Math. Proc. Camb. Phil. Soc. (1995), **118**, 303
Printed in Great Britain

Locally compact groups with closed subgroups open and p-adic

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(*Received* 14 *March* 1994; *revised* 2 *August* 1994)

1. *Introduction*

An open subgroup U of a topological group G is always closed, since U is the complement of the open set $\bigcup\{Ug \mid g \notin U\}$. An arbitrary closed subgroup C of G is almost never open, unless G belongs to a small family of exceptional groups. In fact, if G is a locally compact abelian group in which every non-trivial subgroup is open, then G is the additive group Δ_p of p-adic integers or the additive group Ω_p of p-adic rationals (cf. Robertson and Schreiber[**5**], proposition 7). The fact that Δ_p has interesting properties as a topological group has many roots. One is that its character group is the Prüfer group $\mathbb{Z}_{p^\infty}$, which makes it unique inside the category of compact abelian groups. But even within the bigger class of not necessarily abelian compact groups the p-adic group Δ_p is distinguished: it is the only one all of whose non-trivial subgroups are isomorphic (cf. Morris and Oates-Williams[**2**]), and it is also the only one all of whose non-trivial closed subgroups have finite index (cf. Morris, Oates-Williams and Thompson[**3**]).

If one abandons not only the requirement of commutativity, but also that of compactness – staying, however, within the realm of local compactness – then the situation deteriorates dramatically. We shall show here that some rather monstrous locally compact groups exist. In fact, for any sufficiently large prime p we shall produce a locally compact torsion-free group $\mathscr{M}$ with dense algebraic commutator group such that each non-trivial proper closed subgroup of $\mathscr{M}$ is open and isomorphic to Δ_p. There will be a centre $\mathscr{Z}$ which is one of these subgroups, containing all proper closed normal subgroups.

We shall obtain the group $\mathscr{M}$ as a highly non-trivial central extension of a p-adic group Δ_p by the Ivanov–Ol'shanskiĭ monster, a simple group with p conjugacy classes in which all proper non-trivial subgroups have order p (see [**1**]). This will illustrate how badly the methods of Morris, Oates-Williams and Thompson in characterizing Δ_p through topological group properties must fail in the absence of compactness, because their focus was on the factor group of G modulo the centre $Z(G)$. This factor group turned out to be finite with all proper subgroups cyclic. For sufficiently large exponent, the Ivanov–Ol'shanskiĭ group will be an unsurmountable obstruction, and our construction shows that this will not be any different in the structure theory of locally compact groups.

2. *Construction of certain central extensions*

Let G be a topological group and Z a topological abelian group. We make the following definitions.

Definition 2·1. (1) A group $\tilde{G}$ is a *central extension* of G by Z if $\tilde{G}$ is a topological group containing a central subgroup $\tilde{Z}$ isomorphic (as a topological group) to Z such that $\tilde{G}/\tilde{Z} \cong G$. This implies, in particular, that there is an exact sequence of topological groups

$$1 \to Z \xrightarrow{j} \tilde{G} \xrightarrow{q} G \to 1,$$

and if we add the information that j is an embedding and q a quotient morphism this is an equivalent formulation of the definition.

(2) $\tilde{G}$ is a *centrally compact extension* of G by Z if it is a central extension with a compact abelian group Z.

[Note that our definition of central extension differs from that used by some people.]

Central extensions are easy to come by. The group $\tilde{G} = Z \times G$ will always be a central extension of G by Z, but this is the one from which one expects the least information. Therefore it is called *trivial.*

There is an easy way of constructing centrally compact extensions once simple central extensions are given. The new ones will not be trivial if the given ones are not.

LEMMA 2·2. *Suppose that Γ is a topological group with a closed central subgroup Z. Let $\alpha: Z \to C$ be a morphism of Z into an additively written compact abelian group. Let $\Delta = \{(-\alpha(z), z) \in C \times \Gamma : z \in Z\}$ and set*

$$\tilde{G} = \frac{C \times \Gamma}{\Delta}, \quad \tilde{C} = \frac{C \times Z}{\Delta} = \frac{(C \times \{1\})\Delta}{\Delta}, \quad \Gamma^* = \frac{\alpha(Z) \times \Gamma}{\Delta}.$$

