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VARIETIES OF TOPOLOGICAL GROUPS, LIE GROUPS AND SIN-GROUPS

BY

KARL H. HOFMANN (DARMSTADT),
SIDNEY A. MORRIS (WOLLONGONG, NEW SOUTH WALES)
AND MARKUS STROPPEL (DARMSTADT)

In this paper we answer three open problems on varieties of topological groups by invoking Lie group theory. We also reprove in the present context that locally compact groups with arbitrarily small invariant identity neighborhoods can be approximated by Lie groups.

Recall that a class $\mathfrak{V}$ of topological groups closed under the operations of forming subgroups, quotient groups and arbitrary products is called a *variety of topological groups*. The following problems on varieties were formulated in Colloquium Mathematicum [10]:

P1249. Theorem 7 of [10] says that if Ω is any class of topological groups and if G is a Hausdorff group in the variety $\mathfrak{V}(\Omega)$ generated by Ω then G is a subgroup of a (possibly infinite) product of Hausdorff quotient groups of closed subgroups of finite products of members of Ω. If Ω consists of abelian topological groups, this can be simplified. In that case G is a subgroup of a product of Hausdorff quotient groups of finite products of members of Ω. *Is this true in the absence of commutativity*?

P1250. *If Ω is a class of Lie groups and G is a Lie group in the variety $\mathfrak{V}(\Omega)$ generated by Ω, is it true that G is a quotient group of a subgroup of a* finite *product of members of Ω*?

P1251. *Let $\mathfrak{V}$ denote the variety of topological groups generated by the class of topological groups having a compact neighborhood of the identity which is invariant under all inner automorphisms. If G is a locally compact group in $\mathfrak{V}$, does it have a compact invariant identity neighborhood*?

Problem 1249 was first posed in [3], Problem 1250 in [9] as P897, and Problem 1251 in [11].

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These questions remained unsettled, and we shall answer 1250 and 1251 in the affirmative and 1249 in the negative. Our entire discussion involves Lie groups.

For dealing with 1251 we shall use an old result of Iwasawa (see 2.3 below) which says that a locally compact group G has a compact invariant identity neighborhood if and only if G allows a proper homomorphism (see 0.1) into an SIN-group, i.e., a group having arbitrarily small invariant identity neighborhoods. We work in a mildly category-theoretical framework, which necessarily leads us outside the class of locally compact groups.

In Section 3 we shall show that each locally compact SIN-group is a strict projective limit of Lie groups.

Problem 1249 will be disposed of by a counterexample, and in Section 5 we deal with 1250.

0. Proper morphisms. We collect information on proper maps. A topological space is called *compact* if it has the Heine–Borel property.

0.1. Definition. A map $f : X \to Y$ between topological spaces is called *proper* if it is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

Bourbaki [2, p. 115, Théorème 1] points out that this condition is equivalent to the following:

For each space Z the map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed.

Furthermore, this condition implies that $f^{-1}(K)$ is compact for every compact subset K of Y (loc. cit., p. 118, Prop. 6). The converse holds if Y is locally compact (loc. cit., p. 119, Prop. 7). For the converse, it suffices in fact that Y is a Hausdorff k-space.

Let $\mathrm{T}\mathcal{G}$ denote the category of all topological groups and continuous homomorphisms.

0.2. Lemma. *Consider a surjective* $\mathrm{T}\mathcal{G}$*-morphism $\varphi : A \to B$ and the following two conditions*:

(i) *φ is a quotient morphism.*
(ii) *φ is a closed map.*

Then (ii) *implies* (i), *and if* $\ker\varphi$ *is compact*, *the two conditions are equivalent.*

Proof. The map φ is a quotient morphism if and only if a saturated subset $X = \varphi^{-1}(\varphi(X))$ of A is closed exactly when $\varphi(X)$ is closed. This is clearly the case if φ is a closed continuous map.

