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Locally compact groups, residual Lie groups, and varieties generated by Lie groups

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Abstract

The concept of approximating in various ways locally compact groups by Lie groups is surveyed with emphasis on pro-Lie groups and locally compact residual Lie groups. All members of the variety of Hausdorff groups generated by the class of all finite dimensional real Lie groups are residual Lie groups. Conversely, we show that every *locally compact* member of this variety is a pro-Lie group. For every locally compact residual Lie group we construct several better behaved residual Lie groups into which it is equidimensionally immersed. We use such a construction to prove that for a locally compact residual Lie group G the component factor group G/G_0 is residually discrete.

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Introduction

In the investigation of the structure of locally compact (Hausdorff) topological groups there are several basic strategies:

- *Reduce the problem to a question on Lie groups for which a rich theory is available.*
- *Investigate special classes of locally compact groups and present explicit structure theorems in terms of direct or semidirect products of well-known components and extensions.*

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- *Investigate projective limits of locally compact groups (sometimes locally compact again, sometimes not).*

Examples of special classes of locally compact groups are *maximally almost periodic* groups (MAP-groups, i.e., groups which permit an injective morphism into a compact group), groups possessing a *compact invariant neighborhood* of the identity (IN-groups), groups possessing arbitrarily *small invariant neighborhoods* of the identity (SIN)-groups, groups G with a center (*Zentrum*) Z such that G/Z is compact (Z-groups). Since the sixties, a host of those classes have been investigated, and a good source of information is the work by Grosser and Moskowitz (see [7]).

As regards the first and more general strategy, the best types of groups are the groups satisfying the following property:

(1) *G is a topological group and for every identity neighborhood U there is a compact normal subgroup N contained in U such that G/N is a Lie group.*

Such a group G is called *a pro-Lie group*. Note that pro-Lie groups are locally compact.

We list a few sample pieces of information known about these:

(A) *If G is a locally compact group with identity component G_0 such that G/G_0 is compact, then G is a pro-Lie group* [14, Theorem 4.6]. *In particular, every locally compact group contains open pro-Lie subgroups* [14, Corollary 4.5].

(B) *Every Z-group is a pro-Lie group* [7]. *In particular, all compact and all abelian groups are pro-Lie groups.*

(C) *Every SIN-group is a pro-Lie group* [7], *see also* [11].

The characterisation (1) of pro-Lie groups suggests an equivalent reformulation. Indeed, let $N \trianglelefteq G$ denote the statement "N is a normal subgroup of G". Then with each topological group G we can canonically associate the set

$$\mathcal{N}(G) = \{N \trianglelefteq G\colon\ G/N \text{ is a Lie group}\}$$

and observe that $G \in \mathcal{N}(G)$. Note that $N \in \mathcal{N}(G)$ implies that N is closed. We shall observe (see Proposition 3.2(iii)) that $\mathcal{N}(G)$ is a filter basis. We must recall what it means that a filterbasis in a topological space converges to a point (see Section 1 below, first paragraph). In terms of this terminology we have that (see Remark 3.3)

(2) *a topological group G is a pro-Lie group if and only if it is a locally compact group such that* $\lim \mathcal{N}(G) = 1$.

The concept of "approximating G by Lie groups" expressed in (1) and (2) can be expanded so as to encompass a properly larger class than that of pro-Lie groups. Indeed we shall say that a topological group G is a *residual Lie group* if

(3) *the class of morphisms of topological groups from G into Lie groups separates the points of G.*

Obviously, every pro-Lie group is a residual Lie group, but the groups in Example 0.3 below are residual Lie groups, but not pro-Lie groups. Every MAP-group is a residual Lie group (see Proposition 3.5 below), but the Lie groups which are not MAP form a vast majority of all Lie groups, beginning with the 2-dimensional nonabelian solvable group, the simple noncompact $\mathrm{Sl}(2, \mathbb{R})$, and the three-dimensional nilpotent Heisenberg group.

The following condition on a topological group G is close to (3):

(4) *The filter* $\mathcal{N}(G)$ *satisfies* $\bigcap \mathcal{N}(G) = \{1\}$.

Clearly (4) implies (3), and if G is a locally compact group, then the two conditions are equivalent (see Proposition 3.5 below). Thus *a locally compact group G is a residual Lie group if and only if* $\bigcap \mathcal{N}(G) = \{1\}$. Therefore, the difference between pro-Lie groups and locally compact residual Lie groups is fully captured by the difference between the two statements

$$\lim \mathcal{N}(G) = 1 \quad \text{versus} \quad \bigcap \mathcal{N}(G) = \{1\}.$$

The concept of a residual Lie group is very much in the spirit of another general strategy for approaching the structure theory of topological groups via that of Lie groups or other special classes of well known groups, namely, the strategy of *varieties* borrowed from universal algebra (see [16]):

- *Investigate the smallest class of groups containing all Lie groups and being closed under the formation of subgroups, quotient groups modulo closed normal subgroups, and arbitrary products.*

A class of topological groups is called a *variety of Hausdorff groups* if it is closed under the formation of subgroups, quotient groups modulo closed normal subgroups, and arbitrary products.

Since we are restricting our attention to Hausdorff topological groups, we note that a variety of Hausdorff groups is not a variety in the original sense of the word used for example in [16]. However, a variety in our present sense consists of all of the Hausdorff groups in a variety in the sense of [16].

Let us abbreviate various operations applied to a class Ω of topological groups as follows

P: the formation of all finite products of members of Ω with the product topology,

S: the passing to a subgroup of a member of Ω endowing the subgroup with the induced topology,

$\overline{\text{S}}$: the passing to a closed subgroup of a member of Ω endowing the subgroup with the induced topology,

Q: the forming of quotient groups of a member of Ω modulo a closed normal subgroup endowing the quotient group with the quotient topology,

C: the formation of the cartesian product of arbitrary families of members of Ω, endowing the product with the product topology.

Then the variety of Hausdorff groups $\mathfrak{V}(\Omega)$ generated by Ω is obtained as $\mathfrak{V}(\Omega) = \text{SCQ}\overline{\text{S}}\text{P}(\Omega)$ (cf. [16, Theorem 7]). If each member G of a class Ω' has sufficiently many morphisms $G \to L$ into a member L of a class Ω to separate the points, then each member of the classes $\text{S}(\Omega')$, $\text{C}(\Omega')$ (and thus certainly of $\text{P}(\Omega')$) has the same property. The class Λ of finite dimensional real Lie groups satisfies $\text{Q}\overline{\text{S}}\text{P}(\Lambda) = \Lambda$. Hence

$$\mathfrak{V}(\Lambda) = \text{SC}(\Lambda),$$

and it follows that every member of $\mathfrak{V}(\Lambda)$ is a residual Lie group.

We shall see in Example 0.5 that the additive group of any Banach space is a residual Lie group. However, no infinite dimensional Banach space belongs to $\mathfrak{V}(\Lambda)$. This is proved in [11, 5.4]; recall that the additive group B of a Banach space has no small subgroups. Compare also [15]. We shall see in Proposition 5.3 that every *locally compact* group in $\mathfrak{V}(\Lambda)$ is a pro-Lie group, and we shall see examples of locally compact residual Lie groups which are not pro-Lie groups. The residual Lie groups in Example 0.3 are not pro-Lie groups, hence they are not in $\mathfrak{V}(\Lambda)$.

A closer look at the filter basis $\mathcal{N}(G)$ of a topological group G reveals two canonical constructions attached to G. In order to deal with these constructions in a systematic fashion we may (and should) consider an arbitrary filter basis $\mathcal{F}$ of closed normal subgroups of G. One can always pass to the quotients $G/\bigcap\mathcal{F}$ and $N/\bigcap\mathcal{F}$; most information on the structure of G contained in the G/N is captured by this set-up. We shall therefore assume, for the following discussion, that

$$\bigcap\mathcal{F} = \{1\}.$$

(Yet the example of IN-groups (see Example 0.3 below) shows that not all interesting properties are stable under extensions. Thus our assumption entails a certain loss of generality even in the case of $\mathcal{F} = \mathcal{N}(G)$. One can still satisfactorily deal with that situation, however.)

Firstly, the filter generated by all sets $UN = NU$ where U ranges through the identity neighborhoods of G with respect to the given group topology $\mathcal{O}$ and N through $\mathcal{F}$ generates the filter of identity neighborhoods of a second group topology $\mathcal{O}_\mathcal{F}$ on G which, in general, is coarser than the given one. (Inspect Example 0.3 in this regard!) Generally, it need not be a complete group topology. In fact, it need not be locally precompact even if G is locally compact. We shall see that

– *for a locally compact group $(G, \mathcal{O})$ the group $(G, \mathcal{O}_\mathcal{F})$ is locally precompact if and only if for some member $N \in \mathcal{F}$ and all members $M \in \mathcal{F}$ with $M \subseteq N$ the factor group N/M is compact.* (See Proposition 1.3.)

Secondly, and more importantly, we attach to G a new group $G_\mathcal{F}$, namely the projective limit $\lim_{N\in\mathcal{F}} G/N \subseteq \prod_{N\in\mathcal{F}} G/N$ and a canonical dense injection $\gamma: G \to G_\mathcal{F}$ (see Lemma 2.1). The significance of the group topology $\mathcal{O}_\mathcal{F}$ on G mentioned above is the following: γ induces an isomorphism of topological groups from $(G, \mathcal{O}_\mathcal{F})$ to $\gamma(G)$ (see Lemma 2.1(v)), and $G_\mathcal{F}$ emerges as a sort of completion. It is literally a completion if G and thus all G/N are locally compact and hence complete. In this fashion we know that

– *for a locally compact group G the group $G_\mathcal{F}$ is a locally compact group if and only if $\bigcap\mathcal{F} = \{1\}$ and there is a member N of $\mathcal{F}$ such that for all members $M \in \mathcal{F}$ with $M \subseteq N$ the groups N/M are compact.* (See Proposition 2.2 below.)

An instructive example is the Lie group $G = \mathbb{R}$ with the filter basis $\mathcal{F}$ of all subgroups of $\mathbb{Z}$ except $\{0\}$. (See the comments following Theorem 4.2 below.) Then each G/N is isomorphic to the circle group and $G_\mathcal{F}$ is the compact solenoid which is the character group of the discrete group $\mathbb{Q}$.

A *one-parameter subgroup* of a topological group G is a morphism of topological groups $X:\mathbb{R} \to G$. We denote the set of one-parameter subgroups of G by $\mathfrak{L}(G)$. Clearly $\mathfrak{L}$ is the object portion of a functor from the category of topological groups and continuous group morphisms to the category of sets. On the full subcategory of Lie groups, there is much additional structure attached to $\mathfrak{L}$. From [12] we recall the following definition:

Assume that $f:G \to H$ is a morphism of topological groups. We say that f is an *equidimensional immersion* if f is injective and for any one-parameter subgroup $X:\mathbb{R} \to H$ of H there is a one-parameter subgroup $X':\mathbb{R} \to G$ of G such that $X = f \circ X'$. Equivalently, f is injective and induces a bijection $\mathfrak{L}(f):\mathfrak{L}(G) \to \mathfrak{L}(H)$.

In the preceding example, $\gamma:\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is an equidimensional immersion. Note that here $\mathcal{F} \neq \mathcal{N}(G)$. We shall show in Theorem 4.2 that

– *For any locally compact residual Lie group G the canonical morphism $\gamma:G \to G_{\mathcal{N}(G)}$ is an equidimensional immersion.*

Another example is the discrete group (hence Lie group) $G = \mathbb{Z}^{\mathbb{N}}$ with the filter basis $\mathcal{F}$ of all partial products with finite co-rank. Then $G_{\mathcal{F}}$ is $\mathbb{Z}^{\mathbb{N}}$ with the product topology. Hence $G_{\mathcal{F}}$ is nondiscrete complete metric separable totally disconnected but not locally compact.

There is the question when $\gamma:G \to G_{\mathcal{F}}$ is an isomorphism. This is settled by the following result (irrespective of local compactness!):

– *If G is a complete topological group then the morphism $\gamma:G \to G_{\mathcal{F}}$ is an isomorphism of topological groups if and only if* $\lim \mathcal{F} = 1$. (See Proposition 2.3.)