Then

(i) *$\tilde{G}$ is a centrally compact extension of $G \overset{\text{def}}{=} \Gamma/Z$ by C.*
(ii) *$\sigma: \Gamma \to \tilde{G}$, $\sigma(g) = (0, g)\Delta$ is a morphism with image Γ^* and kernel* $\ker \alpha$.
(iii) *$\tilde{G} = \tilde{C}\Gamma^*$ and $\tilde{C} \cap \Gamma^* \cong$* $\operatorname{im} \alpha$.
(iv) *If the image of α is dense in C, then Γ^* is dense in $\tilde{G}$.*

Proof. (i) The subgroup Δ is essentially the graph of the morphism $-\alpha\colon Z\to C$ in the closed subgroup $C\times Z$ of $C\times\Gamma$ and is therefore closed in $C\times\Gamma$, and it is clearly central. Hence $\tilde{G}=C\times\Gamma/\Delta$ is a Hausdorff topological group.

We note $(c,z)\in(C\times\{1\}\cap\Delta$ if and only if $z=1$ and $c=-\alpha(z)$ if and only if $(c,z)=(0,1)$. Hence the map $c\mapsto(c,1)\colon C\to\tilde{C}$ is an isomorphism of compact groups. Also,

$$\tilde{G}/\tilde{C}\cong\frac{C\times\Gamma}{C\times Z}\cong\Gamma/Z\cong G.$$

(ii) The map σ is a well-defined morphism with image

$$\Gamma^*=\frac{(\{0\}\times\Gamma)\,\Delta}{\Delta}=\frac{\alpha(Z)\times\Gamma}{\Delta}.$$

We note $1=\sigma(g)=(0,g)\,\Delta$ if and only if $(0,g)\in\Delta$ if and only if $(0,g)=(-\alpha(z),z)$ for some $z\in Z$ and this is the case if and only if $g\in\ker\alpha$.

(iii) It is clear that $\tilde{G}=\tilde{C}\Gamma^*$. The intersection $\tilde{C}\cap\Gamma^*$ is $(C\times Z)\cap(\alpha(Z)\times\Gamma)=\alpha(Z)\times Z$ modulo Δ. From the proof of (i) we recall $(\alpha(Z)\times\{1\})\cap\Delta=\{(0,1)\}$ and note

$$\frac{\alpha(Z)\times Z}{\Delta}=\frac{(\alpha(Z)\times\{1\})\,\Delta}{\Delta}\cong\frac{\alpha(Z)\times\{1\}}{((\alpha(Z)\cap\{1\})\cap\Delta)}\cong\alpha(Z).$$

Thus $\tilde{C}\cap\Gamma^*\cong\operatorname{im}\alpha$ as asserted.

(iv) From the definition of Γ^* we know

$$\Gamma^*=\frac{\alpha(Z)\times\Gamma}{\Delta}.\quad\text{If}\quad C=\overline{\alpha(Z)},\quad\text{then}\quad\tilde{G}=\frac{C\times\Gamma}{\Delta}=\bar{\Gamma}^*.\quad\blacksquare$$

It is useful to have some notation.

Notation 2·3. The group $\tilde{G}$ constructed from Γ, α and C as in Lemma 2·2 will be denoted by $\Gamma[\alpha]$ and q_α will denote the morphism from $\Gamma[\alpha]$ to G which takes $(c,g)\,\Delta$ to gZ.

Note that q_α is a quotient morphism with kernel $\tilde{C}$.

Remark 2·4. Suppose that H is a closed subgroup of $\Gamma[\alpha]$. Then

(i) $q_\alpha(H)$ is closed in G,
(ii) $H\cap\tilde{C}$ is a closed central subgroup of $\tilde{C}\cong C$ such that $q_\alpha(H)\cong H/H\cap\tilde{C}$.

In other words, H is a centrally compact extension of $q_\alpha(H)\leqslant G$ by $H\cap\tilde{C}$.

Proof. (i) The subgroup $\tilde{C}$ of $\Gamma[\alpha]$ is central and compact. Hence $H\tilde{C}$ is a closed subgroup containing H. But $H\tilde{C}=q_\alpha^{-1}q_\alpha(H)$. Since q_α is a quotient morphism, $q_\alpha(H)$ is closed in G.

(ii) The injective continuous morphism $h(H\cap\tilde{C})\mapsto h\tilde{C}\colon H/H\cap\tilde{C}\to H\tilde{C}/\tilde{C}$ is an isomorphism because $H\cap\tilde{C}$ is compact. Since $q_\alpha(H)\cong H\tilde{C}/\tilde{C}$ the assertion follows. $\blacksquare$

The most interesting special case which we shall consider in the sequel is that for which α is injective and the image of α is dense in C. In other words, C is a compactification of Z.

