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LIMIT LAWS FOR WIDE VARIETIES OF TOPOLOGICAL GROUPS

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ABSTRACT. In 1977, Taylor introduced limit laws as natural analogues for topological algebras of algebraic laws for abstract algebras, and showed, in analogy to Birkhoff's theorem, that a class of topological algebras is a wide variety if and only if it is the class of models $\text{Mod}(\Sigma \cup \Theta)$ for some set Σ of algebraic laws and some perhaps proper class Θ of limit laws. A wide variety is a class of topological algebras closed under the formation of subalgebras, products and continuous homomorphic images. This paper is concerned specifically with wide varieties of topological groups, and limit laws in topological groups. The main contributions are as follows. (1) As a step towards determining whether 'perhaps proper class' above can be strengthened to 'set', a simple necessary and sufficient condition is derived for a wide variety to require a set rather than a proper class of limit laws. (2) Two closely related families of wide varieties, $T(\kappa)$ and $B(\kappa)$, for κ an infinite cardinal, are studied in detail. The varieties $T(\kappa)$ have played an important role in the theory to date, while the $B(\kappa)$ are studied here for the first time. Detailed information about both families is obtained. In particular (i) a wide sub-variety of $T(\kappa)$ requires only a set of limit laws in addition to those defining $T(\kappa)$, and (ii) $B(\kappa)$ is defined by a set of limit laws. (3) A detailed analysis is given of certain simple limit laws in locally compact abelian groups.

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1. Introduction.

The notions of a *variety* and a *wide variety* of topological groups were introduced by Morris [8, 9, 10] and Taylor [14], respectively, and have been studied extensively since (see [11] and its bibliography). (Taylor in fact deals with varieties and wide varieties of general topological algebras.) A wide variety of topological groups is a class of topological groups closed under the formation of subgroups, arbitrary products, and continuous homomorphic images; if the latter condition is weakened to closure under quotients, then the class is called a variety.

A major focus of Taylor's work was the attempt to characterise varieties and wide varieties by suitable generalisations of the algebraic laws which are well known to characterise abstract varieties of groups (see [13] and [2], for example). In the topological case, this attempted characterisation has been completely successful only in the case of wide varieties. We outline the relevant ideas.

Suppose that D is a directed set and V is any set. Then a *limit law* (with respect to D and V) is a formal expression $[\tau_d \to e]$, where d runs through the elements of D, and each τ_d is a term in the first order theory of groups which has V as its (not necessarily countable) set of variables. Given a topological group G and a variable valuation $\phi : V \to G$, such a law *is satisfied* (or *holds*) *with respect to* ϕ *in* G if the net $\tau_d[\phi]$ converges to the identity e of G, where for any term τ, $\tau[\phi]$ is the term assignment of τ in G with respect to ϕ; we may then write $G \models [\tau_d \to e][\phi]$. A law $[\tau_d \to e]$ *is satisfied* (or *holds*) *in* G (or G *models* the law) if $[\tau_d \to e]$ is satisfied with respect to every valuation ϕ; we may then write $G \models [\tau_d \to e]$.

It is often convenient to adopt the equivalent view in which, given D and V as before, a limit law is a formal expression $[\tau_d \to e]$, where the τ_d are elements of the (abstract) free group $F(V)$ on V, and in which the place of valuations ϕ is taken by maps ϕ of the free basis V of $F(V)$ into G and their canonical extensions to homomorphisms from $F(V)$ to G.

If Σ is a set of algebraic laws and Θ is a class of limit laws, then it is easy to see that the class of topological groups which model all the laws in Σ and Θ, which we denote by $\mathrm{Mod}(\Sigma \cup \Theta)$, is a wide variety. (We make the assumption that algebraic laws are over some countably infinite set of variables, fixed once and for all.) In an analogue of Birkhoff's famous theorem, Taylor [14] proves a converse to the above statement: for any wide variety $\mathcal{V}$, there are a set Σ of algebraic laws and a (perhaps proper) class Θ of limit laws such that $\mathcal{V} = \text{Mod}(\Sigma \cup \Theta)$.

Two general problems have prompted our work in this paper. The first is the question: Can '(perhaps proper) class' above be strengthened simply to 'set'? The second problem is, loosely, to find interesting, non-trivial limit laws—for example, laws satisfied in well-understood groups, or sets or classes of laws characterising specific wide varieties of interest.

We do not have a full solution to the first problem above. In Section 2, however, we give a simple necessary and sufficient condition for a variety $\mathcal{V}$ to require a set (rather than a proper class) of limit laws for its definition (Theorem 2.3). The condition is that there exists a cardinal $\mu \equiv \mu_{\mathcal{V}}$ with the property that, for all topological groups G, G is in $\mathcal{V}$ if and only if all subgroups of G of cardinality less than or equal to μ are in $\mathcal{V}$. Our proof is essentially a refinement of the proof of Taylor's characterisation of wide varieties.

Section 3 contains results pertaining to both the problems above. The family of wide varieties $T(\kappa)$, introduced by Morris [10], have played an important role in the theory to date. For κ an infinite cardinal, $T(\kappa)$ is the class of topological groups in which every neighbourhood of the identity e contains a normal subgroup of index less than κ. In Section 3, we note a number of elementary but useful facts about $T(\kappa)$, and then show (Theorem 3.6) that a wide sub-variety of $T(\kappa)$ requires only a set of limit laws in addition to those defining $T(\kappa)$. We do not know whether $T(\kappa)$ itself requires a proper class of laws for its definition.

In Section 3, we also consider a new family of wide varieties $B(\kappa)$, closely related to the varieties $T(\kappa)$. For any κ, $B(\kappa)$ is the class of κ-*precompact* topological groups—those topological groups G with the property that G can be covered by fewer than κ translates of any open set. We introduce a new class of special limit laws, the *uniform limit laws*, and we show (Theorem 3.16) that $B(\kappa)$ is defined by a set of such laws.

Sections 4 and 5 are concerned with the second of the problems above. Section 4 goes some way towards explaining why few simple laws in well known topological groups have so far been discovered. The most thoroughly investigated class of topological groups is the class of locally compact abelian groups, and the simplest limit laws are the *sequential laws*—those indexed by the directed set of natural numbers. The main result of

Section 4 (Theorem 4.11) is that non-totally disconnected locally compact abelian groups have no non-trivial sequential laws (in a sense made precise in Section 4). Examples are given to show that abelian topological groups may have non-trivial sequential laws if they are totally disconnected, or if they are not locally compact.

Finally, in Section 5, we give a miscellany of results and examples, dealing with the relations between some of the varieties studied in previous sections, with laws satisfied in particular topological groups, and with sets of limit laws defining some specific wide varieties.

DEFINITIONS AND NOTATION.

If $\mathcal{V}$ is a wide variety (or variety) and X is a Tychonoff space, then the *free topological group in* $\mathcal{V}$ *on* X [8], if it exists, is a topological group $F_{\mathcal{V}}(X) \in \mathcal{V}$ and an embedding $i : X \to F_{\mathcal{V}}(X)$, such that $i(X)$ algebraically generates $F_{\mathcal{V}}(X)$, and such that for any topological group $G \in \mathcal{V}$ and any continuous map $\phi : X \to G$, there is a continuous homomorphism $\Phi : F_{\mathcal{V}}(X) \to G$ such that $\phi = \Phi \circ i$.

