseteq Y \cdot U$. Further, if V is a neighborhood of 1 in G with $V^{-1} = V$ and $V^4 \subseteq U$ then the set Y is uniformly V-discrete in G.*

Proof. Straightforward recursive construction. □

Lemma 6.5. *If $\{O_n\colon n < \omega\}$ is a symmetric open basis at the identity of a topological group G and for each $n < \omega$ the set $F_n \subseteq G$ satisfies $F_n \cdot O_n = G$, then $F = \bigcup_{n<\omega} F_n$ is dense in G.*

Proof. If W is a nonempty open subset of G, then there exists an $x \in G$ and $n < \omega$ such that $x \cdot O_n \subseteq W$. Then $x \in y \cdot O_n$, for some $y \in F_n$, whence $y \in x \cdot O_n^{-1} = x \cdot O_n \subseteq W$. So we have $y \in W \cap F_n \subseteq W \cap F \neq \emptyset$. □

Theorem 6.6. *Every metrizable topological group G has a suitable set. Further, if G is not compact, it has a closed suitable set.*

Proof. By Theorem 6.3, it suffices to prove that a metrizable topological group G has a suitable set. Let $\{V_n\colon n \in \omega\}$ be a base at the identity in G satisfying $V_0 = G$, $V_{n+1}^4 \subset V_n$ and $V_n^{-1} = V_n$ for each $n \in \omega$. By Lemma 6.4, for every $n \in \omega$ we can define a subset $F_n = \{x_{n,\alpha}\colon \alpha < \gamma_n\}$ of V_n so that

(i) $V_n \subseteq F_n \cdot V_{n+1}$;

(ii) $x_{n,\beta} \notin x_{n,\alpha} \cdot V_{n+1}$ whenever $\alpha < \beta < \gamma_n$.

Put $S = \bigcup_{n\in\omega} F_n$. We claim that the set S is suitable for G. First, we prove that

$$\langle S\rangle \cdot V_n = G \quad \text{for each } n \in \omega. \tag{1}$$

Indeed, it suffices to show that

$$F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot V_{n+1} = G \quad \text{for each } n \in \omega. \tag{2}$$

We prove (2) by induction on n. The equality $F_0 \cdot V_1 = G$ follows from (i). If (2) is valid for some $n \in \omega$, then (i) implies that

$$F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot F_{n+1} \cdot V_{n+2} \supseteq F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot V_{n+1} = G.$$

This proves (2), and hence (1). Since the sets V_n form a base at the identity of G, the definition of S, (1) and Lemma 6.5 imply together that $\langle S\rangle$ is dense in G. It remains to show that S is closed and discrete in $G\setminus\{1\}$. Let $x \in G$ be arbitrary, $x \neq 1$. There exists $n \in \omega$ such that $x \notin \overline{V_n}$. Since $F_k \subseteq V_k$ for each $k \in \omega$, we have $F_k \cap (G \setminus \overline{V_n}) = \emptyset$ whenever $k > n$. Therefore, $G \setminus \overline{V_n}$ can intersect only the sets $F_0, F_1, \dots, F_n$. From (ii) and $V_{n+2}^4 \subseteq V_{n+1}$ it follows that the set F_n is uniformly V_{n+2}-discrete in G, and hence is closed discrete in G; $n \in \omega$. Therefore, the union $F = F_0 \cup F_1 \cup \dots \cup F_n$ is a closed discrete subset of G and there exists an open neighborhood W of x in G whose intersection with F contains at most one point. Clearly, the neighborhood $O = W \cap (G \setminus \overline{V_n})$ of x has the property $|O \cap S| \leqslant 1$. This proves that S is closed and discrete in $G \setminus \{1\}$.

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References

[1] A.V. Arhangel'skii, Classes of topological groups, Russian Math. Surveys 36 (1981) 151–174.
[2] J. Cleary and S.A. Morris, Generators for locally compact groups, Proc. Edinburgh Math. Soc. 36 (1993) 463–467.
[3] W.W. Comfort, Topological groups, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 1143–1263.
[4] W.W. Comfort and K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pacific J. Math. 16 (1966) 483–496.
[5] W.W. Comfort and T. Soundararajan, Pseudocompact group topologies and totally dense subgroups, Pacific J. Math. 100 (1982) 61–84.
[6] A. Douady, Cohomologie des groupes compacts totalement discontinus, Séminaire Bourbaki, Exposé 189, Secrétariat Math. (Paris, 1960); Reprint: Benjamin, New York, 1966.
[7] E.K. van Douwen, The product of two countably compact topological groups, Trans. Amer. Math. Soc. 262 (1980) 417–427.
[8] R. Engelking, General Topology (Heldermann, Berlin, 1989).
[9] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand, New York, 1960).
[10] J.P.L. Hardy, S.A. Morris and H.B. Thompson, Applications of the Stone–Čech compactification to free topological groups, Proc. Amer. Math. Soc. 55 (1976) 160–164.
[11] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis I, Grundl. Math. Wissensch. 115 (Springer, Berlin, 1963).

[12] K.H. Hoffmann and S.A. Morris, Weight and $\mathfrak{c}$, J. Pure Appl. Algebra 68 (1990) 181–194.
[13] K.H. Hoffmann and S.A. Morris, Free compact groups III: Free semisimple compact groups, in: J. Adámek and S. MacLane, eds., Proceedings of the Categorical Topology Conference, Prague, August 1988 (World Scientific Publishers, 1989) 208–219.
[14] K.H. Hoffmann and S.A. Morris, Generators on the arc component of compact connected groups, Math. Proc. Cambridge Phil. Soc. 113 (1993) 479–486.
[15] K.H. Hoffmann and S.A. Morris, Finitely generated connected locally compact groups, Seminar Sophus Lie 2 (1992) 123–134.
[16] K.H. Hoffmann and S.A. Morris, The Structure of Compact Groups: A Primer for Students—A Handbook for Experts (Walter de Gruyter, New York, to appear).
[17] K. Iwasawa, On solvable extensions of algebraic number fields, Ann. of Math. 58 (1953) 548–572.
[18] H. Koch, Galoissche Theorie der p-Erweiterung (VEB Deutscher Verlag der Wissenschaften, Berlin, 1970).
[19] J. Mack, S.A. Morris and E.T. Ordman, Free topological groups and the projective dimension of a locally compact abelian group, Proc. Amer. Math. Soc. 40 (1973) 303–308.
[20] A.A. Markov, On free topological groups, C. R. (Doklady) Acad. Sci. URSS (N.S.) 31 (1941) 299–301; Bull. Acad. Sci. Sér. Math. [Izvestiya Akad. Nauk SSSR] 9 (1945) 3–64; (in Russian, English summary); English transl.: Amer. Math. Soc. Transl. 30 (1950) 11–88; reprinted in: Amer. Math. Soc. Transl. 8 (1) (1962) 195–273.
[21] S.A. Morris, Free abelian topological groups, Proc. Internat. Conf. Categorical Topology, Toledo, Ohio (Heldermann, Berlin, 1984) 375–391.
[22] S.A. Morris, Varieties of topological groups, Colloq. Math. 46 (1982) 147–165.
[23] N.T. Varopoulos, A theorem on the continuity of locally compact groups, Proc. Cambridge Philos. Soc. 60 (1964) 449–463.
[24] R.C. Walker, The Stone–Čech Compactification, Ergeb. Math. Grenzgeb. 83 (Springer, Berlin, 1974).
[25] A. Weil, Sur les Espaces à Structure Uniforme et sur la Topologie Générale, Publ. Math. Univ. Strasbourg (Hermann, Paris, 1937).