In particular,

– *for a locally compact group G the following statements are equivalent:*

(1) *G is a pro-Lie group.*

(2) $\lim \mathcal{N}(G) = 1$.

(3) $\gamma:G \to G_{\mathcal{N}(G)}$ *is an isomorphism of topological groups.*

Many of our examples illustrate filter bases $\mathcal{F}$ with $\bigcap \mathcal{F} = \{1\}$ which fail to satisfy $\lim \mathcal{F} = 1$. It may be worthwhile to observe at least one case where the trivial intersection suffices for the filter basis to converge:

– *If $\mathcal{F}$ is a filter basis of closed normal and* connected *subgroups of a locally compact group G intersecting trivially, then* $\lim \mathcal{F} = 1$. (See Lemma 2.5.)

As a consequence we shall derive

– *Every locally compact residual Lie group G is the strict projective limit* $\lim_{N \in \mathcal{N}(G)} G/N_0$ *of finite dimensional locally compact groups.* (See Lemma 4.3.)

In many respects, finite dimensional locally compact groups are very close to Lie groups. Example 0.6 shows that they do not have to be pro-Lie groups. In the applications finite dimensional groups often serve almost as well as Lie groups (see for example Stroppel [17]). The link between finite dimensional locally compact groups and Lie groups is investigated in [12] in terms of equidimensional immersions.

While it is inherent in the concept of a variety that $G \in \mathfrak{V}(\Lambda)$ implies $G/G_0 \in \mathfrak{V}(\Lambda)$ it is by no means obvious that for a residual Lie group also G/G_0 is a residual Lie group. (In fact, the class of all residual Lie groups does not form a variety, see Proposition 5.1.)

We shall show that this is nevertheless the case. In fact we shall prove the following result (Theorem 4.6, Corollary 4.9):

Theorem. *Assume that G is a locally compact residual Lie group and define $\Gamma \stackrel{\mathrm{def}}{=} \lim_{N\in\mathcal{N}(G)} G/(G_0\cap N)$. Then Γ is a residual Lie group and there is an equidimensional dense immersion $G \to \Gamma$ such that*

(i) *$\Gamma_0 \cap N$ is compact for all sufficiently small $N \in \mathcal{N}(\Gamma)$.*

(ii) *$\Gamma/\Gamma_0 \cong G/G_0$.*

(iii) *In G as well as in Γ the identity component is the intersection of all open* normal *subgroups.*

Claim (iii) will be established for Γ first, the assertion on G is then a consequence. Thus this result illustrates the use of the techniques and constructions that are introduced in this paper.

A topological group is called residually discrete, if the filter of open normal subgroups intersects in $\{1\}$. Thus

– *for a locally compact residual Lie group G, the factor group G/G_0 is residually discrete.*

The investigation of locally compact residual Lie groups is thus, among other things, reduced to that of locally compact totally disconnected residual Lie groups. Example 0.3 shows that there are totally disconnected residual Lie groups which are not pro-Lie groups.

0. Basic definitions and examples

In this paper, we consider *Hausdorff* topological groups. We remark that there is some reason to include non-Hausdorff groups when discussing varieties of topological groups, see [16].

Definition 0.1. (i) A topological group G is called a *residual Lie group* if the continuous group homomorphisms into Lie groups separate the points of G.

(ii) G is called a pro-Lie group if every identity neighborhood of G contains a normal compact subgroup N such that G/N is a Lie group.

Example 0.2. Let L be a nontrivial compact Lie group and let $\mathbb{Z}$ act automorphically on $P \stackrel{\mathrm{def}}{=} L^{\mathbb{Z}}$ by the shift. Set $G = P \rtimes \mathbb{Z}$. Then $N = P \times \{0\}$ is a compact normal pro-Lie subgroup and G/N is discrete, and thus a Lie group. But G is not a pro-Lie group (and, therefore, not an SIN-group by [7]. Yet G is an IN-group).

Example 0.3. Let D be a discrete group, and A a compact nontrivial group of automorphisms of D. Then the compact group $A^{\mathbb{N}}$ acts automorphically on the discrete group $V = D^{(\mathbb{N})}$ via $(a_n)_{n\in\mathbb{N}} \cdot (d_n)_{n\in\mathbb{N}} = (a_n d_n)_{n\in\mathbb{N}}$. Set $G = V \rtimes A^{\mathbb{N}}$. Then $N = V \times \{1\}$ is a discrete normal subgroup, thus in particular a normal Lie subgroup. The factor group

$G/N \cong A^{\mathbb{N}}$ is compact and therefore a pro-Lie group. But G is not a pro-Lie group (and, therefore, not an SIN-group by [11]). It is not an IN-group.

However, the continuous homomorphisms

$$f_k : G \to D \rtimes A, \quad f_k\big(((d_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}})\big) = (d_k, a_k),$$

separate the points of G. Thus, if $D \rtimes A$ is a residual Lie group, then G is a residual Lie group. This is certainly the case if A is finite. If D is maximally almost periodic, i.e., if the natural morphism $\beta_D : D \to \alpha(D)$ into the Bohr compactification of D is injective and if A is finite, then the action of A on D extends to an action of A on $\alpha(D)$ and the map

$$\varepsilon : G \to \alpha(D)^{\mathbb{N}} \rtimes A^{\mathbb{N}}, \quad \varepsilon\big(((d_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}})\big) = \big((\beta(d_n))_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}\big),$$

is injective and thus G is an MAP-group.

If D is infinite, then $D^{\mathbb{N}} \rtimes A^{\mathbb{N}}$ is not even locally compact.

Special cases: (a) $D = \mathbb{Z}(3)$, $A = \{1, -1\}$, $D \rtimes A \cong S_3$ yields the smallest example.

(b) $D = \mathbb{Z}$, $A = \{1, -1\}$; $D \rtimes A$ is the infinite dihedral group. As an abelian group, D is an MAP-group, and so G is a locally compact, totally disconnected MAP-group. Let $\mathcal{E}$ denote the filter of all cofinite subsets of $\mathbb{N}$ and $\mathcal{F}$ the filter basis of all closed normal subgroups

$$N_S = \big\{((d_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}) \in G\colon\ (\forall m \in \mathbb{N})\ (m \notin S) \Rightarrow (d_n = 0 \text{ and } a_n = 1)\big\},$$

where $S \in \mathcal{E}$. Then $\lim_{S\in\mathcal{E}} G/N_S \cong D^{\mathbb{N}} \rtimes A^{\mathbb{N}}$ is not locally compact.

(c) $D = \mathbb{Z}(p^\infty)$, the discrete Prüfer group for the prime number p, A the group of units of the ring $\mathbb{Z}_p$ of p-adic integers. (We record that the additive group of $\mathbb{Z}_p$ is isomorphic to the character group of D and that $\mathbb{Z}_p$ may be identified with the endomorphism ring of D. Also note that $A = \mathbb{Z}_p \setminus p\mathbb{Z}_p$.) Then G is locally compact totally disconnected, it is not a pro-Lie group but is a residual Lie group. Similar conclusions hold as in case (b). Example (b) above can be found in [1, Example 2]. For the general construction of examples of this sort see also [2, p. 837].

Example 0.4. Let $\mathbb{Q}_p$ denote the locally compact field of p-adic rationals and set $G = \mathrm{Sl}(2, \mathbb{Q}_p)$. Then G is a locally compact totally disconnected group (in fact a p-adic Lie group). Its center is $Z = \{1, -1\}$ and $\mathrm{PSl}(2, \mathbb{Q}_p)$ is a simple locally compact totally disconnected group. It has arbitrarily small compact open subgroups, but no nontrivial normal subgroups whatsoever. Thus it is not a residual Lie group.

Example 0.5. Let G be the additive group of a real locally convex topological vector space. Then G is a residual Lie group. It is locally compact if and only if $\dim G < \infty$.

For a proof we need only note that the continuous linear functionals separate the points according to the Hahn–Banach theorem and that the image of every functional is a Lie group. We recall that a Hausdorff topological vector space (over a locally compact nondiscrete division ring) is locally compact if and only if it is finite dimensional.

Let E denote a vector subspace of the topological dual of G. Let $\mathcal{F}$ denote the filter basis of all closed subspaces of finite codimension which are annullators of finite dimensional subspaces of E. Then the projective limit (cf. the introduction to Section 2) $\widetilde{G} = \lim_{N\in\mathcal{F}} G/N$ is the completion of G with respect to the weak $\sigma(E,G)$-topology.

If G is the topological dual of a locally convex vector space in the weak-$*$ topology, then $G \cong \widetilde{G}$.

In general we see that there are *vast* residual Lie groups which are far from being locally compact. Our discussion will concentrate on *locally compact residual Lie groups.*

Example 0.6. Let G denote the universal covering group of $\mathrm{Sl}(2,\mathbb{R})$. Then the center Z of G is infinite cyclic, and G/Z is simple. Let $\mathcal{N}(G)^\times$ be the filterbasis of all nontrivial subgroups of Z, i.e., the filterbasis of all nontrivial closed normal subgroups. This filter basis has trivial intersection but it does not converge to 1 in G because none of its members is compact. The group $\lim_{N\in\mathcal{N}(G)^\times} G/N$ is a locally compact semisimple group of dimension 3 with a center isomorphic to the universal zero dimensional compactification of $\mathbb{Z}$, which is isomorphic to $\prod_{p \text{ is prime}} \mathbb{Z}_p$.

Lemma 0.7. *Let G be a residual Lie group and C a compact subset not containing the identity. Then there exist a Lie group L and a continuous group homomorphism $f: G \to L$ such that $C \cap \ker f = \emptyset$.*

Proof. For every $c \in C$, we find a Lie group L_c and a continuous group homomorphism $f_c: G \to L_c$ such that $f_c(c) \neq 1$. By the continuity of f_c there is a whole neighborhood U_c of c in G such that $1 \notin f_c(U_c)$. The compactness of C permits us to find elements $c_1, \ldots, c_n \in C$ such that $C \subseteq \bigcup_{j=1}^n U_{c_j}$. Then $1 \notin \bigcup_{j=1}^n f_{c_j}(U_{c_j})$. We set $L = \prod_{j=1}^n L_{c_j}$ and define $f: G \to L$ by $f(g) = (f_{c_1}(g), \ldots, f_{c_n}(g))$. Then $1 \notin f(C)$ which is equivalent to our assertion. □

Proposition 0.8. *Every residual Lie group G satisfies the following condition:*

(M) *For any compact subset $C \subseteq G$ not containing the identity there is a closed identity neighborhood V in G containing a normal subgroup of G which is, in addition, the largest of all subgroups contained in V.*

Condition (M) *implies*

(L) *For any compact subset $C \subseteq G$ not containing the identity there is an identity neighborhood V in G such that*

$$(\forall g \in G,\ H \leqslant G) \quad H \subseteq V \Rightarrow gHg^{-1} \cap C = \emptyset.$$

Proof. By Lemma 0.7 we find $f: G \to L$ with $\ker f \cap C = \emptyset$. Since L is a Lie group there is a closed identity neighborhood W in which $\{1\}$ is the only subgroup of L. Let $V = f^{-1}(W)$. Then $\ker f$ is the largest subgroup of G among those contained in V. The remainder is trivial. □

For the validity of condition (L) in a residual Lie group compare [2, Theorem 1.3 and its corollary].

1. Filter bases of normal subgroups and group topologies

We begin with background information. Let $\mathcal{B}$ be a filter basis on a set X. Let $\langle\mathcal{B}\rangle = \{Y \subseteq X\colon (\exists B \in \mathcal{B})\ B \subseteq Y\}$ denote the filter generated by $\mathcal{B}$ on X. On a topological space X let $\mathcal{U}_X(x)$ denote the filter of all neighborhoods of x in X. Recall that a filter basis $\mathcal{F}$ on a topological space *converges to* $x \in X$, written $\lim \mathcal{F} = x$ if for every neighborhood U of x there is an $F \in \mathcal{F}$ such that $F \subseteq U$. This is equivalent to saying that $\mathcal{U}_X(x) \subseteq \langle\mathcal{F}\rangle$.

Recall that a filter basis $\mathcal{B}$ on a group G generates the neighborhood filter of 1 with respect to a group topology $\mathcal{O}$ if and only if

(I) $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})\quad VV^{-1} \subseteq U$.