It is shown in [8] that $F_{\mathcal{V}}(X)$ exists if and only if some member of $\mathcal{V}$ contains X as a subspace. Provided that $\mathcal{V}$ contains a non-indiscrete group, the free topological group in $\mathcal{V}$ on every discrete space exists. If the latter is the case, then for a topological group G, we have $G \in \mathcal{V}$ if and only if G is the continuous homomorphic image of the free topological group in $\mathcal{V}$ on a discrete space with the same cardinality as G. We make substantial use below of varietal free topological groups on discrete spaces, and it is convenient to exclude from further consideration those wide varieties which contain only indiscrete topological groups.

We frequently use the fact that the class of abstract groups which occur, with some topology, in a wide variety $\mathcal{V}$ forms an abstract variety in the standard sense of [13] and [2]. It is clear that the underlying group of a varietal free topological group is the abstract varietal free group on the underlying set of the space concerned.

For any set, topological space or topological group X, we use $X_{\mathfrak{I}}$ and $X_{\mathfrak{D}}$ to denote X with the indiscrete and discrete topology, respectively, considered as a topological space or topological group, as appropriate.

It is often convenient to use cardinal numbers as representative sets of their own cardinality; for example, we frequently use $F(\lambda)$ as a representative free group of rank λ, for a cardinal λ. The cardinality of a set X is denoted by $|X|$.

If $(X,\mathcal{U})$ is a uniform space and $U \in \mathcal{U}$, then $U[x]$ denotes $\{y \in X : (x,y) \in U\}$.

2. Laws Defining Wide Varieties.

For any cardinal λ, let $\mathcal{D}_\lambda$ be a set comprising exactly one representative from each isomorphism class of directed sets of cardinality less than or equal to 2^λ. For any wide variety $\mathcal{V}$, let $\Theta_\lambda^{\mathcal{V}}$ be the collection of nets in the abstract free group $F(\lambda)$ which are indexed by directed sets from $\mathcal{D}_\lambda$ and whose images converge to e in the varietal free group $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ under the canonical homomorphism from $F(\lambda)$. Clearly, $\Theta_\lambda^{\mathcal{V}}$ is a set. Also let $\Theta^{\mathcal{V}}$ be the proper class $\bigcup_\lambda \Theta_\lambda^{\mathcal{V}}$, where the union is over all cardinals λ. Finally, let $\Sigma^{\mathcal{V}}$ be the set of algebraic laws satisfied by the groups in $\mathcal{V}$. We write Θ_λ, Θ and Σ for $\Theta_\lambda^{\mathcal{V}}$, $\Theta^{\mathcal{V}}$ and $\Sigma^{\mathcal{V}}$, respectively, when the context permits.

2.1 Lemma. *If $G \in \mathrm{Mod}(\Sigma \cup \Theta_\lambda)$ and $|G| \leq \lambda$, then $G \in \mathcal{V}$.*

Proof. Let f be any surjection from λ to G. Since G models Σ, f extends algebraically to a surjective homomorphism $\hat{f} : F_{\mathcal{V}}(\lambda_{\mathfrak{D}}) \to G$, so since $\mathcal{V}$ is closed under the formation of continuous images, it will suffice if we prove that $\hat{f}$ is continuous at e. Therefore it suffices to show that if $\langle w_N : N \in \mathcal{N}\rangle$ is a net in $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ which is indexed by the directed set $\mathcal{N}$ of neighbourhoods at e and satisfies $w_N \in N$ for all $N \in \mathcal{N}$, then $\hat{f}(w_N)$ converges to e in G. Now the set of neighbourhoods at e in $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ has cardinality at most 2^λ, so we may assume without loss of generality that $\mathcal{N}$ is in $\mathcal{D}_\lambda$. Under the natural homomorphism from $F(\lambda)$ to $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$, choose arbitrary pre-images $\{v_N\}$ of the $\{w_N\}$. Then clearly the law $[v_N \to e]$ is in Θ_λ, so G models that law, and therefore $\hat{f}(w_N)$ converges to e in G as required. $\square$

Taylor's characterisation of wide varieties now follows.

2.2 Theorem. *If $\mathcal{V}$ is a wide variety, then $\mathcal{V} = \mathrm{Mod}(\Sigma \cup \Theta)$.*

Proof. Write $\mathcal{V}' = \mathrm{Mod}(\Sigma \cup \Theta)$. To show that $\mathcal{V} \subseteq \mathcal{V}'$, it suffices to show that $F_{\mathcal{V}}(\lambda_{\mathfrak{D}}) \in \mathcal{V}'$, for all cardinals λ, since every group in $\mathcal{V}$ is the continuous image of such a group. Certainly each $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ models the laws in Σ. Fix a cardinal μ; we shall show that $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ models the laws in Θ_μ. Let

$i : \mu \to F_{\mathcal{V}}(\mu_{\mathfrak{D}})$ be the canonical injection. Given any map $f : \mu \to F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$, there is a continuous homomorphism $\hat{f} : F_{\mathcal{V}}(\mu_{\mathfrak{D}}) \to F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ such that $\hat{f} \circ i = f$. Now by definition the laws in Θ_μ hold in $F_{\mathcal{V}}(\mu_{\mathfrak{D}})$ under i, and so the continuity of $\hat{f}$ ensures that they also hold in $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ under $\hat{f}$. Since f is arbitrary, $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ models the laws in Θ_μ. Thus $F_{\mathcal{V}}(\lambda_{\mathfrak{D}})$ models all the laws in Θ, and so $\mathcal{V} \subseteq \mathcal{V}'$.

Conversely, if $G \in \mathcal{V}'$, then $G \in \mathrm{Mod}(\Sigma \cup \Theta_\lambda)$, where $\lambda = |G|$, and we have $G \in \mathcal{V}$, by Lemma 2.1. Thus $\mathcal{V}' \subseteq \mathcal{V}$, and the proof is complete. □

We now derive simple necessary and sufficient conditions for a wide variety to be defined by a set, rather than a proper class, of limit laws (together with a set of algebraic laws, as usual).

2.3 Theorem. *If $\mathcal{V}$ is a wide variety, then the following conditions are equivalent*:

(i) *$\mathcal{V}$ is defined by Σ and a set of limit laws*;
(ii) *there is a cardinal μ such that $\mathcal{V} = \mathrm{Mod}(\Sigma \cup \bigcup_{\lambda \le \mu} \Theta_\lambda)$*;
(iii) *there is a cardinal μ such that for all topological groups G, $G \in \mathcal{V}$ if and only if every subgroup H of G satisfying $|H| \le \mu$ is in $\mathcal{V}$.*

Proof. Clearly (ii) implies (i). We show first that (i) implies (iii). Thus, suppose that $\mathcal{V} = \mathrm{Mod}(\Sigma \cup \Lambda)$, for some set Λ of limit laws. Let X be the set of variables which appear in $\Sigma \cup \Lambda$, and let $\mu = |X| + \aleph_0$. Suppose that G is a topological group all of whose subgroups of cardinality less than or equal to μ are in $\mathcal{V}$, and let $f : X \to G$ be a variable valuation. Then the subgroup H of G generated by $f(X)$ is of cardinality less than or equal to μ, is therefore in $\mathcal{V}$, and therefore models $\Sigma \cup \Lambda$. Thus any law in $\Sigma \cup \Lambda$ holds in H under the valuation f, and therefore also holds in G under f. Since this is the case for every valuation f, we have $G \in \mathcal{V}$, and so (iii) holds.