3. *An algebraic construction of a central extension*

In this section we are dealing with (discrete) groups only. Suppose that a group G is given and that we have a free resolution

$$1 \to N \xrightarrow{j} F(X) \xrightarrow{q} G \to 1$$

with N normal in $F(X)$. By Schreier's Theorem, N is free. Note that $N = \{1\}$ if and only if q is an isomorphism. The group $[N, F(X)]$ is normal in $F(X)$ and contained in N. If $n \in N$ and $k \in F(X)$, then $knk^{-1} \in n[N, F(X)]$, that is, the inner automorphisms of $F(X)$ induce the identity on $N/[N, F(X)]$. As a consequence, if $[N, F(X)] \leqslant K \leqslant N$, then K is normal in $F(X)$.

Now we make the following hypothesis:

Hypothesis $(*)$. There is a free resolution of G such that $N/[N, F(X)]$ has a quotient isomorphic to $\mathbb{Z}$.

This is the case in particular if $N/[N, F(X)]$ is free abelian.

If $(*)$ is satisfied, then there is a morphism $f: N \to \mathbb{Z}$, such that f is constant on the conjugacy classes under inner automorphisms of $F(X)$.

Now we set $\tilde{G} = F(X)/K$ and $\tilde{Z} = N/K$ where $K = \ker f$. Then

$$G \cong \frac{F(X)}{N} \cong \frac{F(X)/K}{N/K} = \frac{\tilde{G}}{\tilde{Z}}.$$

We summarize:

PROPOSITION 3·1. *If*

$$1 \to F(Y) \xrightarrow{j} F(X) \xrightarrow{q} G \to 1$$

is a free resolution of a group G satisfying Hypothesis $(*)$, then there is a central extension

$$1 \to \mathbb{Z} \to \tilde{G} \to G \to 1$$

and maps $f: F(Y) \to \mathbb{Z}$, $\phi: F(X) \to \tilde{G}$ such that there is a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & K \arrow[r, "\mathrm{id}"] \arrow[d] & K \arrow[d] & & \\
1 \arrow[r] & F(Y) \arrow[r, "j"] \arrow[d, "f"'] & F(X) \arrow[r, "q"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \tilde{G} \arrow[r, "\tilde{q}"] \arrow[d] & G \arrow[r] & 1 \\
 & 1 & 1. & &
\end{tikzcd}$$

We record some information about the commutator group $\tilde{G}'$.

LEMMA 3·2. *Under the hypotheses of* 2·1:

(i) $$\tilde{G}/\tilde{G}' \cong \frac{F(X)}{F'(X)K} \quad \text{and} \quad \tilde{G}/(\tilde{G}'\tilde{Z}) \cong G/G'.$$

(ii) *If G is perfect, that is, equals its own commutator group G', then*

$$\frac{\tilde{G}}{\tilde{G}' \cap \tilde{Z}} = \frac{\tilde{G}'}{\tilde{G}' \cap \tilde{Z}} \times \frac{\tilde{Z}}{\tilde{G}' \cap \tilde{Z}}.$$

Proof. (i) We note

$$\tilde{G}' = \frac{F'(X)K}{K}.$$

Hence

$$\tilde{G}/\tilde{G}' = \frac{F(X)/K}{F'(X)K/K} = F(X)/F'(X)K.$$

Also

$$\tilde{G}/(\tilde{G}'\tilde{Z}) \cong \frac{\tilde{G}/\tilde{Z}}{(\tilde{G}'\tilde{Z})/\tilde{Z}} \cong G/G'.$$

(ii) If G is perfect, then (i) implies that $\tilde{G} = \tilde{G}'\tilde{Z}$. Thus, by the Second Isomorphism Theorem,

$$\frac{\tilde{G}}{\tilde{G}' \cap \tilde{Z}} = \frac{\tilde{G}'}{\tilde{G} \cap \tilde{Z}} \times \frac{\tilde{Z}}{\tilde{G}' \cap \tilde{Z}}. \quad \blacksquare$$

Due to a construction of Ivanov, using techniques of Ol'shanskiĭ, there is a monstrous group M of exponent p whose properties are described in the following theorem:

THEOREM 3·3. *For any sufficiently large prime p, there is a group M with the following properties*:

(i) *Every proper non-trivial subgroup of M has order p.*
(ii) *There are p conjugacy classes.*

Proof. See [**1**]. $\blacksquare$

The following properties of the Ivanov–Ol'shanskiĭ monster are consequences.