(II) $(\forall U \in \mathcal{B},\, g \in G)(\exists V \in \mathcal{B})\quad gVg^{-1} \subseteq U$.

We shall denote the filter of identity neighborhoods of a topological group G briefly by $\mathcal{U} = \mathcal{U}_G$.

Let $\mathcal{F}$ denote a filter basis of closed normal subgroups of G. Set

$$\mathcal{U}_{\mathcal{F}} = \{UN\colon\ U \in \mathcal{U} \text{ and } N \in \mathcal{F}\}.$$

Since $N \in \mathcal{F}$ is normal we have $UN = NU$. It is readily verified that $\mathcal{U}_{\mathcal{F}}$ satisfies (I) and (II) above. It therefore defines a group topology $\mathcal{O}_{\mathcal{F}}$ which is coarser than or equal to the given group topology. Note that $\mathcal{O}_{\mathcal{F}}$ is a Hausdorff topology if and only if $\bigcap \mathcal{F} = \{1\}$. Our interest in $\mathcal{O}_{\mathcal{F}}$ will become clear in Lemma 2.1 below.

The proof of the following lemma is straightforward.

Lemma 1.1. *The following statements are equivalent:*

(1) $\mathcal{O} = \mathcal{O}_{\mathcal{F}}$.

(2) $\mathcal{U} \subseteq \langle\mathcal{F}\rangle$.

(3) $\lim \mathcal{F} = 1$.

Lemma 1.2. *Let $M \in \mathcal{F}$. Both $\mathcal{O}$ and $\mathcal{O}_{\mathcal{F}}$ induce on G/M the same quotient topology.*

Proof. A basic identity neighborhood of G/M in the quotient topology of $\mathcal{O}$ is (a) UM/M with $U \in \mathcal{U}$. A basic identity neighborhood of G/M in the quotient topology of $\mathcal{O}_{\mathcal{F}}$ is (b) UNM/M with $U \in \mathcal{U}$ and $N \in \mathcal{F}$. The sets of type (a) and (b) are equal as soon as $N \subseteq M$. □

Proposition 1.3. *Assume that G is a topological group with respect to a topology $\mathcal{O}$ and let $\mathcal{F}$ be a filter basis of closed normal subgroups such that $\bigcap \mathcal{F} = \{1\}$. Consider the following statements:*

(1) *$\mathcal{O}_{\mathcal{F}}$ is locally precompact.*

(2) *For some $N \in \mathcal{F}$ and all $M \in \mathcal{F}$ with $M \subseteq N$ the quotient group N/M is precompact.*

Then (1) *implies* (2), *and if G is locally precompact then* (1) *and* (2) *are equivalent.*

Proof. (1) implies (2): By (1) there is an $\mathcal{O}_{\mathcal{F}}$-precompact subspace UN with $U \in \mathcal{U}$ and $N \in \mathcal{F}$. In particular, N is $\mathcal{O}_{\mathcal{F}}$-precompact. Let $M \in \mathcal{F}$, $M \subseteq N$. Then by Lemma 1.2, N/M is precompact in G/M (with respect to the unique quotient topology).

Assume now that G is locally precompact. We show that (2) implies (1): Let $U \in \mathcal{U}$ be a precompact $\mathcal{O}$-identity neighborhood and let N be as in (2). We claim that UN is $\mathcal{O}_{\mathcal{F}}$-precompact. Let $V \in \mathcal{U}$ and $M \in \mathcal{F}$ with $M \subseteq N$. Pick $W \in \mathcal{U}$ so that $WW \subseteq V$. Since U is precompact there is a finite subset $E_U \subseteq U$ such that $U \subseteq WE_U$. Now select a $W' \in \mathcal{U}$ so that $xW'x^{-1} \subseteq W$ for all $x \in E_U$, and also $W' \subseteq W$. Then $xW' \subseteq Wx$ for $x \in E_U$. Since N/M is precompact, there is a finite subset E_N such that $N \subseteq W'ME_N = W'E_NM$. Set $E = E_UE_N$. Then $UN \subseteq WE_UW'E_NM \subseteq WWE_UE_NM \subseteq VEM = VME$. This proves the claim. □

The closure of a subset A of a topological space $(X, \mathcal{O})$ will be denoted by $\mathrm{cl}_{\mathcal{O}}(A)$. The identity component of a topological group G is denoted by G_0. A topological group G is called *almost connected* if G/G_0 is compact.

For the following we recall that for a topological group G the set of all one-parameter subgroups $X : \mathbb{R} \to G$ is denoted by $\mathfrak{L}(G)$. The function $\exp : \mathfrak{L}(G) \to G$ is defined by $\exp X = X(1)$. If G is a Lie group, then $\mathfrak{L}(G)$ may be interpreted as the usual Lie algebra of G. (Cf. [12].)

Lemma 1.4. *Assume that G is a locally compact group.*
(a) *The identity arc-component G_a is $\langle \exp \mathfrak{L}(G) \rangle$.*
(b) *If $q : G \to H$ is a quotient morphism, then the following statements hold:*
  (i) *The image $q(G_0)$ is dense in H_0.*
  (ii) *If q is a closed mapping, or if G_0 contains the kernel of q, then $q(G_0) = H_0$.*
  (iii) *In any case, $H_a = q(G_a)$.*
  (iv) *If connectivity on H is arc connectivity, then $q(G_0) = H_0$.*
  (v) *If H is a Lie group, then $q(G_0) = H_0$.*
  (vi) *If G is totally disconnected, then H is totally disconnected.*

Proof. (a) Every locally compact connected group is a pro-Lie group [14]. Then by the Theorem of Iwasawa [13] there is a compact connected group C and images of one-parameter subgroups $E_1, \dots, E_k$, all isomorphic to $\mathbb{R}$, such that $G_0 = CE_1 \cdots E_k$ and that this decomposition is topologically direct. By the Theorem of Scheerer and Hofmann [10], $C = C'A$ where the commutator subgroup C' of C is a closed, semisimple group satisfying $C' = \exp \mathfrak{L}(C')$, and where A is a compact connected abelian semidirect complement of C'. By Dixmier's Theorem [6], $A_a = \langle \exp \mathfrak{L}(A) \rangle$. Thus $G_a \supseteq \langle \exp \mathfrak{L}(G) \rangle \supseteq A_aC'E_1 \cdots E_k$. Since

$$(a, c, e_1, \dots, e_k) \mapsto ace_1 \cdots e_k : A \times C' \times E_1 \times E_k \to G_0$$

is a homeomorphism, we have $A_aC'E_1 \cdots E_k = G_a$. Thus claim (a) is proved.

(b) We prove assertion (vi) first. Assume that G is totally disconnected. Then there exists a neighborhood basis $\mathcal{B}$ at 1 such that $\mathcal{B}$ consists of open subgroups of G. Since

q is an open continuous mapping, the set $\{q(B)\colon B \in \mathcal{B}\}$ is a neighborhood basis at 1 in H. Now every $q(B)$ is open and closed in H, whence H is totally disconnected.

In order to prove assertion (i), we consider the q-preimage X of the closure of $q(G_0)$ in H. We have to show that $q(X) = H_0$. It is clear that $q(X) \subseteq H_0$. The quotient $G/X \cong (G/G_0)/(X/G_0)$ is totally disconnected by (vi). But $G/X \cong H/q(X)$, whence $q(X) \supseteq H_0$.

(ii) If q is a closed mapping, or if G_0 contains the kernel of q, then $q(G_0)$ is closed in H, whence $q(G_0) = H_0$.

(iii) It has been proved in [12, 1.3] that q maps $\exp \mathfrak{L}(G)$ onto $\exp \mathfrak{L}(H)$. By Part (a) above this implies $q(G_a) = H_a$, as asserted.

(iv) We have $q(G_a) = H_a$ by (iii); thus $H_a = H_0$ implies $q(G_0) \subseteq H_0 = q(G_a) \subseteq q(G_0)$.

(v) In any manifold, connectivity is arc-connectivity. This therefore holds for any Lie group, and thus (v) follows at once from (iv). □

Note that the conclusion $q(G_0) = H_0$ for a quotient morphism $q\colon G \to H$ does not hold without some restriction on q. For example, consider the **a**-adic solenoid, as constructed in [8, 10.12]: Let C be the group of **a**-adic integers. Then C is a compact totally disconnected group with a dense cyclic subgroup generated by some $c \in C$. Form the direct product $C \times \mathbb{R}$, and factor out the discrete subgroup Z that is generated by the pair $(c, 1)$. Then the quotient $S := (C \times \mathbb{R})/Z$ is a connected compact group. However, the image of the connected component of $C \times \mathbb{R}$ is only the arc component of S, a proper subgroup. Note also that $C \times \mathbb{R}$ is a locally compact abelian group, and that it is almost connected (that is, the quotient by the connected component is compact). Thus $C \times \mathbb{R}$ belongs to a class of topological groups that is considered to be the nicest class next to the class of abelian Lie groups: In fact, each locally compact almost connected abelian group is a projective limit of abelian Lie groups.

If we drop the assumption that G is locally compact, even Lemma 1.4(b)(vi) is false: Every topological group is a quotient of a totally disconnected topological group, see [18].

Proposition 1.5. *Let $\mathcal{F}$ be a filter basis of closed normal subgroups of a topological group G such that $\bigcap \mathcal{F} = \{1\}$. Then the following conclusions hold:*

(i) $\bigcap_{N\in\mathcal{F}} \mathrm{cl}_{\mathcal{O}_\mathcal{F}}(G_0N) = \mathrm{cl}_{\mathcal{O}_\mathcal{F}} G_0$.

(ii) $\bigcap_{N\in\mathcal{F}} \mathrm{cl}_{\mathcal{O}_\mathcal{F}}(G_0N) = \bigcap_{N\in\mathcal{F}} \mathrm{cl}_{\mathcal{O}}(G_0N)$.

(iii) *If G is locally compact and G/N is a Lie group then $G_0N \in \mathcal{O}$.*

(iv) *If all G_0N are closed in $(G, \mathcal{O})$, which is certainly the case if all $G_0N \in \mathcal{O}$, then $\bigcap_{N\in\mathcal{F}} G_0N = \mathrm{cl}_{\mathcal{O}_\mathcal{F}} G_0$.*

Proof. (i) We have

$$\mathrm{cl}_{\mathcal{O}_\mathcal{F}} G_0 \subseteq \bigcap_{N\in\mathcal{F}} \mathrm{cl}_{\mathcal{O}_\mathcal{F}}(G_0N) \subseteq \bigcap_{(U,N)\in\mathcal{O}\times\mathcal{F}} \mathrm{cl}_{\mathcal{O}_\mathcal{F}}(G_0UN).$$

Since in a topological group the closure of a subset A is computed as $\bigcap_{U\in\mathcal{U}} AU$, the last set is equal to $\mathrm{cl}_{\mathcal{O}_\mathcal{F}} G_0$ and the assertion follows.

(ii) Clearly, since $\mathcal{O}_\mathcal{F} \subseteq \mathcal{O}$ the right hand side is contained in the left hand side. Conversely, let $x \in \bigcap_{N\in\mathcal{F}} \mathrm{cl}_{\mathcal{O}_\mathcal{F}}(G_0N)$. Then for each $(U,N) \in \mathcal{U}\times\mathcal{F}$ there are elements $g_{(U,N)} \in G_0$, $u_{(U,N)} \in U$ and $n_{(U,N)} \in N$, such that $x = g_{(U,N)}u_{(U,N)}n_{(U,N)}$. Now

$$g'_{(U,N)} \overset{\mathrm{def}}{=} u^{-1}_{(U,N)}g_{(U,N)}u_{(U,N)} \in G_0 \quad \text{and} \quad \lim_{(U,N)\in\mathcal{U}\times\mathcal{F}} u_{(U,N)} = 1 \quad \text{w.r.t. } \mathcal{O}.$$

Hence

$$\lim_{(U,N)\in\mathcal{U}\times\mathcal{F}} g'_{(U,N)}n_{(U,N)} = \lim_{(U,N)\in\mathcal{U}\times\mathcal{F}} u^{-1}_{(U,N)}x = x \quad \text{w.r.t. } \mathcal{O}.$$

It follows that $x \in \mathrm{cl}_\mathcal{O}(G_0N)$ for all $N \in \mathcal{F}$ and the assertion follows.