Suppose that (iii) holds, and write $\mathcal{V}' = \mathrm{Mod}(\Sigma \cup \bigcup_{\lambda \le \mu} \Theta_\lambda)$. If $G \in \mathcal{V}$, then G models $\Sigma \cup \Theta$, by the previous theorem, and so $G \in \mathcal{V}'$. Conversely, if $G \in \mathcal{V}'$, then every subgroup of G of cardinality less than or equal to μ is in $\mathrm{Mod}(\Sigma \cup \Theta_\mu)$, and so, by Lemma 2.1, is in $\mathcal{V}$. By (iii), we therefore have $G \in \mathcal{V}$. Thus (ii) holds, and the proof is complete. □

3. The Varieties $T(\kappa)$ and $B(\kappa)$.

We recall from Section 1 that, for any infinite cardinal κ, $T(\kappa)$ is the wide variety of topological groups in which every neighbourhood of e contains a normal subgroup of index less than κ. Clearly $T(\kappa)$ contains all the discrete groups of cardinality less than κ, and contains all indiscrete groups. In particular, $T(\kappa)$ requires no algebraic laws, and we have $T(\kappa) = \mathrm{Mod}(\Delta(\kappa))$ for some class $\Delta(\kappa)$ containing only limit laws. We begin by noting some straightforward results on $T(\kappa)$.

3.1 Proposition. *Let G be a topological group, and suppose that for each neighbourhood N of e, H_N is a normal subgroup of G such that $H_N \subseteq N$. Then G is topologically isomorphic to a subgroup of the product $G_{\mathfrak{I}} \times \prod_N (G/H_N)$. In particular, if $\mathcal{V}$ is a wide variety and if $G_{\mathfrak{I}} \in \mathcal{V}$ and $G/H_N \in \mathcal{V}$ for each N, then $G \in \mathcal{V}$.*

Proof. It is straightforward to check that the map from G into the product defined using the identity mapping from G to $G_{\mathfrak{I}}$ and the projections of G into the G/H_N is an embedding. □

3.2 Proposition. *Let $\mathcal{V}$ be a wide variety. Then $\mathcal{V}$ is generated by a set of topological groups if and only if $\mathcal{V} \subseteq T(\kappa)$ for some infinite cardinal κ.*

Proof. If $\mathcal{V}$ is generated by a set of groups, then clearly $\mathcal{V} \subseteq T(\kappa)$ for any cardinal κ greater than the cardinalities of all the generating groups. Conversely, suppose that $\mathcal{V} \subseteq T(\kappa)$ for some κ. Now by Proposition 3.1, $\mathcal{V}$ is generated by the class containing all its groups of cardinality less than κ and all its indiscrete groups. The class of groups of cardinality less than κ may clearly be reduced to a set of groups comprising one representative of each its topological isomorphism classes. Also, the underlying abstract groups of the indiscrete groups in $\mathcal{V}$ form an abstract variety, and so they are generated algebraically by a single group [13]. Adding this group, given the indiscrete topology, to the earlier set therefore gives a set of topological groups which generates $\mathcal{V}$, as required. □

Notice that the proof of the converse part of the preceding proposition yields the following result.

3.3 Proposition. *If $\mathcal{V}$ is a wide variety such that $\mathcal{V} \subseteq T(\kappa)$ for some cardinal $\kappa > \aleph_0$, then $\mathcal{V}$ is generated by its groups of cardinality less than κ. In* *particular, $T(\kappa)$ (for $\kappa > \aleph_0$) is the wide variety generated by the topological groups of cardinality less than κ.* □

It is not difficult to show that if $\mathcal{V}$ is a wide variety and G is a group in the abstract variety underlying $\mathcal{V}$, then there is a (unique) finest group topology with which G occurs in $\mathcal{V}$. In fact, let $\mathcal{H}$ be a set of topological groups containing a representative of each topological isomorphism class of groups in $\mathcal{V}$ of cardinality at most $|G|$. Further, let $\{(H_\alpha, h_\alpha)\}$ be the set of all pairs consisting of a group $H_\alpha \in \mathcal{H}$ and an (abstract) homomorphism $h_\alpha : G \to H_\alpha$. Then there is a natural algebraic isomorphism of G into the product $\prod_\alpha H_\alpha$ induced by all the maps h_α. It is easy to check that the topology induced on G by this embedding is as claimed above, and indeed is the unique strongest topology on G making all homomorphisms from G into groups in $\mathcal{V}$ continuous. (This argument is a straightforward generalisation of one used in Theorem 9.13 of [3].)

In the case of the variety $T(\kappa)$, we can obtain a particularly simple description of the topology just mentioned.

Fix an infinite cardinal κ, and let $\mathcal{V}$ be any wide variety of the form $\mathcal{V} = \mathrm{Mod}(\Delta(\kappa) \cup \Sigma) = T(\kappa) \cap \mathrm{Mod}(\Sigma)$, for some set Σ of algebraic laws. Let G be any member of $\mathcal{V}$, and let $\mathcal{K}$ be the set of kernels of all the homomorphisms from G into groups in $\mathcal{V}$ of cardinality less than κ. Note that $\mathcal{K}$ is closed under finite intersections: if $f_1 : G \to H_1$ and $f_2 : G \to H_2$ are two such homomorphisms, then $\ker(f_1) \cap \ker(f_2) = \ker(f)$, where f is the obvious homomorphism from G into $H_1 \times H_2$ induced by f_1 and f_2; and clearly $H_1 \times H_2$ is in $\mathcal{V}$ and has cardinality less than κ. It follows by Theorem 4.5 of [5] that the collection of cosets of all the normal subgroups in $\mathcal{K}$ is an open basis for a group topology on G. Denote G, when equipped with this topology, by G'.

We claim that $G' \in \mathcal{V}$, and that the topology of G' contains that of G. Now $G_\daleth \in \mathcal{V}$, since $G_\daleth \in \mathrm{Mod}(\Sigma)$ and $G_\daleth$ satisfies all limit laws. Also, for each neighbourhood N of e in G', there is an open normal subgroup $H_N \subseteq N$ such that the discrete group G/H_N is in $\mathcal{V}$. Hence, by Proposition 3.1, we have $G' \in \mathcal{V}$. Let N be a neighbourhood of e in G with its original topology. Since $G \in T(\kappa)$, there is a surjective homomorphism $f : G \to H$, for some abstract group H of cardinality less than κ, such that $\ker(f) \subseteq N$. If we give H the quotient topology with respect to f, then we have $H \in \mathcal{V}$, and so we have $\ker(f)$ open in G'. It follows that the topology of G' contains that of G, and that this topology is therefore that described above.

We can use the above analysis to obtain the following explicit description of the free topological groups on discrete spaces in the varieties $T(\kappa)$.