Conversely, suppose that $\varphi : A \to B$ is a quotient morphism and that $\ker\varphi$ is compact. Let $C \subseteq A$ be any closed subset. Then $\varphi^{-1}(\varphi(C)) =$

$C \ker\varphi$ is closed. But then also $\varphi(C)$ is closed since φ is a quotient morphism. ∎

0.3. Proposition. *For a* T$\mathcal{G}$*-morphism* $f : G \to H$ *the following conditions are equivalent*:

(1) $f : G \to H$ *is proper.*

(2) *The corestriction* $f : G \to f(G)$ *is a quotient morphism onto a closed subgroup of* H *and* $\ker f$ *is compact.*

Proof. By Definition 0.1, condition (1) says that $\ker f$ is compact and that f is a closed map. Now f is closed if and only if $f(G)$ is closed and the corestriction $f : G \to f(G)$ is a closed map. Hence (1) is equivalent to

(1′) $\ker f$ *is compact*; *further*, $f(G)$ *is closed*; *and finally*, *the corestriction* $f : G \to f(G)$ *is a closed map.*

Lemma 0.2 now establishes that (1′) and (2) are equivalent. ∎

We record from [2, p. 117, Cor. 3]:

0.4. Proposition. *Assume that* $f_j : X_j \to Y_j$, $j \in J$, *is a family of proper maps between spaces. Then* $\prod_{j\in J} f_j : \prod_{j\in J} X_j \to \prod_{j\in J} Y_j$ *is a proper map.* ∎

1. A categorical lemma. We recall that a category is *complete* if it allows arbitrary limits. In order to verify completeness it suffices to show that arbitrary products and pull-backs exist. As a consequence, a subcategory of T$\mathcal{G}$ is complete if and only if it is closed under the formation of arbitrary products and under the passing to subgroups. The full subcategory of T$\mathcal{G}$ containing all Hausdorff topological spaces will be denoted by $\mathrm{T}_2\mathcal{G}$. A subcategory of $\mathrm{T}_2\mathcal{G}$ is complete if and only if it is closed under the formation of arbitrary products and under the passing to *closed* subgroups.

1.1. Lemma. *Let* $\mathcal{G}$ *be a complete and full subcategory of* $\mathrm{T}_2\mathcal{G}$. *Define* $\mathcal{G}'$ *to be the full subcategory of* $\mathrm{T}_2\mathcal{G}$ *containing all objects* G *of* $\mathrm{T}_2\mathcal{G}$ *for which there is a proper morphism* $f : G \to H$ *such that* H *is an object of* $\mathcal{G}$. *Then* $\mathcal{G}'$ *is also complete.*

Proof. We must show that $\mathcal{G}'$ admits products and is closed under the formation of closed subgroups. Assume that $\{G_j : j \in J\}$ is a family of objects of $\mathcal{G}'$. By definition of $\mathcal{G}'$ there are proper morphisms $f_j : G_j \to H_j$ with $H_j \in \operatorname{ob}\mathcal{G}$. Since $\mathcal{G}$ is complete, $\prod_{j\in J} H_j \in \operatorname{ob}\mathcal{G}$. By Proposition 0.4, the morphism $\prod_{j\in J} f_j : \prod_{j\in J} G_j \to \prod_{j\in J} H_j$ is proper. By the definition of $\mathcal{G}'$ once more it follows that $\prod_{j\in J} G_j \in \operatorname{ob}\mathcal{G}'$.

Next assume that $G \in \operatorname{ob}\mathcal{G}'$ and that A is a closed subgroup of G. There is a proper map $f : G \to H$ with $H \in \operatorname{ob}\mathcal{G}$. Now $f(A)$ is closed in H by Proposition 0.3 and thus $f(A) \in \operatorname{ob}\mathcal{G}$ since $\mathcal{G}$ is complete. The restriction

$f|A : A \to H$ is proper since it is closed and $\ker(f|A) = A \cap \ker f$ is compact. Hence $A \in \operatorname{ob} \mathcal{G}'$. The proof is complete. ∎

2. An application to IN-groups

2.1. Definition. Let **IN** denote the full subcategory of $\mathrm{T}\mathcal{G}$ containing all groups which have a compact neighborhood of the identity which is invariant under all inner automorphisms. The objects of this category are called IN-*groups*.

Also, let **SIN** denote the full subcategory of $\mathrm{T}\mathcal{G}$ containing all groups with arbitrarily small invariant identity neighborhoods. Its objects are called SIN-*groups*.