LEMMA 3·4. *The group M also has the following properties*:

(iii) *Let $\Omega \cong M$ denote the group of inner automorphisms. Take $1 \neq g \in M$ and let $\Omega_g \cong N(g,M)$ (the centralizer of g in M) denote the stabilizer of Ω at g. Then for each $k = 1, \ldots, p-1$, the function $\omega\Omega \mapsto \omega(g^k): \Omega/\Omega_g \to \Omega(g^k)$, is a bijection onto each of the $p-1$ non-trivial conjugacy classes.*
(iv) *M is simple.*

Proof. (iii) The elements $g, g^2, \ldots, g^{p-1}$ are in different conjugacy classes, and $\Omega(g^k) = \{h^k : h \in \Omega(g)\}$. Two different cyclic subgroups of order p intersect in $\{1\}$. Hence for $k \in \{1, \ldots, p-1\}$ the function $x \mapsto x^k : M \to M$ is a bijection permutating the conjugacy classes and mapping one conjugacy class Ω-equivariantly onto its image. In particular, $\Omega_g = \Omega_{g^k}$. Assertion (iii) follows from these facts.

(iv) Suppose that $1 \neq g$ is an element of a normal subgroup N of M. Then $g^k \in N$ for $k = 1, \dots, p-1$. Since N is normal, $\Omega(g^k) \subseteq N$ for $k = 1, \dots, p-1$. Then (iii) implies $G \subseteq N$.

LEMMA 3·5. *The group M has a free resolution satisfying Hypothesis* (*).

Proof. From the proof of theorem 41·2 of [4] we see that the group M is defined by its presentation:

$$M = \langle a_1, a_2 \mid R = 1; R \in \mathscr{R} \rangle, \tag{1}$$

that is, as a quotient of the free group on two generators. To obtain an aspherical presentation of M we have to replace the set of relators $\mathscr{R}$ by a smaller set, $\mathscr{R} \setminus \mathscr{R}_1$, where $\mathscr{R}_1$ contains all relators of the first type of the form A_i^p, where $A_i \neq a_1$, and of the second type of rank $i \geqslant 2$ of the form $S_{i-1,i} A_i^m S_{i-1,i}^{-1} A_{i-1}^{-m}$, where $2 \leqslant m < p$. Every defining relation $R = 1$, where $R \in \mathscr{R}_1$ follows from the set of defining relations $\{R = 1; R \in \mathscr{R} \setminus \mathscr{R}_1\}$, so the group M can also be defined by the presentation

$$M = \langle a_1, a_2 \mid R = 1; R \in \mathscr{R} \setminus \mathscr{R}_1 \rangle. \tag{2}$$

Let $F(X) = F(\{a_1, a_2\})$. Consider the central extension $L = F(X)/[F(X), N]$. The proof of [4], theorem 41·2, shows that the presentation (2) of M is aspherical, and it follows from [4], theorem 31·1, that the group $\bar{N} = N/[F, N]$ is a free abelian group with basis $\{\bar{R} \mid R \in \mathscr{R} \setminus \mathscr{R}_1\}$, where $\bar{R} = R[F(X), N]$. Thus M has a free resolution satisfying Hypothesis (*).

Now we construct $\tilde{M}$ as a central extension of M by Z as in Proposition 3·1 and obtain $f: F(Y) \to \mathbb{Z}$ and $\phi: F(X) \to \tilde{M}$ as in Proposition 3·1.

PROPOSITION 3·6. *For any sufficiently large prime p there is a group $\tilde{M}$ with the following properties*:

(i) *The centre $Z = Z(\tilde{M})$ of M is isomorphic to $\mathbb{Z}$.*

(ii) $\tilde{M}/Z \cong M$.