3.4 Proposition. *Let $\mathcal{V}$ be a wide variety of the form $\mathcal{V} = \mathrm{Mod}(\Delta(\kappa) \cup \Sigma)$. If X is any set, and F is the free group on X in the abstract variety $\mathrm{Mod}(\Sigma)$, then the topology of the varietal free group $F_{\mathcal{V}}(X_{\mathfrak{D}})$ is generated by the normal subgroups of F of index less than κ.*

Proof. By the argument above, the topology of $F_{\mathcal{V}}(X_{\mathfrak{D}})$ is generated by the kernels of all the homomorphisms from F into groups in $\mathcal{V}$ of cardinality less than κ. If G is an arbitrary topological group of cardinality less than κ, and $f : F \to G$ is a homomorphism, then clearly $|f(F)| < \kappa$ and $f(F) \in \mathrm{Mod}(\Sigma)$. Since $T(\kappa)$ contains all topological groups of cardinality less than κ, we therefore have $f(F) \in \mathcal{V}$. Since the kernels of $f : F \to G$ and $f : F \to f(F)$ are identical, the result follows. □

This result can be combined usefully with our description in Section 2 of the laws $\Theta_\lambda^{\mathcal{V}}$ and $\Theta^{\mathcal{V}}$, in the case where $\mathcal{V}$ is $T(\kappa)$. Clearly, the varietal free group $F_{T(\kappa)}(\lambda_{\mathfrak{D}})$ is algebraically the free group $F(\lambda)$, and we have the following result. (Recall from Section 2 that for each cardinal λ, $\mathcal{D}_\lambda$ denotes a set comprising exactly one representative from each isomorphism class of directed sets of cardinality less than or equal to 2^λ.)

3.5 Corollary. *The wide variety $T(\kappa)$ is $\mathrm{Mod}(\Theta^{T(\kappa)})$, where $\Theta^{T(\kappa)} = \bigcup_\lambda \Theta_\lambda^{T(\kappa)}$, and where, for each cardinal λ, $\Theta_\lambda^{T(\kappa)}$ is the set of nets in $F(\lambda)$ which are indexed by directed sets in $\mathcal{D}_\lambda$ and which are eventually in each normal subgroup of $F(\lambda)$ of index less than κ.* □

The following theorem reduces the question whether wide sub-varieties of $T(\kappa)$ require sets or proper classes of limit laws to the question whether $T(\kappa)$ itself does: any wide sub-variety requires only a set of limit laws in addition to the class defining $T(\kappa)$.

3.6 Theorem. *Let $\mathcal{V}$ be a wide variety such that $\mathcal{V} \subseteq T(\kappa)$. Then $\mathcal{V} = \mathrm{Mod}(\Delta(\kappa) \cup \Sigma^{\mathcal{V}} \cup \Theta_\kappa^{\mathcal{V}})$.*

Proof. Write $\mathcal{V}' = \mathrm{Mod}(\Delta(\kappa) \cup \Sigma^{\mathcal{V}} \cup \Theta_\kappa^{\mathcal{V}})$. From Theorem 2.2, we have $\mathcal{V} \subseteq \mathcal{V}'$. To show that $\mathcal{V}' \subseteq \mathcal{V}$, Proposition 3.1 shows that it suffices to prove that every indiscrete group in $\mathcal{V}'$ is in $\mathcal{V}$ and that every group in $\mathcal{V}'$ of cardinality less than κ is in $\mathcal{V}$. Now an indiscrete group in $\mathcal{V}'$ satisfies $\Sigma^{\mathcal{V}}$ and every limit law, and so is in $\mathcal{V}$ by Theorem 2.2. If $G \in \mathcal{V}'$, then $G \in \mathrm{Mod}(\Sigma^{\mathcal{V}} \cup \Theta_\kappa^{\mathcal{V}})$, and so if G has cardinality less than κ, then $G \in \mathcal{V}$ by Lemma 2.1. Therefore $\mathcal{V}' \subseteq \mathcal{V}$. □

The most interesting question about $T(\kappa)$ which we are unable to resolve is whether $T(\kappa)$ is definable by a set, rather than a proper class, of limit laws. We turn to examination of a family of wide varieties related in an obvious fashion to the varieties $T(\kappa)$, and for which we can answer this question. We borrow a definition and two theorems from [7].

3.7 Definition. [7] Let κ be an infinite cardinal. A uniform space $(X, \mathcal{U})$ is *κ-precompact* if for each $U \in \mathcal{U}$ there is a set $\{x_\alpha\}$ of fewer than κ points in X such that $X = \bigcup_\alpha U[x_\alpha]$.

3.8 Theorem. [7] *A uniform space $(X, \mathcal{U})$ is κ-precompact if and only if for each $U \in \mathcal{U}$ there is a partition of X into a collection of fewer than κ sets $\{X_\alpha\}$ such that $X_\alpha \times X_\alpha \subseteq U$ for all α.* □

Let $B(\kappa)$ be the class of topological groups which are κ-precompact in their left uniformity. Then we have the following.

3.9 Theorem. *For each infinite cardinal κ, $B(\kappa)$ is a wide variety.*

Proof. The fact that $B(\kappa)$ is closed under the formation of continuous images and products follows directly from the definition of κ-precompactness, and the fact that $B(\kappa)$ is closed under the taking of subgroups follows easily from the characterisation of κ-precompactness given in Theorem 3.8. □

It is clear that the groups in $T(\kappa)$ are κ-precompact; indeed, κ-precompactness is a natural weakening of the topologico-algebraic condition defining membership of $T(\kappa)$ to a purely uniform condition. We therefore have the following result.

3.10 Theorem. *For each infinite cardinal κ, $T(\kappa) \subseteq B(\kappa)$.* □

We work now with a fixed infinite cardinal κ.

Recall that in a uniform space $(X,\mathcal{U})$, a set D is said to be *uniformly discrete* if there exists $U \in \mathcal{U}$ such that D is *uniformly discrete with respect to* U; that is, such that $U[d] \cap D = \{d\}$, for all $d \in D$. There is an elegant characterisation of κ-precompactness in terms of uniformly discrete subsets.

3.11 Theorem. [7] *A uniform space* $(X,\mathcal{U})$ *is* κ*-precompact if and only if all uniformly discrete subsets of* X *have cardinality less than* κ. □

3.12 Corollary. *A uniform space* $(X,\mathcal{U})$ *is* κ*-precompact if and only if every subspace of cardinality exactly* κ *is* κ*-precompact.* □

It is easy to see, further, that a topological group is κ-precompact if and only if every subgroup of cardinality exactly κ is κ-precompact, and Theorem 2.3 therefore gives a set of limit laws which define $B(\kappa)$. We go on, however, to describe a particularly simple set of such laws.

Let S be a fixed set of cardinality exactly κ, and let $\mathcal{U}_\kappa(S)$ be the coarsest uniformity on S which makes every map from S into all uniform spaces of cardinality strictly less than κ uniformly continuous. Equivalently, $\mathcal{U}_\kappa(S)$ is the coarsest uniformity on S which makes every map from S into all discrete uniform spaces of cardinality less than κ uniformly continuous. This latter description allows us to see straightforwardly that $\mathcal{U}_\kappa(S)$ has a basis consisting of all equivalence relations on S with strictly fewer than κ equivalence classes. (Note that the topology induced on S by $\mathcal{U}_\kappa(S)$ is discrete, though $\mathcal{U}_\kappa(S)$ is not the discrete uniformity.)