(iii) If $f: G \to L$ is a quotient morphism of a locally compact group onto a Lie group, then $f(G_0) = L_0$ by Lemma 1.4(b)(v). If $N = \ker f$, this means $G_0N = f^{-1}(L_0)$. Then G_0N is open, since L_0 is open.

(iv) is a consequence of (i) and (ii). □

Lemma 1.6. *Let $\mathcal{F}$ denote a filter basis of closed sets in a topological group G and let C be a compact subspace. Then $C\bigcap\mathcal{F} = \bigcap_{F\in\mathcal{F}} CF$.*

Proof. Clearly, the left hand side is contained in the right hand side. Now let $g \in \bigcap_{F\in\mathcal{F}} CF$. Then for each $F \in \mathcal{F}$ there are elements $c_F \in C$ and $f_F \in F$ such that $g = c_F f_F$. Since C is compact there is a subnet $(c_{F_j})_{j\in J}$ of $(c_F)_{F\in\mathcal{F}}$ with a limit $c \in C$. Now $f \overset{\mathrm{def}}{=} \lim_{j\in J} f_{F_j} = \lim_{j\in J} c^{-1}_{F_j} g = c^{-1}g$ exists. Also, $f_{F_j} \in F$ as soon as j is so large that $F_j \subseteq F$. Hence $f \in F$ for all F and thus $f \in \bigcap\mathcal{F}$. □

Lemma 1.7. *Let $\mathcal{F}$ denote a filter basis of closed subgroups in a topological group G such that $\bigcap\mathcal{F} = \{1\}$, and let H be a closed subgroup. Assume that there is an $N \in \mathcal{F}$ such that $H \cap N$ is compact. Then $H = \bigcap_{F\in\mathcal{F}} HF$.*

Proof. We set $C = H \cap N$ and apply Lemma 1.6 to the filter basis $\{M \in \mathcal{F}\colon M \subseteq N\}$. Thus we obtain

$$C = \bigcap_{N\supseteq M\in\mathcal{F}} CM. \tag{1.1}$$

We set $H_1 = \bigcap_{N\supseteq M\in\mathcal{F}} HM$. Clearly $H \subseteq \bigcap_{F\in\mathcal{F}} HF \subseteq H_1$. Let

$$C_1 \overset{\mathrm{def}}{=} H_1 \cap N = \bigcap_{N\supseteq M\in\mathcal{F}} (HM \cap N). \tag{1.2}$$

The modular law for groups asserts that $HM \cap N = CM$. Thus (1.1) implies

$$\bigcap_{N\supseteq M\in\mathcal{F}} (HM \cap N) = \bigcap_{N\supseteq M\in\mathcal{F}} CM = C. \tag{1.3}$$

From (1.2) and (1.3) we conclude $C_1 = C$. Hence $H_1 \cap N = H \cap N$. By the modular law again, we have $H(H_1 \cap N) = H_1 \cap HN = H_1$. Thus we obtain $H_1 = H(H_1 \cap N) = H(H \cap N) = H$, and the assertion follows. □

2. Filter bases of closed normal subgroups and projective limits

Let $\mathcal{F}$ be a filterbasis of closed normal subgroups of a topological group G. Then for $N \subseteq M$ in $\mathcal{F}$ there is a natural homomorphism $f_{MN}: G/N \to G/M$ given by $f(gN) = gM$. Since $G/M \cong (G/N)/(M/N)$, this morphism is a quotient morphism. The f_{MN} form an inverse system whose projective limit we denote by

$$G_{\mathcal{F}} = \lim_{N \in \mathcal{F}} G/N.$$

Recall that

$$G_{\mathcal{F}} = \left\{ (g_N N)_{N \in \mathcal{F}} \in \prod_{N \in \mathcal{F}} G/N \colon (\forall M, N \in \mathcal{F}) \quad (N \subseteq M) \Rightarrow g_N M = g_M M \right\}.$$

A comprehensive theory of projective limits of topological groups is given in [3, Chapter III, Section 7]. There is a natural morphism $\gamma: G \to G_{\mathcal{F}}$, $\gamma(g) = (gN)_{N \in \mathcal{F}}$. Also, for each $N \in \mathcal{F}$ the projection induces the *limit map* $f_N: G_{\mathcal{F}} \to G/N$, $f_N((g_M M)_{M \in \mathcal{F}}) = g_N N$. Let $\widetilde{N} = \ker f_N$, and let $\widetilde{\mathcal{F}} = \{\widetilde{N} \colon N \in \mathcal{F}\}$.

Lemma 2.1. *With the notation introduced in the preceding paragraph we have:*

(i) *Every identity neighborhood of $G_{\mathcal{F}}$ contains one of the form $f_N^{-1}(UN/N)$ with U an identity neighborhood in G.*

(ii) $\ker \gamma = \bigcap \mathcal{F}$, *and if $q_G: G \to G/\bigcap \mathcal{F}$ denotes the quotient morphism and $\gamma': G/\bigcap \mathcal{F} \to G_{\mathcal{F}}$, $\gamma'(g \bigcap \mathcal{F}) = \gamma(g)$, then γ' is injective and $\gamma = \gamma' \circ q_G$.*

(iii) $\overline{\operatorname{im} \gamma} = G_{\mathcal{F}}$.

(iv) *$f_N: G_{\mathcal{F}} \to G/N$ is open and $G_{\mathcal{F}}/\widetilde{N} \cong G/N$.*

(v) *γ induces an isomorphism $(G, \mathcal{O}_{\mathcal{F}})/\ker \gamma \to \gamma(G)$ of topological groups.*

Proof. (i) Let $\widetilde{U}$ be an identity neighborhood of $G_{\mathcal{F}}$. Then there is a basic open identity neighborhood V in $\prod_{N \in \mathcal{F}} G/N$ such that $V \cap G_{\mathcal{F}} \subseteq \widetilde{U}$. As V is basic there is a finite set $\{N_1, \ldots, N_k\} \subseteq \mathcal{F}$ such that we have open identity neighborhoods $U_1, \ldots, U_k$ in G being $N_1, \ldots, N_k$-saturated, respectively, such that $V = \prod_{N \in \mathcal{F}} V_N$ with

$$V_N = \begin{cases} U_j/N_j, & \text{for } N = N_j,\ j = 1, \ldots, k, \\ G/N, & \text{otherwise.} \end{cases}$$

Since $\mathcal{F}$ is a filterbasis, there is an $M \in \mathcal{F}$ with $M \subseteq \bigcap_{j=1}^{k} N_j$. Now $U_M = \bigcap_{j=1}^{k} U_j$ is an open M-saturated identity neighborhood such that $U_M N_j/N_j \subseteq U_j/N_j$ for $j = 1, \ldots, k$. Let $W = \prod_{N\in\mathcal{F}} W_N$ such that

$$W_N = \begin{cases} U_M/M, & \text{for } N = M, \\ G/N, & \text{otherwise.} \end{cases}$$

We put $\widetilde{W} \stackrel{\text{def}}{=} W \cap G_{\mathcal{F}} \subseteq V \cap G_{\mathcal{F}} \subseteq \widetilde{U}$. Now $\widetilde{W} = f_M^{-1}(U_M M/M)$, and this proves the claim.

(ii) is clear.

(iii) Let $\widetilde{g} = (g_N N)_{N\in\mathcal{F}} \in G_{\mathcal{F}}$ and let $\widetilde{U}$ be an identity neighborhood in $G_{\mathcal{F}}$. By (i) we find an identity neighborhood U of G and an $M \in \mathcal{F}$ such that $f_M^{-1}(UM/M) \subseteq \widetilde{U}$. Now $\gamma(g_M) = (g_M N)_{N\in\mathcal{F}}$ and

$$f_M\big(\gamma(g_M)^{-1}\widetilde{g}\big) = f_M\big((g_M^{-1} g_N N)_{N\in\mathcal{F}}\big) = g_M^{-1} g_M M = M,$$

the identity of G/M. Thus $\gamma(g_M) \in \widetilde{g}\widetilde{M} \subseteq \widetilde{g}\widetilde{U}$.

(iv) Since every identity neighborhood of $G_{\mathcal{F}}$ contains one of the form $f_N^{-1}(UN/N)$ with $U \in \mathcal{U}$ it follows that f_N is open. Since f_N is surjective and has the kernel $\widetilde{N}$, the remainder follows.

(v) Let $U \in \mathcal{U}$ and $N \in \mathcal{F}$. Then $\gamma^{-1} f_N^{-1}(UN/N) = UN$. Since the $f_N^{-1}(UN/N) = UN$ form a basis of the filter of identity neighborhoods of $G_{\mathcal{F}}$ by (i), the map γ induces a continuous and open surjective morphism $(G, \mathcal{O}_{\mathcal{F}}) \to \gamma(G)$, and this directly implies the assertion. □

Proposition 2.2. *Let $\mathcal{F}$ denote a filter basis of closed normal subgroups of a locally compact group G. Then the following statements are equivalent:*

(1) $G_{\mathcal{F}}$ is locally compact.

(2) There is an $N \in \mathcal{F}$ such that for all $M \in \mathcal{F}$ with $M \subseteq N$ the factor group N/M is compact.

If these conditions are satisfied, then there is a member $N \in \mathcal{F}$ such that $\widetilde{N} = \ker f_N$ is compact and satisfies $G_{\mathcal{F}}/\widetilde{N} \cong G/N$.

Proof. First we note that all G/N, $N \in \mathcal{F}$ are locally compact and hence are complete. Thus $P \stackrel{\text{def}}{=} \prod_{N\in\mathcal{F}} G/N$ is complete, and since $G_{\mathcal{F}} = \lim_{N\in\mathcal{F}} G/N$ is closed in P, we have $G_{\mathcal{F}}$ is complete.

Next we observe that we can assume that $\bigcap \mathcal{F} = \{1\}$; otherwise we replace G by $G/\bigcap \mathcal{F}$ and all $N \in \mathcal{F}$ by $N/\bigcap \mathcal{F}$.

Now assertions (iii) and (v) of Lemma 2.1 show that $G_{\mathcal{F}}$ is the completion of $(G, \mathcal{O}_{\mathcal{F}})$. Hence $G_{\mathcal{F}}$ is locally compact if and only if $(G, \mathcal{O}_{\mathcal{F}})$ is locally precompact. By Proposition 1.3 this is the case if and only if (2) holds.

Now assume that (1) and (2) are satisfied. Let $\widetilde{U}$ be a compact identity neighborhood of $G_{\mathcal{F}}$. Then by Lemma 2.1(i), $\widetilde{U}$ contains an identity neighborhood of the form

$f_N^{-1}(UN/N)$ and thus certainly contains $\widetilde{N} = \ker f_N$. Hence $\widetilde{N}$ is compact. Finally $G_{\mathcal{F}}/\widetilde{N} \cong G/N$ by Lemma 2.1(iv). □

The preceding proposition was proved in [12, Theorem 4.1].

Example 0.3(b) shows how Condition (2) of Proposition 2.2 may be violated. There we find $G_{\mathcal{F}} = \lim_{N\in\mathcal{F}} G/N \cong \mathbb{Z}^{\mathbb{N}} \rtimes \{-1, 1\}^{\mathbb{N}}$ in the product topology. This group is completely metrizable and separable but not locally compact. (The underlying space is homeomorphic to the space of irrational numbers in the topology induced from that of $\mathbb{R}$, see [4, Section 6, Example 7, p. 143].)

Proposition 2.3. *Let $\mathcal{F}$ denote a filter basis of closed normal subgroups of a topological group G. Consider the following statements:*

(1) *$\gamma : G \to G_{\mathcal{F}}$ is an isomorphism of topological groups.*

(2) $\lim \mathcal{F} = 1$.

Then (1) *implies* (2) *and if G is a complete topological group, then they are equivalent.*

Proof. (1) implies (2): Let U be an identity neighborhood of G. Then $\gamma(U)$ is an identity neighborhood of $G_{\mathcal{F}}$ by (1). By Lemma 2.1(i) it contains an identity neighborhood of the form $f_M^{-1}(VM/M)$. Thus $M = \gamma^{-1}(f_M^{-1}(1)) \subseteq \gamma^{-1}(f_M^{-1}(VM/M)) \subseteq U$. Since U was an arbitrary neighborhood of 1 in G, we have shown that $\mathcal{F}$ converges to 1.