2.2. Remark. **SIN** is a complete subcategory of $\mathrm{T}\mathcal{G}$.

Proof. Clearly **SIN** is closed under the formation of arbitrary products and the passage to subgroups. ∎

In [7], Iwasawa proved the following theorem:

2.3. Theorem. *For a locally compact Hausdorff group G the following statements are equivalent*:

(1) *There is a compact neighborhood of the identity which is invariant under all inner automorphisms of G*; *that is*, *G is an* IN-*group.*

(2) *There is a compact normal subgroup N of G such that G/N has arbitrarily small compact invariant identity neighborhoods*; *that is*, *G/N is an* SIN-*group.* ∎

For a generalization of this theorem see Theorem 2.5 of Grosser and Moskowitz [5].

2.4. Corollary. *For a locally compact Hausdorff group G*, *the following statements are equivalent*:

(1) *G is an* IN-*group.*

(2) *There is a proper morphism $f : G \to H$ with $H \in \operatorname{ob} \mathbf{SIN}$.*

Proof. Assume (1) is true. Combining Proposition 0.3 and Theorem 2.3 we see that the map $f : G \to G/N$ is a proper morphism and G/N is an SIN-group.

Assume (2) is true. Since $f(G)$ is a quotient of G, by Proposition 0.3 we know that $f(G)$ is locally compact. Let V be an invariant identity neighborhood in H such that $V \cap f(G)$ is compact. Then $U = f^{-1}(V)$ is compact since $\ker f$ is compact. It is clearly invariant. ∎

2.5. Definitions. The full subcategory of all topological groups G for which there is a proper morphism $f : G \to H$ with $H \in \operatorname{ob} \mathbf{SIN}$ will be denoted by **KSIN**. The objects of **KSIN** are called KSIN-*groups*. Let $\mathbf{KSIN}_2$ denote the full subcategory containing all Hausdorff objects.

From the above we obtain directly:

2.6. Remark. A locally compact Hausdorff group is an IN-group if and only if it is a KSIN-group. ∎

2.7. PROPOSITION. $\mathbf{KSIN}_2$ *is a complete subcategory of* $\mathrm{T}_2\mathcal{G}$ *containing* $\mathrm{T}_2\mathcal{G} \cap \mathbf{IN}$.

Proof. This follows immediately from 1.1 and 2.2. ∎

Recall from the introduction that the variety $\mathfrak{V}$ of topological groups is generated by all IN-groups. Considered as a full subcategory of $\mathrm{T}\mathcal{G}$, it is the smallest full complete subcategory of $\mathrm{T}\mathcal{G}$ containing **IN**. Thus it is contained in **KSIN**. Therefore:

2.8. COROLLARY. *If a locally compact Hausdorff group belongs to* $\mathfrak{V}$ *then it is an* IN-*group.* ∎

This settles P1251 of [10] in the affirmative.

3. SIN-groups are Lie-projective. Grosser and Moskowitz [5] proved structure theorems for locally compact SIN-groups (among other things). We reprove in this section that every SIN-group can be approximated by Lie groups ([5], 2.11(1); see also [1], Theorem 1.2, Corollary 1).

3.1. DEFINITION. We say that a topological group G is a *pro-Lie* group if for every identity neighborhood U there is a compact normal subgroup N contained in U such that G/N is a Lie group. ∎

In other words, a pro-Lie group is a strict projective limit of Lie groups. Note that pro-Lie groups are always locally compact. Each locally compact Hausdorff group which is compact modulo its identity component is a pro-Lie group. A pro-Lie group without (arbitrarily) small subgroups is a Lie group [8].

3.2. PROPOSITION. *Let G be a pro-Lie group and $A \subseteq \operatorname{Aut} G$ a group of automorphisms. Assume that G has arbitrarily small A-invariant neighborhoods of the identity. Then G has arbitrarily small normal subgroups N such that G/N is a Lie group and N is A-invariant.*

Proof. Let W be any identity neighborhood of G. We pick a compact normal subgroup M contained in W such that G/M is a Lie group. This is possible because G is a pro-Lie group. Then

$$N := \bigcap_{\alpha \in A} \alpha(M)$$

is the largest A-invariant closed normal subgroup of G contained in M. Obviously, $N \subseteq W$.