3.13 Theorem. *Let* $(X,\mathcal{U})$ *be a uniform space, then* $(X,\mathcal{U})$ *is* κ*-precompact if and only if every map from* $(S,\mathcal{U}_\kappa(S))$ *to* $(X,\mathcal{U})$ *is uniformly continuous.*

Proof. First, suppose that $(X,\mathcal{U})$ is κ-precompact, and let $\phi : S \to X$ be an arbitrary map. Given $U \in \mathcal{U}$, there is, by Theorem 3.8, a partition of X into fewer than κ subsets $\{X_\alpha\}$ such that $X_\alpha \times X_\alpha \subseteq U$ for all α. If we define $S_\alpha = \phi^{-1}(X_\alpha)$ for all α, then the equivalence relation $\bigcup_\alpha S_\alpha \times S_\alpha$ is in $\mathcal{U}_\kappa(S)$, and clearly $\bigcup_\alpha S_\alpha \times S_\alpha \subseteq (\phi \times \phi)^{-1}(U)$, so that ϕ is uniformly continuous.

Second, suppose that every map from S to X is uniformly continuous. We wish to show that X is κ-precompact, and by Corollary 3.12, it will suffice if we prove this when $|X| = \kappa$. For such an X, fix any bijection $\phi : S \to X$. Then ϕ is uniformly continuous, by hypothesis. Therefore, for any $U \in \mathcal{U}$, the set $(\phi \times \phi)^{-1}(U)$ is a member of $\mathcal{U}_\kappa(S)$. Therefore there is an equivalence relation R on S with fewer than κ equivalence classes $\{R_\alpha\}$ such that $R = \bigcup_\alpha R_\alpha \times R_\alpha \subseteq (\phi\times\phi)^{-1}(U)$. Then the sets $X_\alpha = \phi(R_\alpha)$ form a partition of X into fewer than κ sets, and we clearly have $X_\alpha \times X_\alpha \subseteq U$. By Theorem 3.8, X is κ-precompact, as required. □

We introduce a new class of limit laws, in terms of which we will describe the variety $B(\kappa)$.

3.14 Definition. Let $(X,\mathcal{U})$ be a uniform space. A *uniform limit law* on $(X,\mathcal{U})$ is a law of the form $[x_U^{-1}y_U \to e]$, where U runs through the directed set $\mathcal{U}$, and where $(x_U, y_U) \in U$ for all $U \in \mathcal{U}$.

3.15 Theorem. *Let G be a topological group and let $\phi : S \to G$ be any mapping. Then ϕ is uniformly continuous with respect to $\mathcal{U}_\kappa(S)$ and the left uniformity of G if and only if every uniform limit law on $(S,\mathcal{U}_\kappa(S))$ holds in G under ϕ. In particular, all mappings from S to G are uniformly continuous if and only if all uniform limit laws on S hold in G.*

Proof. First, suppose that ϕ is uniformly continuous and let $[x_U^{-1}y_U \to e]$ be a uniform limit law on S. Let N be a neighbourhood of e in G. Then the set $V = \{(g,h) : g^{-1}h \in N\}$ is a member of the left uniformity of G, and so there is a $U_0 \in \mathcal{U}_\kappa(S)$ such that $(\phi \times \phi)(U_0) \subseteq V$. Therefore, for all $U \in \mathcal{U}_\kappa(S)$ satisfying $U \subseteq U_0$, we have $(\phi(x_U), \phi(y_U)) \in V$, that is, $\phi(x_U)^{-1}\phi(y_U) \in N$. Hence $\phi(x_U)^{-1}\phi(y_U) \to e$.

Second, suppose that ϕ is not uniformly continuous. Then there exists a set V in the left uniformity of G such that $(\phi \times \phi)^{-1}(V) \notin \mathcal{U}_\kappa(S)$. Therefore, for each $U \in \mathcal{U}_\kappa(S)$ we can choose $(x_U, y_U) \in U$ such that $(\phi(x_U), \phi(y_U)) \notin V$. But there is a neighbourhood N of e in G such that $\{(g,h) : g^{-1}h \in N\} \subseteq V$, and it follows that $\phi(x_U)^{-1}\phi(y_U) \notin N$. Hence the uniform limit law $[x_U^{-1}y_U \to e]$ fails to hold in G under ϕ. □

Now set Υ_κ equal to the set of all uniform limit laws on $(S,\mathcal{U}_\kappa(S))$. Our main result on the varieties $B(\kappa)$ is as follows; the proof is immediate from Theorems 3.13 and 3.15.

3.16 Theorem. *For every infinite cardinal κ, $B(\kappa) = \mathrm{Mod}(\Upsilon_\kappa)$.* □

4. Sequential Laws in Abelian Topological Groups.

In this section, we discuss abelian groups exclusively, and it is convenient to think of the individual terms of a limit law as elements of the free abelian group $A(X)$ on the set X of variables involved, since clearly, given a variable valuation $\phi : X \to G$, where G is abelian, the extension of ϕ to a homomorphism from $F(X)$ to G may be factored through the natural homomorphism from $F(X)$ onto $A(X)$. Despite the abelian context, we shall continue to use multiplicative notation, except in specific abelian groups whose operations are traditionally written additively.

We may, as convenient, represent any element w of $A(X)$ either as a reduced word of the form $x_1^{a_1} \dots x_n^{a_n}$, where the x_i are distinct elements of X and the a_i are non-zero integers, or as a formal product $\prod_{x \in X} x^{a_x}$, in which only finitely many of the integers a_x are non-zero. For any variable x, the *exponent of* x *in* w is the integer a_x in the above product. We say that x *occurs non-trivially in* w if $a_x \neq 0$, and that it *occurs trivially* otherwise. These concepts are obviously well defined.

We call a limit law a *sequential limit law* if its index set is the directed set of natural numbers $\mathbf{N}$.

We say that an abelian topological group G *has large powers* if there exists a neighbourhood U of the identity such that for all $g \in G$ and for all $n \in \mathbf{N}$ with $n \neq 0$, there is an $h \in G$ such that $gh^n \notin U$. For example, if G is divisible and is not indiscrete, then G has large powers, though as the following easily proved lemma shows, this sufficient condition is not necessary.

4.1 Lemma. *The groups* $\mathbf{R}$, $\mathbf{Z}$ *and* $\mathbf{T}$ *have large powers.* □

4.2 Lemma. *If an abelian topological group* G *has large powers and* $[\tau_n \to e]$ *is a sequential law satisfied in* G*, then the set of variables occurring non-trivially in the* τ_n *is finite.*

Proof. Let X be the countable set of variables which occur in the τ_n, and consider the set of variables in X which occur non-trivially. Suppose that this set is infinite, and define a map $\phi : X \to G$ by induction, as follows. For each $n \in \mathbf{N}$, only finitely many variables occur non-trivially in $\tau_1, \dots, \tau_n$. Let $n_1 < n_2 < \dots$ be an infinite sequence defined by the condition that for all i, some variable occurs non-trivially in τ_{n_i} but occurs only trivially in all of $\tau_{n_1}, \dots, \tau_{n_{i-1}}$; such a sequence clearly exists.