Assume that G is a complete topological group. We show then that (2) implies (1). Since G is Hausdorff, $\bigcap \mathcal{F} \subseteq \bigcap \mathcal{U}_G = \{1\}$ in view of (2). Hence γ is injective by Lemma 2.1(ii). Now let $(g_N N)_{N\in\mathcal{F}} \in G_{\mathcal{F}}$. Assume that $U \in \mathcal{U}_G$. Then by (2) there is a $K \in \mathcal{F}$ such that $K \subseteq U$. Then N, $M \subseteq K$ implies $g_M g_K^{-1} \in K$ and $g_N g_K^{-1} \in K$ and thus $g_M g_N^{-1} \in K \subseteq U$. Hence $(g_N)_{N\in\mathcal{F}}$ is a Cauchy net on G. Since G is complete, $g = \lim_{N\in\mathcal{F}} g_N$ exists. If $N, M \in \mathcal{F}$, then $M \subseteq N$ implies $g_M \in g_N N$, and since N is closed, it follows that $g = \lim_{M\in\mathcal{F}} g_M \in g_N N$. Therefore

$$\gamma(g) = (gN)_{N\in\mathcal{F}} = (g_N N)_{N\in\mathcal{F}},$$

which shows that γ is surjective. Thus γ is bijective. From Lemma 2.1(v) and 1.1 as well as from hypothesis (2) we conclude that γ is a homeomorphism. □

Lemma 2.4. *Let $\mathcal{F}$ be a filterbasis of closed subsets of a Hausdorff topological space and assume that $\mathcal{F}$ contains an element which is a compact set. Then the following statements are equivalent:*

(1) $\lim \mathcal{F} = x$.

(2) $\bigcap \mathcal{F} = \{x\}$.

Proof. (1) implies (2): Since X is Hausdorff, we have $\bigcap \mathcal{F} \subseteq \bigcap \mathcal{U}_X(x) = \{x\}$.

(2) implies (1): Let $N \in \mathcal{F}$ be compact and assume $\bigcap \mathcal{F} = \{x\}$. Let U be any open neighborhood of x. Now $\{M \in \mathcal{F}\colon\ M \subseteq N\}$ is a filter basis of compact sets intersecting in $\{x\}$. There is at least one of these M contained in U, for if not, then $\{M \setminus U\colon\ N \supseteq M \in \mathcal{F}\}$ is a filterbasis of compact sets which, accordingly, has a point

g in its intersection. On the one hand, $g \in \bigcap \mathcal{F} = \{x\}$, on the other $g \in G \setminus U$, a contradiction. □

Lemma 2.5. *Let $\mathcal{F}$ denote a filter basis of closed normal* connected *subgroups of a locally compact topological group G such that $\bigcap \mathcal{F} = \{1\}$. Then* $\lim \mathcal{F} = 1$.

Proof. Let H be an almost connected open subgroup of G; such groups always exist [14, Corollary 4.5]. Since H is open, we have $H \cap N = N$ for every $N \in \mathcal{F}$; recall that the elements of $\mathcal{F}$ are assumed to be connected. It is therefore no loss of generality to assume that G is almost connected. Let U be an identity neighborhood of G. Then there is a compact normal subgroup K of G contained in U such that G/K is a Lie group [14, Theorem 4.6]. For $M \in \mathcal{F}$ we obtain a closed normal subgroup MK of G and thus a closed normal subgroup MK/K of the Lie group G/K. By Lemma 1.7 we have $\bigcap_{M \in \mathcal{F}} MK = (\bigcap \mathcal{F})K = K$. Thus, $\bigcap_{M \in \mathcal{F}} MK/K = \{1\}$ in G/K. The closed connected subgroups of a connected Lie group satisfy the descending chain condition and all MK/K are connected since all $M \in \mathcal{F}$ are connected. Hence, eventually, $MK/K = K/K$ and thus $M \subseteq K$, eventually. By Lemma 2.4, $\lim \mathcal{F} = 1$ follows. □

Proposition 2.6. *Let $\mathcal{F}$ denote a filter basis of closed normal subgroups of a locally compact topological group G. Assume that $\bigcap \mathcal{F} = \{1\}$. Then the following conditions are equivalent:*

(1) $\lim \mathcal{F} = 1$.

(2) *$\mathcal{F}$ contains a compact member.*

(3) *$\mathcal{F}$ contains a member which is contained in an almost connected open subgroup, and eventually all $M \in \mathcal{F}$ are almost connected.*

Proof. (1) implies (2): Since G is locally compact we find a compact identity neighborhood U in G. By (1) there is an $N \in \mathcal{F}$ such that $N \subseteq U$. Then N is compact.

(2) implies (1): Lemma 2.4 proves this assertion.

(2) implies (3) is trivial.

(3) implies (2): Define $\mathcal{G} = \{M_0 \colon M \in \mathcal{F}\}$. Since M_0 is characteristic in M and $(M \cap N)_0 \subseteq M_0 \cap N_0$, the set $\mathcal{G}$ is a filterbasis of closed normal connected subgroups of G. Clearly $\bigcap \mathcal{G} = \{1\}$. Now Lemma 2.5 and (3) imply $\lim \mathcal{G} = 1$. In particular, there is an $M \in \mathcal{F}$ such that M_0 is compact. But M/M_0 is compact eventually by (3). Hence M is compact, eventually. □

With a filter basis of closed normal subgroups one can canonically associate other filter bases of closed normal subgroups.

Definition 2.7. Assume that $\mathcal{F}$ is a filter basis of closed normal subgroups of a topological group G. We let $\mathcal{F}_0$ denote the filter basis of all identity components N_0 of members $N \in \mathcal{F}$, and $\mathcal{F}|G_0$ the filter basis of all $G_0 \cap N$, $N \in \mathcal{F}$.

Clearly, for each $N \in \mathcal{F}$ we have $N_0 \subseteq G_0 \cap N$. Thus $\langle \mathcal{F}|G_0 \rangle \subseteq \langle \mathcal{F}_0 \rangle$.

We first discuss the filter basis $\mathcal{F}_0$.

Proposition 2.8. *Let G be a locally compact group and $\mathcal{F}$ a filter basis of closed normal subgroups such that $\bigcap \mathcal{F} = \{1\}$. Then the following conclusions hold:*

(i) *The canonical map*

$$\gamma_0 : G \to G_{\mathcal{F}_0} \stackrel{\text{def}}{=} \lim_{N \in \mathcal{F}} G/N_0$$

is an isomorphism.

(ii) *For every $N \in \mathcal{F}$ the quotient morphism $q_N : G/N_0 \to G/N$, $q_N(gN_0) = gN$ has a totally disconnected kernel $\Delta_N \stackrel{\text{def}}{=} N/N_0$. The morphism*

$$\prod_{N \in \mathcal{F}} q_N : \prod_{N \in \mathcal{F}} G/N_0 \to \prod_{N \in \mathcal{F}} G/N$$

induces a quotient morphism $q : G_{\mathcal{F}_0} \to G_{\mathcal{F}}$ such that

$$\begin{array}{ccc} G & \xrightarrow{\text{id}_G} & G \\ {\scriptstyle \gamma_0}\downarrow & & \downarrow{\scriptstyle \gamma} \\ G_{\mathcal{F}_0} & \xrightarrow[q]{} & G_{\mathcal{F}} \end{array}$$

commutes.

Proof. (i) By Lemma 2.5, $\lim \mathcal{F}_0 = 1$. Then the assertion follows from Proposition 2.3.

(ii) Since $G/N \cong (G/N_0)/(N/N_0)$ the map q_N is indeed a quotient morphism, clearly Δ_N is totally disconnected. Because G and therefore all G/N_0 and G/N are locally compact, $\prod_{N \in \mathcal{F}} q_N$ also is a quotient morphism. The remainder is straightforward. □

Secondly we discuss the filter basis $\mathcal{F}|G_0$. We begin with a few lemmas.

Lemma 2.9. *Let G be a Lie group with finitely many components and D a discrete normal subgroup. Then there is a unique natural number $r(D)$ and a finite subgroup $F(D)$ of D which is central in G_0 such that $A(D) \stackrel{\text{def}}{=} G_0 \cap D$ is isomorphic to $\mathbb{Z}^{r(D)} \times F(D)$ and $D/A(D)$ is finite. The group $F(D)$ is characteristic in $A(D)$.*

Proof. There is a bijection from $D/G_0 \cap D$ onto $DG_0/G_0 \leqslant G/G_0$. Thus $D/G_0 \cap D$ is finite. Now $A(D) = G_0 \cap D$ is a discrete normal subgroup of G_0 and thus is central in G_0. Thus $A(D)$ is contained in a connected abelian closed Lie subgroup (see [9, Chapter XVI, Section 1, Theorem 1.2, p. 189]). We conclude that $A(D)$ is finitely generated. If $r(D)$ is the torsion free rank of $A(D)$, then $A(D) \cong \mathbb{Z}^{r(D)} \times F(D)$ where $F(D)$ denotes the characteristic maximal finite subgroup of $A(D)$. □

Lemma 2.10. *Under the assumptions and with the notation of 2.9 assume that $\mathcal{F}$ is a filter base of normal subgroups of G contained in D such that $\bigcap \mathcal{F} = \{1\}$. Then there is an $N \in \mathcal{F}$ such that*

(i) *$G_0 \cap N \cong \mathbb{Z}^{r(N)}$, and that*

(ii) *all $M \in \mathcal{F}$ contained in N have finite index in N.*

Proof. The set of natural numbers

$$r(M) \in \{0, \dots, r(D)\}, \quad M \in \mathcal{F},$$

has a minimum $r(N_1)$. Now let $N \in \mathcal{F}$ with $N \subseteq N_1$. Then $r(N) = r(N_1)$, $A(N) = G_0 \cap N \subseteq G_0 \cap N_1 = A(N_1)$ and $F(N) \subseteq F(N_1)$. Since $F(N_1)$ is finite and

$$\bigcap_{N_1 \supseteq N \in \mathcal{F}} F(N) \subseteq \bigcap \mathcal{F} = \{1\}$$

there is an $N \subseteq N_1$ such that $F(N) = \{1\}$. This implies that $A(N) \cong \mathbb{Z}^{r(N)}$. Now $N \supseteq M \in \mathcal{F}$ implies that $A(N)/A(M)$ is finite. Since

$$N/A(N) \cong \big(N/A(M)\big)/\big(A(N)/A(M)\big)$$

is finite we obtain that $N/A(M)$ must be finite. But then

$$N/M \cong \big(N/A(M)\big)/\big(M/A(M)\big)$$

is finite, too. □

Lemma 2.11. *Let G be a locally compact almost connected group and $\mathcal{F}$ a filter basis of closed subgroups such that $\bigcap \mathcal{F} = \{1\}$. Then there is an $N \in \mathcal{F}$ such that N/M is compact for all $M \in \mathcal{F}$ contained in N.*

Proof. The group G is a pro-Lie group [14, Theorem 4.6]. Let K be a compact normal subgroup of G such that G/K is a Lie group. Then there is an open identity neighborhood U of G such that any subgroup of G contained in U is contained in K. By Lemma 2.5 we find $P \in \mathcal{F}$ such that $P_0 \subseteq K$. Then $D = PK/K$ is a discrete normal subgroup of the Lie group G/K which has finitely many components. By Lemma 2.10 there is an $N \in \mathcal{F}$ contained in P such that for all $M \in \mathcal{F}$ such that $M \subseteq N$ we have that $NK/MK \cong (NK/K)/(MK/K)$ is finite. Since K is compact then MK/M is compact. Hence NK/M is compact. Therefore the subgroup N/M is compact. The claim is proved. □

This lemma applies, in particular, to the identity component of a locally compact group and yields the following result:

Proposition 2.12. *Let G be a locally compact group and let $\mathcal{F}$ be a filter basis of closed subgroups such that $\bigcap \mathcal{F} = \{1\}$. Then the group*

$$G_{\mathcal{F}|G_0} = \lim_{N \in \mathcal{F}} G/(G_0 \cap N)$$

is locally compact.