Since G/M has no small subgroups we find an identity neighborhood V_M in G/M in which the only subgroup is trivial. Let V_G be the inverse image of V_M in G. Then all closed subgroups K of G which are contained in V_G are contained in M. Now let U be an A-invariant identity neighborhood of G which is contained in V_G. We consider the A-invariant set

$$V = \bigcap_{\alpha \in A} \alpha(V_G).$$

Since U is invariant, we have $U \subseteq V$; that is, V is an invariant identity neighborhood containing N and contained in V_G. Let H be any closed subgroup of G contained in V and containing N. We claim $H = N$. Proof: Let $\alpha \in A$. Then $\alpha^{-1}(H) \subseteq \alpha^{-1}V = V \subseteq V_G$, and thus $\alpha^{-1}(H) \subseteq M$ by the definition of V_G. Then $H \subseteq \alpha(M)$. Since $\alpha \in A$ was arbitrary, we conclude $H \subseteq \bigcap_{\alpha \in A} \alpha(M) = N \subseteq H$. Hence $H = N$ as was claimed.

Therefore, G/N has no small subgroups and thus is a Lie group [8]. ∎

3.3. COROLLARY. *Assume that G is an* SIN-*group containing a normal subgroup N such that*

(i) *G/N is a Lie group*, *and*
(ii) *N is a pro-Lie group.*

Then G is a pro-Lie group.

Proof. If U is an invariant neighborhood of the identity in G then $U \cap N$ is an invariant neighborhood of the identity in N which is also invariant under inner automorphisms $I_g : N \to N$ defined by $I_g(n) = gng^{-1}$ with $g \in G$. Thus Proposition 3.2 applies with $A = \{I_g : g \in G\}$ and shows that N contains arbitrarily small compact subgroups M such that N/M is a Lie group and which are G-invariant. Thus these M are normal in G and G/M is an extension of the Lie group N/M by the Lie group G/N, and thus is a Lie group [4].

3.4. EXAMPLES. (i) Let L be a compact Lie group and let $\mathbb{Z}$ act automorphically on $P := L^{\mathbb{Z}}$ by the shift. Set $G = P \rtimes \mathbb{Z}$. Then $N = P \times \{0\}$ is a pro-Lie group and G/N is discrete, thus a Lie group. But G is not pro-Lie (and, therefore, not an SIN-group). So the SIN condition cannot be dropped in 3.3.

Let $Q := L^{(\mathbb{Z})}$ denote the subgroup of all $(g_n)_{n \in \mathbb{Z}} \in P$ of finite support. Then $Q \rtimes \mathbb{Z}$ is a dense subgroup of the KSIN-group G which is not itself a KSIN-group. Therefore, the objects of **KSIN** do not form a variety. Note, however, that the class of Hausdorff groups is closed under the formation of arbitrary products, Hausdorff quotient groups, and *closed* subgroups.

(ii) Let F be a finite field and M its multiplicative group. Then the compact group $M^{\mathbb{N}}$ acts automorphically on the discrete vector space $V = F^{(\mathbb{N})}$ via $(m_n)_{n\in\mathbb{N}} \cdot (f_n)_{n\in\mathbb{N}} = (m_n f_n)_{n\in\mathbb{N}}$. Set $G = V \rtimes M^{\mathbb{N}}$. Then $N = V \times \{1\}$ is discrete, thus is a Lie group, and $G/N \cong M^{\mathbb{N}}$ is compact and therefore a pro-Lie group. But G is not a pro-Lie group (and, therefore, not an SIN-group, as we shall see in Theorem 3.6 below). ∎

These examples show that, in general, pro-Lie groups do not behave very well under extension. In the presence of the SIN hypothesis, however, we shall see shortly in 3.6 that the likes of Example 3.4(ii) cannot occur.

3.5. LEMMA. *For a locally compact totally disconnected group G the following statements are equivalent*:

(1) *G is an* SIN-*group.*

(2) *G has arbitrarily small compact open normal subgroups.*

(3) *G is a strict projective limit of discrete groups* (*all bonding maps being surjective and having compact kernels*).

Proof. (2)⇒(1) is obvious.

(1)⇒(2). Assume that G is an SIN-group. Let W be a compact invariant identity neighborhood. Since G is totally disconnected, we find a compact open subgroup V contained in W. Since G is an SIN-group we find a compact invariant identity neighborhood U contained in V. Now set $N = \bigcap_{g\in G} gVg^{-1}$. Then N is clearly a compact normal subgroup contained in $\bigcap_{g\in G} gWg^{-1} = W$. On the other hand, N contains $\bigcap_{g\in G} gUg^{-1} = U$. Hence N is open. Thus G has arbitrarily small compact open normal subgroups.