Let U be the neighbourhood of the identity in G given by the large powers property. Assign $\phi(x)$ arbitrarily for all x occurring non-trivially in $\tau_1, \dots, \tau_{n_1 - 1}$. If $x_1, \dots, x_k$, for some $k > 0$, are the variables which occur non-trivially in τ_{n_1} but only trivially in all earlier terms of the sequence, set $\phi(x_j) = e$ for $j = 1, \ldots, k-1$, and, using the large powers property, set $\phi(x_k) = h$ for some h such that $\tau_{n_1}[\phi] \notin U$ (or, strictly, $\tau_{n_1}[\phi^*] \notin U$ for an arbitrary extension ϕ^* of the so far partially defined ϕ to a mapping on all of X). Continuing this process, and finally assigning arbitrary values to all variables which occur only trivially in the τ_n, we obtain ϕ defined on all X such that $\tau_{n_i} \notin U$ for all i, a contradiction. □

We say that an abelian topological group G is *sequentially trivial* if every sequential law $[\tau_n \to e]$ satisfied in G, where the terms τ_n are regarded as elements of $A(X)$ for a suitable X, has the property that for some $N \in \mathbf{N}$, every variable in X occurs trivially in τ_n for all $n > N$; that is, τ_n is eventually the identity in $A(X)$. The following result is easily proved.

4.3 Proposition. *Let G be a topological group.*

(i) *If any subgroup of G is sequentially trivial, then so is G.*
(ii) *If any continuous homomorphic image of G is sequentially trivial, then so is G.* □

4.4 Theorem. *The group $\mathbf{Z}$ is sequentially trivial.*

Proof. Fix a sequential law $[\tau_n \to e]$ satisfied in $\mathbf{Z}$, and let X be the set of variables which occur in the τ_n. By Lemmas 4.1 and 4.2, only a finite number of variables, say $x_1, \ldots, x_k$, occur non-trivially in the τ_n. Fix any $i \in \{1, \ldots, k\}$. By considering the variable valuation $\phi_i : X \to G$ which maps x_i to $1 \in \mathbf{Z}$ and maps all other variables to $0 \in \mathbf{Z}$, we see that we must have a convergent sequence $a_n^i \to 0$ in $\mathbf{Z}$, where $\langle a_n^i \rangle$ is the sequence of exponents of x_i in the τ_n. As $\mathbf{Z}$ is discrete, we therefore have $a_n^i = 0$ for all $n > N_i$, for a suitable N_i. Then for all n larger than $N_1, \ldots, N_k$, we have $a_n^i = 0$ for $i = 1, \ldots, k$, which is as required. □

Proposition 4.3 now gives the following result.

4.5 Corollary. *Any abelian topological group with a subgroup topologically isomorphic to $\mathbf{Z}$ is sequentially trivial.* □

4.6 Corollary. *The group $\mathbf{R}$ is sequentially trivial.* □

We wish next to show that the circle group $\mathbf{T}$ is sequentially trivial. For this we need some preliminary results.

4.7 Lemma. *Let U be any measurable subset of $\mathbf{T}$ with normalised 1-dimensional measure $\lambda \leq 1$, let V be any arc in $\mathbf{T}$ of measure $\mu \leq 1$, and let n be a positive integer. Then the measure ν of the set $\{x \in V : x^n \in U\}$ satisfies $(\mu - 1/n)\lambda \leq \nu \leq (\mu + 1/n)\lambda$.*

Proof. For some measurable subset Θ of $[0,1)$, we can express U as $\{e^{2\pi i\theta} : \theta \in \Theta\}$. Then it is straightforward to verify that

$$\{x \in \mathbf{T} : x^n \in U\} = \cup_{j=0}^{n-1} e^{2\pi ij/n} U_n ,$$

where $U_n = \{e^{2\pi i\theta} : n\theta \in \Theta\}$. Now the right hand side above is a union of n translates of U_n, which respectively lie in the disjoint arcs $\{e^{2\pi i(\theta+j)/n} : \theta \in [0,1)\}$, $j = 0, \ldots, n-1$. Hence it is clear that if $k/n \leq \mu \leq (k+1)/n$ for some integer $k \geq 0$, then $k\lambda/n \leq \nu \leq (k+1)\lambda/n$, and it follows that $(\mu - 1/n)\lambda \leq \nu \leq (\mu + 1/n)\lambda$ as required. □

4.8 Lemma. *Let U be an open arc in $\mathbf{T}$ of measure $\lambda < 1$, and let $\langle a_n \rangle$ be any unbounded sequence of integers. Then the measure of the set $M = \{z \in \mathbf{T} : z^{a_n} \in U$ for all $n \in \mathbf{N}\}$ is zero.*

Proof. The sequence $\langle a_n \rangle$ is either unbounded above or below. We assume the former (an analogous argument applies in the other case).

If for each $n \in \mathbf{N}$ we define $M_n = \{z \in \mathbf{T} : z^{a_n} \in U\}$, then we have $M = \bigcap_{n\in\mathbf{N}} M_n$. We shall construct a sequence $\langle n_k \rangle$ such that $\bigcap_{k\in\mathbf{N}} M_{n_k}$ has measure zero, which is clearly sufficient for the result. More specifically, fix any c satisfying $\lambda < c < 1$. Then we shall construct $\langle n_k \rangle$ so that $a_{n_k} > 0$ for all k, and so that, if ν_k denotes the measure of $\bigcap_{l\leq k} M_{n_l}$, we have $\nu_{k+1} \leq c\nu_k$ for all k.

First, set $n_1 = 1$. Next, suppose that for some k, $n_1, \ldots, n_k$ have been chosen so that $a_{n_l} > 0$ for $l = 1, \ldots, k$, and $\nu_{l+1} \leq c\nu_l$ for $l = 1, \ldots, k-1$. We now show how to choose n_{k+1}. Note that for all n, M_n is a finite union of (disjoint) open arcs of $\mathbf{T}$, and that therefore so is the set $I_k = \bigcap_{l\leq k} M_{n_l}$. Thus we can express I_k as $\bigcup_i A_i$, where $\{A_i\}$ is some finite set of disjoint open arcs. Let $\epsilon > 0$ be the smallest measure of any A_i, and choose n_{k+1} so that $a_{n_{k+1}} \geq \lambda/\epsilon(c - \lambda)$. Write the measure of A_i as α_i. Then by Lemma 4.7, the measure β_i of $\{z \in A_i : z^{a_{n_{k+1}}} \in U\}$ satisfies

$$\begin{aligned}\beta_i &\leq (\alpha_i + 1/a_{n_{k+1}})\lambda \leq (\alpha_i + \epsilon(c-\lambda)/\lambda)\lambda \\ &\leq (\alpha_i + \alpha_i(c-\lambda)/\lambda)\lambda = c\alpha_i.\end{aligned}$$

But since I_k is the disjoint union of the A_i, we have $\nu_k = \sum_i \alpha_i$, so $\nu_{k+1} = \sum_i \beta_i$, and therefore we have $\nu_{k+1} \leq c\nu_k$, as desired. Hence, by induction, a sequence $\langle n_k \rangle$ exists as asserted.