Proof. We apply the preceding Lemma 2.11 with G_0 in place of G and with $\mathcal{F}|G_0$ in place of $\mathcal{F}$, and then conclude the assertion from Proposition 2.2. □

One-parameter subgroups

A *one-parameter subgroup* of a topological group G is a morphism of topological groups $X:\mathbb{R} \to G$. We denote the set of one-parameter subgroups of G by $\mathfrak{L}(G)$. Clearly $\mathfrak{L}$ is the object portion of a functor from the category of topological groups and continuous group morphisms to the category of sets. On the full subcategory of Lie groups, there is much additional structure attached to $\mathfrak{L}$. From [12] we recall the following definition:

Definition 2.13. Assume that $f:G \to H$ is a morphism of topological groups. We say that f is an *equidimensional immersion* if f is injective and for any one-parameter subgroup $X:\mathbb{R} \to H$ of H there is a one-parameter subgroup $X':\mathbb{R} \to G$ of G such that $X = f \circ X'$. Equivalently, f is injective and induces a bijection $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$.

Lemma 2.14. *Let $f:G \to H$ be a morphism of locally compact groups. Then we have the following conclusions:*

(i) *If f is a quotient morphism then every one-parameter subgroup of H lifts to one in G; that is, $\mathfrak{L}(f):\mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective.*

(ii) $\ker f$ *is totally disconnected if and only if $\mathfrak{L}(f)$ is injective.*

Proof. (i) was shown in [12, Lemma 1.3].

(ii) Assume that $\ker f$ is totally disconnected and let $X, Y:\mathbb{R} \to G$ be two one-parameter subgroups such that

$$f \circ X = \mathfrak{L}(f)(X) = \mathfrak{L}(f)(Y) = f \circ Y.$$

Equivalently, for all $r \in \mathbb{R}$ we have $f(X(r)Y(r)^{-1}) = 1$. Then we have the continuous function $r \mapsto X(r)Y(r)^{-1}:\mathbb{R} \to \ker f$. Since $\mathbb{R}$ is connected and $\ker f$ is totally disconnected, this function is constant and for $r = 0$ takes the value 1. Hence $X = Y$.

Conversely, assume that $\mathfrak{L}(f)$ is injective. Now $\mathfrak{L}(f)(\mathfrak{L}(\ker f)) = \{0_H\}$ where $0_H:\mathbb{R} \to H$ is the constant one-parameter subgroup. Since $\mathfrak{L}(f)$ is injective, $\mathfrak{L}(\ker f) = \{0_G\}$ follows. But $\ker f$ is a locally compact group. Since every connected locally compact group possesses nontrivial one-parameter subgroups we conclude that $\mathfrak{L}(\ker f) = \{0_G\}$ is equivalent to $(\ker f)_0 = \{1\}$. The assertion follows. □

Theorem 2.15. *Let G be a locally compact group and let $\mathcal{F}$ be a filter basis of closed normal subgroups. Then $\gamma: G/\bigcap\mathcal{F} \to G_{\mathcal{F}}$ is an equidimensional immersion.*

Proof. By Lemma 2.1(ii) γ is injective, and we may assume that $\bigcap\mathcal{F} = \{1\}$. The kernel N/N_0 of the quotient morphism $q_N:G/N_0 \to G/N$ is totally disconnected . Thus by Lemma 2.14, $\mathfrak{L}(q_N):\mathfrak{L}(G/N_0) \to \mathfrak{L}(G/N)$ is bijective. It is readily observed that the functor $\mathfrak{L}$ preserves products. Hence $\mathfrak{L}(\prod_{N\in\mathcal{F}} q_N)$ defined by

$$\begin{array}{ccc}
\mathfrak{L}\left(\prod_{N\in\mathcal{F}} G/N_0\right) & \xrightarrow{\mathfrak{L}\left(\prod_{N\in\mathcal{F}} q_N\right)} & \mathfrak{L}\left(\prod_{N\in\mathcal{F}} G/N\right) \\
\cong\Big\downarrow & & \Big\uparrow\cong \\
\prod_{N\in\mathcal{F}} \mathfrak{L}(G/N_0) & \xrightarrow[\prod_{N\in\mathcal{F}} \mathfrak{L}(q_N)]{} & \prod_{N\in\mathcal{F}} \mathfrak{L}(G/N)
\end{array}$$

is bijective. Now let $X:\mathbb{R}\to G_{\mathcal{F}}$ be a one-parameter group of $G_{\mathcal{F}}\subseteq\prod_{N\in\mathcal{F}} G/N$. Then there is a unique one-parameter group $X':\mathbb{R}\to\prod_{N\in\mathcal{F}} G/N_0$ such that $\prod_{N\in\mathcal{F}} q_N\circ X' = X$. We are finished if we can show that $X'(r)\in\lim_{N\in\mathcal{F}} G/N_0$ for all $r\in\mathbb{R}$. However, we know that $X'(r) = (\alpha_N(r)N_0)_{N\in\mathcal{F}}$ for a familiy of functions $\alpha_N:\mathbb{R}\to G$. Likewise $X(r) = (\beta_N(r)N)_{N\in\mathcal{F}}$ with

(a) $\alpha_N(r)\in\beta_N(r)N$, and

(b) $(\forall M,\ N\in\mathcal{F},\ M\subseteq N)\quad \beta_M(r)\in\beta_N(r)N$.

We must show that

(c) $(\forall M,\ N\in\mathcal{F},\ M\subseteq N)\quad \alpha_M(r)\in\alpha_N(r)N_0$.

Thus let $M,\ N\in\mathcal{F}$ and $M\subseteq N$. Then, since $r\mapsto\alpha_M(r)M_0:\mathbb{R}\to G/M_0$ and $r\mapsto\alpha_N(r)N_0:\mathbb{R}\to G/N_0$ are continuous, the function

$$d_{M,N}: r\mapsto\alpha_N(r)^{-1}\alpha_M(r)N_0,\ \mathbb{R}\to G/N_0 \quad \text{is continuous.} \tag{2.1}$$

By (a) we have $\alpha_M(r)M = \beta_M(r)M$, whence $\alpha_M(r)N = \beta_M(r)N$ and $\alpha_N(r)N = \beta_N(r)N$. Thus

$$\alpha_N(r)^{-1}\alpha_M(r)\in\beta_N(r)^{-1}\beta_M(r)N. \tag{2.2}$$

From (b) we know $\beta_N(r)^{-1}\beta_M(r)\in N$, whence (2.2) yields

$$(\forall r\in\mathbb{R})\quad \alpha_N(r)^{-1}\alpha_M(r)\in N. \tag{2.3}$$

It follows that

$$(\forall r\in\mathbb{R})\quad \alpha_N(r)^{-1}\alpha_M(r)N_0\in N/N_0. \tag{2.4}$$

Since N/N_0 is totally disconnected and $\mathbb{R}$ is connected, (2.4) implies that the continuous function $d_{M,N}$ in (2.1) is constant. But $\alpha_M(0)\in M_0\subseteq N_0$ and $\alpha_N(0)\in N_0$. Thus $d_{M,N}(0)$ is the identity of N/N_0. This means

$$(\forall r\in\mathbb{R})\quad \alpha_M(r)N_0 = \alpha_N(r)N_0,$$

and this is (c), which we had to show. □

3. Canonical filter bases of normal subgroups

Definition 3.1. (i) For any topological group G we let $\mathcal{N}(G)$ denote the set of all closed normal subgroups such that G/N is a Lie group. Note that $G\in\mathcal{N}(G)$.

(ii) The set of all kernels of a morphism $G\to L$ into some Lie group will be denoted by $\mathcal{L}(G)$. Obviously, $\mathcal{N}(G)\subseteq\mathcal{L}(G)$.

(iii) The set of all open normal subgroups is denoted by $\mathcal{ON}(G)$. Note that $G \in \mathcal{ON}(G)$ and that $\mathcal{ON}(G) \subseteq \mathcal{N}(G)$.

Proposition 3.2. (i) *If $f: G \to L$ is a morphism of a locally compact group G into a Lie group L, then* $\ker f \in \mathcal{N}(G)$. *In particular, for a locally compact group G we have* $\mathcal{L}(G) = \mathcal{N}(G)$.

(ii) *If $f: G \to H$ is a morphism of topological groups onto a Lie group, then* $f^{-1}(H_0) \in \mathcal{ON}(G)$.

If G is locally compact, then $\mathcal{ON}(G) = \{G_0N\colon\ N \in \mathcal{N}(G)\}$.

(iii) $\mathcal{L}(G)$ *and* $\mathcal{ON}(G)$ *are filter bases.*

Proof. (i) Let V denote an open identity neighborhood of L not containing nontrivial subgroups. If N is any subgroup containing $\ker f$ and contained in $f^{-1}(V)$, then $N = \ker f$. Thus $G/\ker f$ is a locally compact group which has no small subgroups and thus is a Lie group. Hence $\ker f \in \mathcal{N}(G)$ by definition.

(ii) If H is a Lie group, then H_0 is open. Hence the continuity of f implies that $f^{-1}(H_0)$ is open. Since f is surjective, $f^{-1}(H_0)$ is normal. If G is locally compact and $N \in \mathcal{N}(G)$, then G_0N is open by Proposition 1.5(iii). Hence $G_0N \in \mathcal{ON}(G)$. Conversely, if $M \in \mathcal{ON}(G)$ then G/M is discrete and thus $G_0 \subseteq M$, whence $M = G_0M$.

(iii) Ad $\mathcal{L}(G)$: Let $M, N \in \mathcal{L}(G)$. Then there exist continuous morphisms $\varphi_M: G \to L_M$ and $\varphi_N: G \to L_N$ with kernel M and N, respectively, into Lie groups L_M and L_N. Now $L_M \times L_N$ is a Lie group, and the morphism $\varphi: G \to L_M \times L_N$, $g \mapsto (\varphi_M(g), \varphi_N(g))$ has kernel $M \cap N$ which is therefore in $\mathcal{L}(G)$.

The assertion on $\mathcal{ON}(G)$ is trivial. □

We note at once that G is a Lie group if and only if $\{1\} \in \mathcal{N}(G)$. The following implications are clear (if G is Hausdorff):

$$\{1\} \in \mathcal{N}(G) \Rightarrow \lim \mathcal{N}(G) = 1 \Rightarrow \bigcap \mathcal{N}(G) = \{1\}.$$

Remark 3.3. For a locally compact group G, the following statements are equivalent:

(1) G is a pro-Lie group.

(2) $\mathcal{N}(G)$ contains a compact set and $\bigcap \mathcal{N}(G) = \{1\}$.

(3) $\lim \mathcal{N}(G) = 1$.

Proof. In view of Lemma 2.4 and Proposition 2.6, the proof is an easy exercise. □

Recall from Definition 0.1 that a topological group G is a *residual Lie group* if the continuous group homomorphisms from G into Lie groups separate the points.

Proposition 3.4. *Assume that G is locally compact and a residual Lie group. Then we have the following conclusions:*

(i) $\bigcap \mathcal{ON}(G) = \mathrm{cl}_{\mathcal{O}_{\mathcal{N}(G)}} G_0$.

(ii) *If there is an* $N \in \mathcal{N}(G)$ *such that the intersection* $G_0 \cap N$ *is compact, then* $\bigcap \mathcal{ON}(G) = G_0$.

Proof. (i) By Proposition 3.2(ii), $\mathcal{ON}(G) = \{G_0N\colon N \in \mathcal{N}(G)\}$. The assertion now follows from 1.5(iv).

(ii) By Lemma 1.7 applied with $H = G_0$ and $\mathcal{F} = \mathcal{N}(G)$ we have $G_0 = \bigcap_{N\in\mathcal{N}(G)} G_0N$. By Proposition 3.2(ii) this is the assertion. □

Proposition 3.5. *Let G be a topological group.*

(a) *Assume that G is a subgroup of some product of Lie groups.*
Then $\lim \mathcal{L}(G) = 1$.

(b) *Assume that G is locally compact and consider the following conditions:*

(A) *G is a residual Lie group.*

(B) $\bigcap \mathcal{N}(G) = \{1\}$.