(3) is simply a reformulation of (2). ∎

3.6. THEOREM. *Every locally compact* SIN-*group is pro-Lie.*

Proof. By Lemma 3.5, the factor group G/G_0 has arbitrarily small compact open normal subgroups. Hence there is a filter basis $\mathcal{F}$ of open normal subgroups N of G such that N/G_0 is compact and $\bigcap \mathcal{F} = G_0$. Take $N \in \mathcal{F}$. Then G/N is discrete, hence is a Lie group. Since N/G_0 is compact, the group N is pro-Lie. Hence Corollary 3.3 applies and shows that G is a pro-Lie group. ∎

3.7. Remark. Under the hypotheses of Theorem 3.6 and with the notation of its proof there are normal subgroups $N \in \mathcal{F}$ such that $N \cong \mathbb{R}^n \times C$ with a unique maximal compact subgroup C of N.

Proof. This follows from Corollary XII.1 in [6] or from Theorem 2.13 in [5]. ∎

3.8. COROLLARY. *Let G be a locally compact group. Then the following conditions are equivalent*:

(1) *G is an* IN-*group.*

(2) *There is a compact normal subgroup N of G such that G/N is an* SIN-*pro-Lie group.*

(3) *There is a compact normal subgroup N of G such that G/N is an* SIN-*Lie group.*

Proof. By Iwasawa's Theorem 2.3, condition (1) is equivalent to

($1'$) *There is a compact normal subgroup N of G such that G/N is an* SIN-*group*.

By Theorem 3.6, Condition ($1'$) is equivalent to Condition (2). Trivially, (3) implies (2). Assume that (2) is satisfied. Then there is a normal closed subgroup M of G containing N such that G/M is a Lie group and M/N is compact. Then M is compact and normal in G and G/M is an SIN-group since G/N is an SIN-group. Thus (3) holds. ∎

3.9. Corollary. *Let G be a topological group containing a normal subgroup N such that*

(i) *G/N is a compact Lie group*, *and*

(ii) *N is a locally compact* SIN-*group.*

Then G is a pro-Lie group.

Proof. Take an N-invariant compact neighborhood V of the identity in N. Find a compact set $K \subseteq G$ such that $G = KN$. Let W denote an identity neighborhood of N. Then there is an identity neighborhood V of N such that $\bigcup_{k\in K} kVk^{-1} \subseteq W$; for if not then for every V there are elements $x_V \in V$ and $k_V \in K$ such that $k_V x_V k_V^{-1} \notin W$; since K is compact, the net k_V has a subnet k_{V_j} in K converging to k, and thus $k_{V_j} x_{V_j} k_{V_j}^{-1}$ converges to $k1k^{-1} = 1$, contradicting the fact that $k_{V_j} x_{V_j} k_{V_j}^{-1}$ stays outside W. Set

$$U = \bigcup_{g\in G} gVg^{-1} = \bigcup_{k\in K\, n\in N} knVn^{-1}k^{-1} = \bigcup_{k\in K} kVk^{-1} \subseteq W$$

since V is N-invariant. Now U is the image of the compact space $V \times K$ under the continuous function $(v,k) \mapsto kvk^{-1}$. Hence it is a compact G-invariant identity neighborhood of N contained in W.

Thus the hypotheses of Theorem 3.6 are satisfied with N in place of G and the set of G-inner automorphisms $I_g : N \to N$, $I_g(n) = gng^{-1}$, in place of A.

Thus N contains arbitrarily small compact subgroups M which are normal in G and are such that N/M is a Lie group. Then G/M is a Lie group. ∎

Note that this corollary does not directly follow from Corollary 3.3 since here we do not assume that the entire group G, but only the subgroup N is an SIN-group. The compactness of G/N is a stronger hypothesis here, however. Example 3.4(ii) shows that the conclusion of 3.9 fails if G/N is only assumed to be a compact pro-Lie group rather than a compact Lie group.

4. Solution to P1249.

P1249 is answered in the negative by the following example.

4.1. EXAMPLE. Let Ω denote the class of all (finite-dimensional) real semisimple centerfree connected Lie groups. This class is closed under finite products and quotients.