It follows that $\bigcap_{k\in\mathbf{N}} M_{n_k}$ has measure zero, and we have the result. □

4.9 Proposition. *Let $[x^{a_n} \to e]$ be a sequential law of which the sequence of exponents $\langle a_n \rangle$ is not eventually* 0. *Then the set $M = \{z \in \mathbf{T} : z^{a_n} \to 1\}$ has measure zero in* $\mathbf{T}$. *In particular,* $\mathbf{T}$ *satisfies no sequential laws of the above kind.*

Proof. Suppose first that the sequence $\langle a_n \rangle$ is bounded. If z is an element of $\mathbf{T}$ of the form $e^{2\pi i\theta}$ for irrational θ, then z^{a_n} is not eventually equal to 1, since $z^{a_n} = 1$ only when $a_n = 0$, and the sequence $\langle a_n \rangle$ is not eventually 0. But the set $\{z^{a_n} : n \in \mathbf{N}\}$ is finite, and so we have $z^{a_n} \not\to 1$. Therefore M contains only z of the form $e^{2\pi i\theta}$ for rational θ, and so is of measure zero. (It is easy to see that M is in fact finite.)

Suppose now that the sequence $\langle a_n \rangle$ is unbounded. Let U be any open arc in $\mathbf{T}$ which contains 1. Then, setting $M_N = \{z \in \mathbf{T} : z^{a_n} \in U$ for all $n > N\}$, we clearly have $M \subseteq \bigcup_{N\in\mathbf{N}} M_N$, so it is sufficient if we prove that each M_N has measure zero. But $M_N = \{z \in \mathbf{T} : z^{b_n} \in U$ for all $n \in \mathbf{N}\}$, if we define $b_n = a_{N+n}$ for all n, and this has measure zero by Lemma 4.8, completing the proof. □

We can now prove the result mentioned earlier.

4.10 Theorem. *The group* $\mathbf{T}$ *is sequentially trivial.*

Proof. Let $[\tau_n \to e]$ be any sequential law satisfied in $\mathbf{T}$. By Lemmas 4.1 and 4.2, the set of variables which occur non-trivially in the τ_n is finite. We wish to show that, for sufficiently large n, all of these variables occur trivially in τ_n, so we suppose the opposite, that there is some variable, x, say, whose sequence of exponents $\langle a_n \rangle$ in the $\langle \tau_n \rangle$ is not eventually 0. But by considering the variable valuations which map all variables in the τ_n, except possibly x, to the identity 1 in $\mathbf{T}$, we see that $[x^{a_n} \to e]$ is a sequential law satisfied in $\mathbf{T}$, and this contradicts Proposition 4.9, giving the result. □

4.11 Theorem. *Every non-totally disconnected, locally compact abelian topological group is sequentially trivial.*

Proof. If G is a group of the type mentioned, then G has $\mathbf{T}$ as a continuous homomorphic image. (The (non-trivial) connected component of the identity in G has sufficiently many characters to separate points. One character of the component must therefore have non-trivial image in $\mathbf{T}$, and this image, being a connected subgroup, must be the whole of $\mathbf{T}$. Finally, this character may be extended to a character on G.) Therefore, by Theorem 4.10 and Proposition 4.3, G is sequentially trivial. □

Some examples show that conditions such as those in the above result are needed. The product $\prod_n \mathbf{Z}_n$ of the discrete finite cyclic groups satisfies the sequential law $[x^{n!} \to e]$, as do the groups of $\mathbf{a}$-adic numbers, denoted in [5] by $\Omega_{\mathbf{a}}$, for any doubly infinite sequence $\mathbf{a} = (\dots, a_{-1}, a_0, a_1, \dots)$ of natural numbers greater than 1. The product group is of course compact and totally disconnected, while the $\mathbf{a}$-adic groups are locally compact, non-compact and totally disconnected. (On the other hand, we have already noted that the discrete group $\mathbf{Z}$, though locally compact and totally disconnected, is sequentially trivial.)

If we remove the assumption of local compactness, then connected abelian topological groups may have non-trivial sequential laws. For example, if we apply to the group $\mathbf{Z}_2$ the construction of Hartman and Mycielski [4] (cf. [1]), we obtain a connected, simply connected (in fact, contractible) metrizable group, which is in addition a topological l_2-manifold, and which satisfies the algebraic law $x^2 = e$. This group therefore also satisfies every sequential law $[x^{a_n} \to e]$ in which the sequence of exponents $\langle a_n \rangle$ is eventually even.

5. Further Results and Examples.

In this section, we discuss a number of results and examples, mostly related to the material of Section 3.

It is convenient for the discussion to define a new family of varieties. For any cardinal κ, we define $S(\kappa)$ to be the class of topological groups in which each neighbourhood of e contains a (not necessarily normal) subgroup of index less than κ. It is easy to see that $S(\kappa)$ is a wide variety. We also denote the wide variety of abelian topological groups by Ab.

We investigate the relations between $T(\kappa)$, $S(\kappa)$ and $B(\kappa)$. First, it is clear that for all κ, we have:

(1) $T(\kappa) \subseteq S(\kappa) \subseteq B(\kappa)$.

It is plausible that in general the inclusion of $T(\kappa)$ in $S(\kappa)$ is strict, though there is at least one exception:

(2) $T(\aleph_0) = S(\aleph_0)$.

This follows immediately from the fact ([5], 4.21(d)) that a subgroup of finite index n in any (abstract) group contains a subgroup which is normal and of index at most $n!$ in the original group.

We examine the inclusion of $S(\kappa)$ in $B(\kappa)$, showing that there are cases where the inclusion is strict and cases where it is not.

(3) When κ equals $\aleph_0$ or $\aleph_1$, we have $S(\kappa) \neq B(\kappa)$.

In fact the circle group $\mathbf{T}$ is an example of a group which lies in $B(\kappa)$ but not in $S(\kappa)$ for both the above values of κ. On the other hand, the work of [6] yields the following result, in which, for any cardinal λ, we denote by λ^+ the cardinal successor of λ.

(4) If κ is any cardinal, then $B(\kappa^+) \subseteq S((\kappa^{\aleph_0})^+) \subseteq B((\kappa^{\aleph_0})^+)$.

In particular:

(5) If κ is such that $\kappa = \kappa^{\aleph_0}$, then $S(\kappa^+) = B(\kappa^+)$.

For any κ such that $\kappa = \kappa^{\aleph_0}$, we therefore clearly have $\mathrm{Ab} \cap T(\kappa^+) = \mathrm{Ab} \cap S(\kappa^+) = \mathrm{Ab} \cap B(\kappa^+)$. The results of [6], however allow an extension. Let SIN be the wide variety of SIN-groups, or locally invariant groups [12]; that is, those groups with arbitrarily small neighbourhoods of e invariant under all inner automorphisms. Then from [6] it follows that:

(6) If κ is such that $\kappa = \kappa^{\aleph_0}$, then $\mathrm{SIN} \cap T(\kappa^+) = \mathrm{SIN} \cap S(\kappa^+) = \mathrm{SIN} \cap B(\kappa^+)$.

Wide varieties generated by single topological groups have been investigated in some detail (see [11], for example). We note the following easy characterisation of one such variety.

5.1 Proposition. *The wide variety generated by* $\mathbf{T}$ *is* $\mathrm{Ab} \cap B(\aleph_0)$.

Proof. Clearly $\mathbf{T}$, and therefore the smallest wide variety containing $\mathbf{T}$, is contained in $\mathrm{Ab} \cap B(\aleph_0)$. On the other hand, any group in $\mathrm{Ab} \cap B(\aleph_0)$ is a precompact abelian group, whose completion is compact and therefore a subgroup of a suitable product of copies of $\mathbf{T}$. □

Of course, by Theorem 3.16, it follows that the wide variety generated by $\mathbf{T}$ is characterised by a set of limit laws, together with a set of algebraic laws defining the abstract variety Ab.