(C) *G is a pro-Lie group.*

(D) *G is topologically isomorphic to a closed subgroup of some product of Lie groups.*

(1) *G is an* SIN-*group.*

(2) *G is maximally almost periodic.*

Then (A) $\Leftrightarrow$ (B) *and* (C) $\Leftrightarrow$ (D), *and* (1) $\Rightarrow$ (C) $\Rightarrow$ (A) $\Leftarrow$ (2). *However,* (A) $\not\Rightarrow$ (C), (A) $\not\Rightarrow$ (2), (C) $\not\Rightarrow$ (2), (2) $\not\Rightarrow$ (C) *and* (C) $\not\Rightarrow$ (1).

Proof. (a) Assume that G is a subgroup of a product $\prod_{j\in J} L_j$ of Lie groups L_j. We claim that $\lim \mathcal{L}(G) = 1$. For a proof consider an identity neighborhood U of G. Then there is a finite subset $F \subseteq J$ and identity neighborhoods U_j of L_j for $j \in F$ such that, with $U_j = L_j$ for $j \in J \setminus F$, we have

$$G \cap \prod_{j\in J} U_j \subseteq U.$$

Let us write $p_F : \prod_{j\in J} L_j \to \prod_{j\in F} L_j$ for the projection onto the partial product which is a Lie group L_F. Set $N_F = G \cap \ker p_F$. Then N_F is a closed normal subgroup of G contained in $\prod_{j\in J} U_j$ and hence in U. Also, the restriction $p_F|G : G \to L_F$ has the kernel N_F which, by Definition 3.1(ii), is contained in $\mathcal{L}(G)$. Since $N_F \subseteq U$ and U was an arbitrary identity neighborhood of G, the claim is proved.

Proof of (b): (B) implies (A) is trivial.

(A) implies (B): Let $1 \neq h \in G$. Then there is a morphism $f : G \to L$ into a Lie group L with $f(h) \neq 1$. Now $N = \ker f \in \mathcal{N}(G)$ by 3.2(i).

(C) implies (D): Every pro-Lie group is a closed subgroup of a product of Lie groups.

(D) implies (C): By (a) above, $\lim \mathcal{L}(G) = 1$. Since G is locally compact, by Proposition 3.2(i) we know $\mathcal{N}(G) = \mathcal{L}(G)$. Remark 3.3 now shows that G is a pro-Lie group.

(2) implies (A): The homomorphisms of a compact group into Lie groups separate points.

(1) implies (C): See [7, 2.11(1)], see also [2,11].

(D) implies (A): The projections of a product separate the points. Hence the morphisms from G into Lie groups separate points.

Since not every residual Lie group is a pro-Lie group as Example 0.3(a) illustrates, (A) does not imply (C). There exist Lie groups that are not maximally almost periodic, for example $\mathrm{Sl}(2,\mathbb{R})$. Thus (A) $\not\Rightarrow$ (2) and (C) $\not\Rightarrow$ (2). Locally compact residual Lie groups that are not pro-Lie groups were constructed in Example 0.3, showing that (A) $\not\Rightarrow$ (C). Lie-groups that are not SIN-groups abound; again, $\mathrm{Sl}(2,\mathbb{R})$ is an example ready at hand. Thus (C) $\not\Rightarrow$ (1). □

Part (b) of Proposition 3.5 shows, among other things that the additive group of a metrizable topological vector space cannot be a subgroup of a product of Lie groups, because the unit ball around 0 cannot contain nontrivial additive subgroups.

The ideas of the proof of (a) can be exploited to yield the following observation (see [11, Lemma 5.2]):

If a Lie group G is a subgroup of a product of topological groups, then it has an isomorphic copy in a finite partial product.

Proposition 3.6. *For a locally compact group G the following statements are equivalent:*

(1) *G/G_0 is a residual Lie group.*

(2) $\bigcap \mathcal{ON}(G) = G_0$.

Proof. (1) implies (2): By Proposition 3.5 we have $\bigcap \mathcal{N}(G/G_0) = \{1\}$. Thus for $g \notin G_0$ there is a closed normal subgroup N containing G_0 but not containing g such that $G/N \cong (G/G_0)/(N/G_0)$ is a totally disconnected Lie group and thus is discrete. Hence N is open and thus $N \in \mathcal{ON}(G)$.

(2) implies (1): Let $g \notin G_0$. Then by (2) there is an open normal subgroup N such that $g \notin N$. The group G/N is discrete, hence is a Lie group. Thus there is a homomorphism from G/G_0 into a Lie group which does not contain gG_0 in its kernel. □

4. Canonical constructs for a residual Lie group

Recall that a group is called a residual Lie group if the morphisms into Lie groups separate its points. In Definition 3.1(ii) we defined the filter basis $\mathcal{N}(G)$.

Theorem 4.1. *Assume that G is a topological group. Then the following statements are equivalent:*

(1) *$G_{\mathcal{N}(G)}$ is locally compact.*

(2) *There is an $N \in \mathcal{N}(G)$ such that for all $M \in \mathcal{N}(G)$ with $M \subseteq N$ the factor group N/M is compact.*

If these conditions are satisfied, then $G_{\mathcal{N}(G)}$ is a pro-Lie group.

Proof. We apply Proposition 2.2 with $\mathcal{F} = \mathcal{N}(G)$ and conclude the equivalence of (1) and (2) as well as the existence, under these conditions, of an $M \in \mathcal{N}(G)$ such that $\widetilde{M} = \ker f_M$ is compact. By Lemma 2.1(i), for every identity neighborhood W of $G_{\mathcal{N}(G)}$ we find an $N \in \mathcal{N}(G)$ such that $N \subseteq M$ and $\widetilde{N} \subseteq W$. Clearly, $\widetilde{N} \subseteq \widetilde{M}$ is compact. Also, $G_{\mathcal{N}(G)}/\widetilde{N} \cong G/N$ by Lemma 2.1(iv). Hence $G_{\mathcal{N}(G)}/\widetilde{N}$ is a Lie group by the definition of $\mathcal{N}(G)$ in Definition 3.1. Hence $G_{\mathcal{N}(G)}$ is a pro-Lie group by Definition 0.1(ii). □

Theorem 4.2. *Let G be a locally compact residual Lie group and assume that it contains an almost connected open* normal *subgroup. Then*

$$G_{\mathcal{N}(G)} = \lim_{N \in \mathcal{N}(G)} G/N$$

is a locally compact pro-Lie group, and $\gamma : G \to G_{\mathcal{N}(G)}$ is an equidimensional immersion. It is an isomorphism of topological groups if and only if G is a pro-Lie group.

Proof. Let H denote an almost connected open normal subgroup of G which exists by hypothesis. Note that $H \in \mathcal{N}(G)$. Now we apply Lemma 2.11 to H and the filter basis $\mathcal{H} = \{N \in \mathcal{N}(G)\colon N \subseteq H\}$. We find an $N \in \mathcal{H}$ such that N/M is compact for all $M \in \mathcal{N}(G)$ with $M \subseteq N$. By Theorem 4.1 the group $G_{\mathcal{N}(G)}$ is a locally compact pro-Lie group. Theorem 2.15 shows that γ is an equidimensional immersion. Now G is complete as a locally compact group. Hence by Proposition 2.3, the immersion γ is an isomorphism if and only if $\lim \mathcal{N}(G) = 1$. By Remark 3.3 this is the case if and only if G is a pro-Lie group. □

We have observed that G is a Lie group if and only if $\{1\} \in \mathcal{N}(G)$, in which case $\gamma : G \to G_{\mathcal{N}(G)}$ is trivially an isomorphism. The set $\mathcal{N}(G)^\times \stackrel{\text{def}}{=} \mathcal{N}(G) \setminus \{\{1\}\}$ may or may not be a filterbase. If $G = \mathbb{R}$, it is not; cf. the remarks at the end of [12, p. 260], for the consequences. If $G = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$, the universal covering group of $\mathrm{Sl}(2,\mathbb{R})$, it is. If $\mathcal{N}(G)^\times$ is a filterbase, then even for a Lie group G, the group $G_{\mathcal{N}(G)^\times}$ is a locally compact non-Lie group; in that case $\gamma : G \to G_{\mathcal{N}(G)^\times}$ is an equidimensional immersion into a finite dimensional group, see [12, Theorem 4.1]. Compare also [17, Section 8].

Example 0.3(b) shows that there are even totally disconnected locally compact residual Lie groups which do not have an almost connected open normal subgroup (which, in the present case, would simply be a compact open normal subgroup). Also, some of the examples in Example 0.3 (namely, those with infinite D) show that there are totally disconnected locally compact separable metric groups for which $G_{\mathcal{N}(G)}$ is not locally compact.

The filter basis $\mathcal{N}(G)_0$

We apply Propositions 2.8 and 2.2 to the filter basis $\mathcal{F} = \mathcal{N}(G)$ and recall that $\mathcal{N}(G)_0$ is the filterbasis of all identity components N_0 of normal subgroups $N \in \mathcal{N}(G)$. Then $\dim G/N_0 = \dim G/N$ and there is a dense equidimensional immersion of a Lie group L which is locally isomorphic to G/N into G/N_0; compare [12, Theorem 4.1].

Theorem 4.3. *Assume that G is a locally compact group with the property that $\bigcap \mathcal{N}(G)_0 = \{1\}$. (This is the case if G is a residual Lie group.) Then* $\lim \mathcal{N}(G)_0 = 1$ *and $\gamma_0 : G \to G_{\mathcal{N}(G)_0}$ is an isomorphism of topological groups; that is, G is a projective limit of finite dimensional locally compact groups.*

Proof. This is a direct consequence of Lemma 2.5 and Proposition 2.3. □

The filter basis $\mathcal{N}(G)|G_0$

Next we consider on G the filter basis

$$\mathcal{N}(G)|G_0 = \{G_0 \cap N\colon\ N \in \mathcal{N}(G)\}.$$

Obviously, if G is a residual Lie group, we have $\bigcap \mathcal{N}(G)|G_0 = \{1\}$. Now Proposition 2.12 and Theorem 2.15 show:

Lemma 4.4. *Let G be a locally compact residual Lie group. Then*

$$G_{\mathcal{N}(G)|G_0} = \lim_{N\in\mathcal{N}(G)} G/(G_0 \cap N)$$

is locally compact and $\gamma' : G \to G_{\mathcal{N}(G)|G_0}$ is an equidimensional immersion.

For simplicity of the notation we abbreviate $\mathcal{N}(G)|G_0$ by $\mathcal{F}$ in the following lemma and its proof. We denote by

$$f_N : G_{\mathcal{F}} \to G/(G_0 \cap N)$$

the natural morphism and set $\widetilde{N} = \ker f_N$ and $N^* = f_N^{-1}(N/G_0 \cap N)$. Then $G/(G_0 \cap N) \cong G_{\mathcal{F}}/\widetilde{N}$, cf. Lemma 2.1(iv).

Lemma 4.5. *Under the hypotheses of Lemma* 4.4 *we have*

(i) $\gamma'^{-1}(\widetilde{N}) = G_0 \cap N$.

(ii) $\gamma'^{-1}(N^*) = N$.

(iii) *The assignment $N \mapsto N^*$ gives a surjection from $\mathcal{N}(G)$ onto $\mathcal{N}(G_{\mathcal{F}})$.*

(iv) $(G_{\mathcal{F}})_0 = \lim_{N\in\mathcal{N}(G)} G_0/(G_0 \cap N)$.

(v) *The map $\gamma^* : G/G_0 \to G_{\mathcal{F}}/(G_{\mathcal{F}})_0$, $\gamma^*(gG_0) = \gamma'(g)(G_{\mathcal{F}})_0$ is an isomorphism of topological groups.*

Proof. (i) Let $g \in G$ be such that $(g(G_0 \cap M))_{M\in\mathcal{N}(G)} = \gamma(g) \in \widetilde{N} = f_N^{-1}(\{G_0 \cap N\})$. This means that $g(G_0 \cap N) = G_0 \cap N$; that is, $g \in G_0 \cap N$.

(ii) is similar.

(iii) Let $N \in \mathcal{N}(G)$, then $N/(G_0 \cap N) \cong N^*/\widetilde{N}$. Then we have isomorphisms $G/N \cong (G/(G_0 \cap N))/(N/(G_0 \cap N)) \cong (G_{\mathcal{F}}/\widetilde{N})/(N^*/\widetilde{N}) \cong G_{\mathcal{F}}/N^*$, whence $N^* \in \mathcal{N}(G_{\mathcal{F}})$. Let $M \in \mathcal{N}(G_{\mathcal{F}})$. Then $\gamma'^{-1}(M)$ is the kernel of the morphism $g \mapsto \gamma'(g)M : G \to G_{\mathcal{F}}/M$. Hence $N \overset{\text{def}}{=} \gamma'^{-1}(M) \in \mathcal{N}(G)$ by Proposition 3.2(i), and (ii) yields $N^* = \gamma'(N) = M$.