(1) Let G be a subgroup of a product of Hausdorff quotients of finite products of members of Ω. In the present case this means that G is a subgroup of a product of simple centerfree real Lie groups. The adjoint representation of a centerfree semisimple Lie group is faithful. It follows at once that G has sufficiently many finite-dimensional representations to separate the points.

(2) Let N be the Heisenberg group of all matrices

$$[x,y;z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x,\, y,\, z \in \mathbb{R},$$

and let $D = \{[0,0;n] : n \in \mathbb{Z}\} \cong \mathbb{Z}$. Then $N \subseteq \mathrm{Sl}(3,\mathbb{R})$ and $\mathrm{Sl}(3,\mathbb{R})$ is a centerfree simple real Lie group. Hence $G = N/D$ is a member of $\mathfrak{V}(\Omega)$. Now a finite linear representation of G has the center Z/D, $Z = \{[0,0;z] : z \in \mathbb{R}\}$, in its kernel (see e.g. [8], pp. 190–192). Therefore G cannot be one of the groups constructed in paragraph (1). ∎

5. Regarding P1250

5.1. DEFINITION. We say that a topological group G has *no small subgroups* (or is an [NSS]-group) if there is an identity neighborhood U such that $\{1\}$ is the only subgroup of G contained in U. ∎

Evidently, a subgroup of an [NSS]-group is an [NSS]-group. A locally compact group is an [NSS]-group if and only if it is a Lie group.

For a product $\prod_{j\in J} G_j$ and a subset $I \subseteq J$ we shall identify $\prod_{j\in I} G_j$ with a subgroup of $\prod_{j\in J} G_j$ in the obvious way and we let $p_I : \prod_{j\in J} G_j \to \prod_{j\in I} G_j$ denote the projection.

5.2. LEMMA. *Assume that G is a subgroup of a product $\prod_{j\in J} G_j$ of topological groups and that N is a normal subgroup of G. Suppose that G/N is an* [NSS]*-group. Then there is a finite subset F of J such that G/N is isomorphic to $p_F(G)/p_F(N)$.*

P r o o f. For any subset $I \subseteq J$ let $N_I = G \cap (N \prod_{j \in J \setminus I} G_j)$. We consider the continuous homomorphism

$$\phi_I : G \to p_I(G)/p_I(N), \qquad \phi_I(g) = p_I(g)p_I(N). \tag{1}$$

We notice

$$\ker \phi_I = \{g \in G : p_I(g) \in p_I(N)\} = G \cap \Big(N \prod_{j \in J \setminus I} G_j\Big) = N_I.$$

Let $\mathcal{U}_j$ denote the set of open identity neighborhoods of G_j. Then the set

$$\mathcal{U}_I = \Big\{ \prod_{j \in I} W_j : W_j \in \mathcal{U}_j \text{ and } (\exists F \subseteq I)\ F \text{ is finite and } j \in I \setminus F \Rightarrow W_j = G_j \Big\}$$

is a basis for the filter of identity neighborhoods of $\prod_{j \in I} G_j$. A basic open identity neighborhood of $p_I(G)/p_I(N)$ is given by

$$V = p_I(N)\Big(p_I(G) \cap \prod_{j \in I} V_j\Big)/p_I(N), \qquad \prod_{j \in I} V_j \in \mathcal{U}_I.$$

We shall show that

$$V = \phi_I\Big(G \cap \prod_{j \in J} U_j\Big), \quad \prod_{j \in J} U_j \in \mathcal{U}_J \quad \text{and} \quad U_j = \begin{cases} V_j & \text{for } j \in I, \\ G_j & \text{for } j \in J \setminus I. \end{cases} \tag{2}$$

In view of the definition of ϕ_I in (1), claim (2) will be proved once we prove

$$p_I\Big(G \cap \prod_{j \in J} U_j\Big) = p_I(G) \cap \prod_{j \in I} V_j. \tag{3}$$