We have no corresponding simple characterisation of the wide variety generated by $\mathbf{Z}$ (with its usual discrete topology), though we do derive some information about this variety in Proposition 5.4, below. First, however, we discuss the wide variety generated by $\mathbf{Z}$ with another topology.

Let $*$ be a singleton topological space. Then Proposition 3.4 shows that $F_{T(\aleph_0)}(*)$ is $\mathbf{Z}$ with the topology $\mathcal{T}$ generated by all the subgroups $n\mathbf{Z}$, for $n = 1, 2, \ldots$. We give several characterisations of the wide variety generated by this group. Recall from Section 3 that $\Delta(\aleph_0)$ denotes a class of limit laws defining $T(\aleph_0)$. Also, let A denote any (finite) set of algebraic laws defining Ab.

5.2 Proposition. *The following varieties are equal:*

(i) *the wide variety generated by* $\mathbf{Z}$ *with the topology* $\mathcal{T}$;
(ii) *the wide variety generated by the discrete finite cyclic groups* $\mathbf{Z}_n$, *for* $n = 1, 2, \ldots$;
(iii) $\mathrm{Mod}(A \cup \Delta(\aleph_0))$; *and*
(iv) $\mathrm{Ab} \cap T(\aleph_0)$.

Proof. The first and second classes are equal because each group $\mathbf{Z}_n$ is a continuous homomorphic image of $(\mathbf{Z}, \mathcal{T})$, and because $(\mathbf{Z}, \mathcal{T})$ is embedded in $\prod_{n \geq 1} \mathbf{Z}_n$. The third and fourth classes are equal because $\mathrm{Mod}(\Psi) \cap \mathrm{Mod}(\Omega) = \mathrm{Mod}(\Psi \cup \Omega)$, for any classes of (algebraic or limit) laws Ψ and Ω. The topological group generating the first class is certainly in the fourth class (as noted above, it is a free topological group in $T(\aleph_0)$). Finally, each abelian group in $T(\aleph_0)$ is, by Proposition 3.1, embedded in the product of an indiscrete abelian group with a product of finite abelian groups, and is thus in a product of groups which lie in the second class. □

The following result, as well as being of interest in own right, allows us to obtain information on the wide variety generated by $\mathbf{Z}$ with the discrete topology.

5.3 Proposition. *There is a limit law (on a countably infinite set of variables) which is satisfied in* $\mathbf{Z}$ *but not in all countable abelian topological groups.*

Proof. Let X be a countably infinite set, and let G be the (abstract) free abelian group on X. Let $\mathcal{D}$ be the set of all finite intersections of kernels of homomorphisms from G to $\mathbf{Z}$; equivalently, $\mathcal{D}$ is the set of all kernels of homomorphisms from G to $\mathbf{Z}^n$, for all finite n. Clearly, $\mathcal{D}$ is a directed set under the superset relation. For each $d \in \mathcal{D}$, select an arbitrary non-zero $w_d \in d$. Then the law $[w_d \to e]$ is satisfied in $\mathbf{Z}$. For if $\phi : X \to \mathbf{Z}$ is a variable valuation, then ϕ extends uniquely to a homomorphism $\Phi : G \to \mathbf{Z}$; if we write $K = \ker \phi$, then $K \in \mathcal{D}$, and it is easy to see that for any $d \in \mathcal{D}$ which is greater than or equal to K in the ordering of $\mathcal{D}$, we have $w_d[\phi] = 0$ in $\mathbf{Z}$.

On the other hand, by construction, the law $[w_d \to e]$ is clearly not satisfied in the (countable abelian) group G equipped with the discrete topology, and the result is proved. $\square$

5.4 Proposition. *Let* $\mathcal{Z}$ *denote the wide variety generated by* $\mathbf{Z}$ *with the discrete topology. Then* $\mathrm{Ab} \cap T(\aleph_0) \subseteq \mathcal{Z} \subseteq \mathrm{Ab} \cap T(\aleph_1)$, *and both inclusions are proper.*

Proof. The first inclusion follows from Proposition 5.2, since $\mathbf{Z}$ with the topology $\mathcal{T}$ is a continuous image of $\mathbf{Z}$ with the discrete topology. The second inclusion is clear. The first inclusion is proper since $\mathbf{Z}$ with the discrete topology is clearly not in $T(\aleph_0)$. The second inclusion is proper by Proposition 5.3, and since $\mathrm{Ab} \cap T(\aleph_1)$ contains all countable abelian topological groups. $\square$

Finally, we prove two results of a miscellaneous nature. In the first of these, we again continue to use multiplicative notation despite the abelian context.

5.5 Proposition. *Let* $\mathcal{B}$ *denote the wide variety generated by those abelian topological groups* G *for which there is an* $n \equiv n_G \in \mathbf{N}$ *such that* $g^n = e$ *for every* $g \in G$. *Then* $\mathcal{B} = \mathrm{Mod}(A \cup \{[x_n^{n!} \to e]\})$.

Proof. It is clear that $\mathcal{B} \subseteq \mathrm{Mod}(A \cup \{[x_n^{n!} \to e]\})$. Conversely, if $G \notin \mathcal{B}$, then by Proposition 3.1, either (i) $G_\jmath \notin \mathcal{B}$, or (ii) there is a neighbourhood

N of e in G such that for all (normal) subgroups H of G lying in N, we have $G/H \notin \mathcal{B}$. However it is easy to see that $\mathcal{B}$ contains all indiscrete abelian groups, so (i) is impossible, and (ii) therefore holds. Now for each $n \in \mathbf{N}$, G^n is a (normal) subgroup of G, and clearly $G/G^n \in \mathcal{B}$. Hence, for the neighbourhood N of e given by (ii), we have $G^n \not\subseteq N$, for each n. In particular, we can find a sequence $\langle g_n \rangle$ of elements of G such that $g_n^{n!} \notin N$ for all n. Therefore the sequence $\langle g_n^{n!} \rangle$ does not converge to e, and so the law $[x_n^{n!} \to e]$ does not hold in G. This proves the reverse inclusion. □

In the last result, we examine a particularly simple limit law, and find the wide variety it generates.

5.6 Proposition. *The sequential limit law* $[x^n \to e]$ *is satisfied by and only by the indiscrete* (*abelian or non-abelian*) *topological groups.*

Proof. Suppose that G is a non-trivial Hausdorff group satisfying the law. Now for any element $a \neq e$ in G, the sequence $\langle a^n \rangle$ converges to e. Therefore, by left-translation, $a^{n+1} \to a$, and it follows that the sequence $\langle a^n \rangle$ converges both to e and to a, a contradiction, since G is Hausdorff. Thus the law is satisfied by no non-trivial Hausdorff group. But if the law is satisfied by any non-trivial non-indiscrete group G, then it is also satisfied in the non-trivial Hausdorff group obtained from G by taking the quotient with respect to the closure of $\{e\}$ in G, and we have shown that this is impossible. The result follows. □

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