(iv) Let $(g_N(G_0 \cap N))_{N \in \mathcal{N}(G)}$ be an element of $(G_{\mathcal{F}})_0$. This implies that $g_N(G_0 \cap N) \in f_N((G_{\mathcal{F}})_0) \subseteq (G/(G_0 \cap N))_0$ for every $N \in \mathcal{N}(G)$. According to Lemma 1.4(b)(ii), we have $(G/(G_0 \cap N))_0 = G_0/(G_0 \cap N)$. Thus $g_N \in G_0$ for every $N \in \mathcal{N}(G)$, and we have proved that $(G_{\mathcal{F}})_0 = \lim_{N \in \mathcal{N}(G)} G_0/(G_0 \cap N)$.

(v) For $N \in \mathcal{F}$ the function $f_N : G_{\mathcal{F}} \to G/(G_0 \cap N)$ is a quotient morphism by Lemma 2.1(iv), and $(G/(G_0 \cap N))_0 = G_0/(G_0 \cap N)$ as was observed in the proof of (iv) above. Since G/N is a Lie group by definition of $\mathcal{F}$, then $G_0/(G_0 \cap N)$, being injected into G/N by $g(G_0 \cap N) \mapsto gN$, is a Lie group by Proposition 3.2(i). Thus by Lemma 1.4(b)(v) the map f_N maps $(G_{\mathcal{F}})_0$ onto $G_0/(G_0 \cap N)$. We abbreviate $G_{\mathcal{F}}/(G_{\mathcal{F}})_0$ by Q. Now f_N induces a quotient morphism

$$F_N : Q \to \big(G/(G_0 \cap N)\big)/\big(G_0/(G_0 \cap N)\big) \to G/G_0$$

defined for $\lambda = (g_M(G_0 \cap M))_{M \in \mathcal{F}} \in G_{\mathcal{F}}$ by $F_N(\lambda(G_{\mathcal{F}})_0) = g_N G_0$. If $M \subseteq N$ in $\mathcal{F}$, then $g_M^{-1} g_N \in G_0 \cap N \subseteq G_0$. Thus $F_M(\lambda(G_{\mathcal{F}})_0) = F_N(\lambda(G_{\mathcal{F}})_0)$. Hence, we have an unambiguously defined morphism $F : Q \to G/G_0$, $F = F_N$ for any $N \in \mathcal{F}$. We notice $(F \circ \gamma^*)(gG_0) = F(\gamma'(g)(G_{\mathcal{F}})_0) = F((g(G_0 \cap M))_{M \in \mathcal{F}}(G_{\mathcal{F}})_0) = gG_0$. Further, $F(\lambda(G_{\mathcal{F}})_0)) = 1$ means $g_N G_0 = F_N(\lambda) = 1$ for all $N \in \mathcal{F}$ and this means $g_N \in G_0$, i.e., $\lambda = (g_M(G_0 \cap M))_{M \in \mathcal{F}} \in \lim_{N \in \mathcal{F}} G_0/(G_0 \cap N) = (G_{\mathcal{F}})_0$. Therefore F is injective and thus is the inverse of γ^*. □

Theorem 4.6. *Let G be a locally compact residual Lie group and set $\Gamma \stackrel{\text{def}}{=} G_{\mathcal{N}(G)|G_0} = \lim_{N \in \mathcal{N}(G)} G/(G_0 \cap N)$. Then*

(i) *$\gamma' : G \to \Gamma$ is an equidimensional dense immersion into a locally compact residual Lie group.*

(ii) *$\Gamma_0 \cap N$ is compact for all sufficiently small $N \in \mathcal{N}(\Gamma)$.*

(iii) *$\Gamma_0 = \bigcap \mathcal{ON}(\Gamma)$; that is, Γ/Γ_0 is a residual Lie group.*

(iv) *$\gamma' : G \to \Gamma$ induces an isomorphism of topological groups $\gamma^* : G/G_0 \to \Gamma/\Gamma_0$, $\gamma^*(gG_0) = \gamma'(g)\Gamma_0$.*

(v) *G/G_0 is a residual Lie group.*

Proof. (i) follows from Lemma 4.4.

(ii) Recall from Lemma 4.5(iii) that $\mathcal{N}(\Gamma) = \{N^* : N \in \mathcal{G}\}$. We have $\Gamma_0 \cap N^* = f_N^{-1}(G_0 \cap N)$ and this group is $\lim_{N \supseteq M \in \mathcal{N}(G)}(G_0 \cap N)/(G_0 \cap M)$. By Lemma 2.11, there is an $N \in \mathcal{N}(G)$ such that $(G_0 \cap N)/(G_0 \cap M)$ is compact for all $N \supseteq M \in \mathcal{N}(G)$. Thus the limit is compact. Hence $\Gamma_0 \cap N^*$ is compact.

(iii) From (ii) and from Corollary 3.4(ii) we know that $\bigcap \mathcal{ON}(\Gamma) = \Gamma_0$.

(iv) follows from Lemma 4.5(v), and (v) is an immediate consequence of (iii) and (iv). □

Definition 4.7. A topological group G is called *residually discrete* if the continuous group homomorphisms into discrete groups separate the points of G.

Obviously, a residually discrete group is a residual Lie group.

Proposition 4.8. *For a locally compact group the following conditions are equivalent:*
(1) *G is a totally disconnected residual Lie group.*
(2) *G is residually discrete.*

Proof. (2) implies (1): Assume that G is residually discrete. For every continuous morphism $f:G \to D$ of D into a discrete group we have $f(G_0) \subseteq D_0 = \{1\}$. Since these f separate the points we have $G_0 = \{1\}$. Thus G is totally disconnected. As noted before, G is trivially a residual Lie group.

(1) implies (2): Let $g \neq 1$ in G. There is a morphism $f:G \to L$ into a Lie group L such that $f(g) \neq 1$. By Proposition 3.2(i) the quotient $G/\ker f$ is a Lie group, and we may therefore assume that f is itself a quotient morphism. Then L is a quotient of a locally compact totally disconnected group. By Lemma 1.4(b)(vi), the quotient L is therefore totally disconnected, and, as a Lie group, is discrete. The assertion follows. □

As a consequence we obtain:

Corollary 4.9. *If G is a locally compact residual Lie group then G/G_0 is a locally compact residually discrete group and G_0 is a pro-Lie group.*

Proof. As a locally compact connected group, G_0 is a pro-Lie group [14]. By Theorem 4.6(iv), G/G_0 is a residual Lie group. By Proposition 4.8, therefore, G/G_0 is residually discrete. □

In many respects, this result reduces questions on locally compact residual Lie groups to the totally disconnected case. Examples 0.2 (with L finite), 0.3, and 0.4 illustrate what may happen in this context. Example 0.2 yields with $L = \mathbb{R}/\mathbb{Z}$ a locally compact group for which $G/G_0 \cong \mathbb{Z}$ is discrete and which is not a residual Lie group while other properties of G are quite decent (G is two step solvable, G_0 is an $\aleph_0$-dimensional torus). Thus there are locally compact groups G such that G/G_0 is discrete, hence certainly residually discrete, and G_0 is a pro-Lie group, but G is not a residual Lie group and thus cannot be injected into a residual Lie group by a continuous homomorphism.

5. Varieties generated by Lie groups

A class of topological groups closed under the operations of forming subgroups, Hausdorff quotient groups and arbitrary products is called a *variety of Hausdorff groups*.

A quite natural variety of Hausdorff groups is the class of all Hausdorff SIN-groups. The classes that we have studied up to this point in this paper, however, do not form varieties of Hausdorff groups. Let Ω_{pro} denote the class of pro-Lie groups, Ω_{res} the class of all residual Lie groups, and Ω_{MAP} the class of all maximally almost periodic groups. Then we have:

Proposition 5.1. *None of the classes Ω_{pro}, Ω_{res} and Ω_{MAP} forms a variety of Hausdorff groups.*

Proof. The additive group of rational numbers (with its usual topology) belongs to $\mathfrak{V}(\Omega_{\text{pro}})$, but is not a pro-Lie group. Thus Ω_{pro} is not a variety of Hausdorff groups.

For every completely regular topological space X there exists the free topological group $F(X)$ over X, see [8, Theorem 8.8], cf. also [5, Section 4]. It is known (cf. [8, proof of Theorem 8.8]) that the morphisms of topological groups from $F(X)$ to (compact) unitary groups separate the points of $F(X)$. Thus $F(X)$ is a residual Lie group, and an MAP-group, as well. However, every Hausdorff group G is completely regular. Since G is a quotient of $F(G)$, cf. [8, Theorem 8.8(iii)], we obtain that $\mathfrak{V}(\Omega_{\text{res}})$ and $\mathfrak{V}(\Omega_{\text{MAP}})$ both equal the class of all Hausdorff groups. Since there are Hausdorff groups that are not residually Lie groups (and thus not maximally almost periodic, see Proposition 3.5), the assertion follows. □

Definition 5.2. Let Λ denote the class of (finite dimensional) real Lie groups and $\mathfrak{V}(\Lambda)$ denote the variety of Hausdorff groups generated by the class Λ.

Although a nontrivial variety of Hausdorff groups is necessarily rather big, $\mathfrak{V}(\Lambda)$ does not include Ω_{res}; for example, no infinite dimensional Banach space belongs to $\mathfrak{V}(\Lambda)$, see [15].

We restate Proposition 3.5 including the use of $\mathfrak{V}(\Lambda)$. Recall the definition of $\mathcal{L}(G)$ from Definition 3.1(ii).

Proposition 5.3. *Let G be a topological group.*

(a) *If $G \in \mathfrak{V}(\Lambda)$, then* $\lim \mathcal{L}(G) = 1$.

(b) *Assume that G is a locally compact group and consider the following conditions:*

(A) *G is a residual Lie group.*

(B) $\bigcap \mathcal{N}(G) = \{1\}$.

(C) *G is a pro-Lie group.*

(D) *G is a member of the variety of Hausdorff groups $\mathfrak{V}(\Lambda)$.*

(1) *G is an* SIN*-group.*

(2) *G is maximally almost periodic.*

Then (A) $\Leftrightarrow$ (B) *and* (C) $\Leftrightarrow$ (D), *and* (1) $\Rightarrow$ (C) $\Rightarrow$ (A) $\Leftarrow$ (2). *However,* (A) $\not\Rightarrow$ (C), (A) $\not\Rightarrow$ (2), (C) $\not\Rightarrow$ (2), (2) $\not\Rightarrow$ (C) *and* (C) $\not\Rightarrow$ (1).

Proof. (a) Since the class of Lie groups is closed under finite products, closed subgroups, and Hausdorff quotients, it follows from Theorem 7 in [16] that the members of $\mathfrak{V}(\Lambda)$ are exactly those groups that are isomorphic to some closed subgroup of a product of Lie groups. Assertion (a) now follows from Proposition 3.5(a).

Proof of (b): We have seen in the proof of (a) above that for locally compact groups, conditions 3.5.b(D) and 5.3.b(D) are equivalent. Then the statements of Proposition 5.3(b) are exactly those of Proposition 3.5(b). □

By the very definition of a variety of Hausdorff groups, if G is in $\mathfrak{V}(\Lambda)$ then G/G_0 is in $\mathfrak{V}(\Lambda)$. The class of residual Lie groups is larger than $\mathfrak{V}(\Lambda)$. It is not immediately clear that if G is a residual Lie group then G/G_0 must be a residual Lie group. However, Corollary 4.9 establishes this fact.

If G is a pro-Lie group, then G/G_0 is a pro-Lie group as well, because in this case the filter basis $\{N/G_0\colon N \in \mathcal{ON}(G)\}$ has a compact member; cf. Propositions 2.6 and 2.3. Conversely, it is shown in [1] that a locally compact group G is a pro-Lie group if and only if G/G_0 is a pro-Lie group and G is an L-group, that is, a group satisfying condition (L) of Proposition 0.8.

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