The left side is clearly in the right side. For a proof of the reverse containment in (3) we consider $h = (h_j)_{j \in J} \in \prod_{j \in J} G_j$ with $p_I(h) \in p_I(G) \cap \prod_{j \in I} V_j$. Then there is a $k = (k_j)_{j \in J} \in \ker p_I = \prod_{j \in J \setminus I} G_j$ such that $g := hk \in G$ and $g_j = h_j \in V_j$ for $j \in I$. Now $g \in G \cap \prod_{j \in J} U_j$ since $U_j = G_j$ for $j \in J \setminus I$, and $p_I(h) = p_I(h)p_I(k) = p_I(hk) = p_I(g) \in p_I(G \cap \prod_{j \in I} U_j)$. This establishes (3) and thus (2). Now by (2) we know that ϕ_I is open. The morphism

$$f_I : G/N_I \to p_I(G)/p_I(N), \qquad f_I(gN_I) = \phi_I(g) = p_I(g)p_I(N), \tag{4}$$

induced by ϕ_I is therefore is an isomorphism of topological groups.

The basic identity neighborhoods of G/N are of the form

$$N\Big(G \cap \prod_{j \in J} U_j\Big)/N, \qquad \prod_{j \in J} U_j \in \mathcal{U}_J.$$

We recall that $\prod_{j \in J} U_j \in \mathcal{U}_J$ means that there exists a finite subset F of J such that $U_j = G_j$ for $j \in J \setminus F$. By the definition of N_F and the modular law,

$$N_F = G \cap \Big(N \prod_{j\in J\setminus F} G_j\Big) = N\Big(G \cap \prod_{j\in J\setminus F} G_j\Big) \subseteq N\Big(G \cap \prod_{j\in J} U_j\Big).$$

Thus $\{N_F/N : F \text{ finite in } J\}$ is a filter basis of normal subgroups of G/N converging to $\{N\}$. Since G/N is an [NSS]-group there is an F such that $N_F/N = \{N\}$, i.e., $N_F = N$. Hence, in view of (4) we have found a finite set $F \subseteq J$ such that $f_F : G/N \to p_F(G)/p_F(N)$ is an isomorphism of topological groups. ∎

Let us abbreviate various operations applied to a class Ω of topological groups as follows:

- P: the formation of the product of a finite family of members of Ω, endowing the product with the product topology,
- C: the formation of the product of an arbitrary family of members of Ω, endowing the product with the product topology,
- S: the passing to a subgroup of a member of Ω endowing the subgroup with the induced topology,
- $\overline{\mathrm{S}}$: the passing to a closed subgroup of a member of Ω endowing the subgroup with the induced topology,
- Q: the forming of quotient groups of a member of Ω modulo a closed normal subgroup endowing the quotient group with the quotient topology.

From Lemma 2 we deduce at once the following observation:

5.3. Proposition. *For a class Ω of topological groups, the members of* $\mathrm{QSC}(\Omega)$ *which do not have small subgroups are contained in* $\mathrm{QSP}(\Omega)$. ∎

The variety $\mathfrak{V}(\Omega)$ generated by Ω is obtained as $\mathfrak{V}(\Omega) = \mathrm{SCQ\overline{S}P}(\Omega)$ (cf. [10], Theorem 7). It follows from 5.3 that the [NSS]-members of $\mathfrak{V}(\Omega)$ are contained in $\mathrm{SPQ\overline{S}P}(\Omega)$. Note that

(a) $\prod_{j\in J} G_j/N_j \cong \prod_{j\in J} G_j/\prod_{j\in J} N_j$, whence $\mathrm{CQ} \subseteq \mathrm{QC}$, and $\mathrm{PQ} \subseteq \mathrm{QP}$.

(b) $G_k \subseteq \prod_{j\in J_k} H_{kj}$, $k \in K$, implies that $\prod_{k\in K} G_k \subseteq \prod_{k\in K,\, j\in J_k} H_{kj}$. Thus $\mathrm{PSP} \subseteq \mathrm{SP}$. Similarly, $\mathrm{P\overline{S}P} \subseteq \mathrm{\overline{S}P}$.

Hence

(c) $\mathrm{PQ\overline{S}P} \subseteq \mathrm{Q\overline{S}P}$, and $\mathrm{SPQ\overline{S}P} \subseteq \mathrm{SQ\overline{S}P}$.

(d) Let H be a closed subgroup of G/N and let $p : G \to G/N$ denote the quotient morphism. Set $K = p^{-1}H$. Then K is a closed subgroup of G and $H = K/N$ (algebraically and topologically). Hence $\mathrm{\overline{S}Q} \subseteq \mathrm{Q\overline{S}}$. Similarly $\mathrm{SQ} \subseteq \mathrm{QS}$.

Since obviously $\mathrm{S}\overline{\mathrm{S}} \subseteq \mathrm{S}$, we have $\mathrm{SPQ}\overline{\mathrm{S}}\mathrm{P} \subseteq \mathrm{QSP}$ and $\overline{\mathrm{S}}\mathrm{PQ}\overline{\mathrm{S}}\mathrm{P} \subseteq \mathrm{Q}\overline{\mathrm{S}}\mathrm{P}$. These observations yield the following result.

5.4. THEOREM. *The class of members of* $\mathfrak{V}(\Omega)$ *which do not have small subgroups is contained in* $\mathrm{SPQ}\overline{\mathrm{S}}\mathrm{P}(\Omega) \subseteq \mathrm{QSP}(\Omega)$. ∎

We have the following corollary:

5.5. THEOREM. *If* Ω *is any class of Hausdorff topological groups and* $\mathfrak{V}(\Omega)$ *the variety of Hausdorff topological groups generated by* Ω, *then every Lie group in* $\mathfrak{V}(\Omega)$ *is contained in* $\mathrm{Q}\overline{\mathrm{S}}\mathrm{P}(\Omega)$.

Proof. A Lie group G has no small subgroups. Hence by Theorem 5.4, if $G \in \mathfrak{V}(\Omega)$ then $G \in \mathrm{SPQ}\overline{\mathrm{S}}\mathrm{P}(\Omega)$. But G is locally compact and therefore is closed in any Hausdorff topological group. Hence $G \in \overline{\mathrm{S}}\mathrm{PQ}\overline{\mathrm{S}}\mathrm{P}(\Omega) \subseteq \mathrm{Q}\overline{\mathrm{S}}\mathrm{P}(\Omega)$ in view of (d) above. ∎

This answers **P1250** in the affirmative. The assumption made in the formulation of the problem that the class Ω should consist of Lie groups is immaterial for the conclusion of Theorem 5.5.

The following example illustrates the fact that one has to be careful to distinguish between embeddings (as closed subgroups) and injective continuous homomorphic images of Lie groups.

5.6. EXAMPLE. Assume that G is the simply connected covering group of the group $\mathrm{Sl}(2,\mathbb{R})$. Let $Z \cong \mathbb{Z}$ denote the center of G and Z_n the subgroup of index n in Z. Set $\Omega = \{G/Z_1, G/Z_2, G/Z_3, \ldots\}$. Then

$$g \mapsto (gZ_n)_{n\in\mathbb{N}} : G \to \prod_{n\in\mathbb{N}} G/Z_n$$

is an injective morphism of topological groups into a member of $\mathfrak{V}(\Omega)$.

However, G is not itself a member of the variety $\mathfrak{V}(\Omega)$. For if it were, then, by Theorem 5.5, it would be in $\mathrm{Q}\overline{\mathrm{S}}\mathrm{P}$. Thus we could find a finite sequence $(n_1, \ldots, n_k)$ of natural numbers and a closed subgroup H of $\prod_{i=1}^k G/Z_{n_i}$ such that G would be a quotient of H. Since quotient morphisms are open, G would be a quotient of H_0, i.e., there would be a quotient morphism $\pi : H_0 \to G$. Let R denote the radical of H_0. Since G is a simple Lie group, $\pi(R) = \{1\}$. Let S be a Levi complement for the radical in H_0. Then S is an analytic subgroup in H_0; a priori, it may not be closed. However, its projection into G/Z_{n_i} is a semisimple analytic subgroup and thus is either $\{1\}$ or G/Z_{n_i}. In any event, the center of S projects into the center of G/Z_{n_i}. Hence the center of S is contained in the finite center $\prod_{i=1}^k Z/Z_{n_i}$ and is therefore finite. Thus the center of $G = \pi(S)$ is finite—a contradiction. ∎

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Fachbereich Mathematik
Technische Hochschule Darmstadt
Schlossgartenstr. 7
D-64289 Darmstadt, Germany
E-mail: hofmann@mathematik.th-darmstadt.de

Faculty of Informatics
University of Wollongong
Wollongong, NSW 2522
Australia
E-mail: s.morris@uow.edu.au

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