(iii) *The group $\tilde{\Omega}$ of inner automorphisms of $\tilde{M}$ allows a homomorphism θ onto Ω such that the quotient map $\tilde{q}: \tilde{M} \to M$ is equivariant in the sense that $\tilde{q}(\psi(m)) = \theta(\psi)(\tilde{q}(m))$ for $\psi \in \tilde{\Omega}$, $m \in \tilde{M}$. An inner automorphism ψ of $\tilde{M}$ implemented by an element $m \in \tilde{M}$ is in the kernel of θ if and only if $[m, h] \in Z$ for all $h \in \tilde{M}$ (with $[m, h] = mhm^{-1} h^{-1}$).*

(iv) *If S is any non-trivial cyclic subgroup, then SZ is cyclic. Either S is contained in Z or S is a subgroup of SZ with an index which is relatively prime to p. In particular, $\tilde{M}$ is torsion free.*

Proof. (i) and (ii) are direct consequences of Proposition 3·1.

(iii) If $m \in \tilde{M}$, let $I_m(m') = mm'm^{-1}$. Let $\tilde{q}: \tilde{M} \to M$ denote the quotient morphism. Then $I_{\tilde{q}(m)}(\tilde{q}(m)) = \tilde{q}(I_m(m'))$. The map $I_m \mapsto I_{\tilde{q}(m)}: \tilde{\Omega} \to \Omega$ is a surjective morphism, and $I_{\tilde{q}(m)} = \mathrm{id}$ if and only if $mhm^{-1}h^{-1} \in Z$ for all $h \in \tilde{M}$.

(iv) By the construction in Lemma 3·5, we see that a_1^p is a free generator of the free abelian group $N/[F(X), N]$. We can thus find a normal subgroup K of $F(X)$ such that $[N, F(X)] \subseteq K \subseteq N$, $N/K \cong \mathbb{Z}$ and $a_1^p K$ is a generator of N/K. Set $g = q(a_1) \in M$ and $g_0 = \phi(a_1) \in \tilde{M}$. Let $m \in \tilde{M}$, $m \neq 1$. Then by Lemma 3·4(iii) we find an $\omega \in \Omega$ and a $k \in \{1, \dots, p-1\}$ such that $\tilde{q}(m) = \omega(g^k)$. By (iii) above there is a $\psi \in \tilde{\Omega}$ such that $\theta(\psi) = \omega$. Then $\tilde{q}(\psi(g_0^k)) = \omega(g^k) = \tilde{q}(m)$. Thus $m = \psi(g_0^k) z$ with a $z \in Z$. Now $a^p K$ is a generator of $\tilde{Z}$ by the choice of $a_1 \in F(X)$. Thus $g_0^p = a_1^p K$ is a generator c of $\tilde{Z}$. There is an integer r such that $z = c^r$ and we have $m^p = \psi(g_0^p)^k z^p = c^k c^{rp} = c^{k+rp}$ with a suitable integer r. Suppose $m^p = 1$. Then $k+pr = 0$. Thus $p \mid k$. Since $k \in \{1, \dots, p-1\}$ this is impossible. Hence $m^p \neq 1$ in the infinite cyclic group Z, and thus m is not a torsion element. Thus $\tilde{M}$ is torsion free. Let S be the cyclic group generated by m. Since $m \neq 1$ we have $S \nsubseteq Z$, that is, $Z \cap S \neq S$. Now ZS is a group containing Z in its centre such that ZS/Z is cyclic of order p. Thus ZS is abelian and generated by at most two generators. By the structure theorem of finitely generated abelian groups, either ZS is infinite cyclic or isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_l$ with $l \neq 0$. This is impossible since $\tilde{M}$ is torsion free. Thus ZS is cyclic.

The group $ZS/(Z \cap S)$ is finite cyclic. If its order is p, then $Z \cap S = Z$ and thus $Z \subseteq S$. If not, then $ZS/(Z \cap S)$ is a direct sum of two cyclic subgroups, one of which is isomorphic to $\mathbb{Z}(p)$. The other one is isomorphic to $Z_l \cong Z/(Z \cap S) \cong ZS/S$, and since the whole group is cyclic, $(l, p) = 1$ follows.

LEMMA 3·7. *Suppose that a group G contains a central subgroup A such that $G = G'A$. Then G' is perfect.*

Proof. Let $g' \in G'$. Then by the definition of the commutator group, there are elements $g_j, h_j \in G$, $j = 1, \dots, n$ such that $g' = [g_1, h_1] \cdots [g_n, h_n]$. Since $G = G'A$ there are elements $g_j', h_j' \in G'$ and $a_j, b_j \in A$ such that $g_j = g_j' a_j$ and $h_j = h_j' b_j$. Now $[g_j, h_j] = [g_j' a_j, h_j' b_j] = [g_j', h_j']$ since a_j and b_j are central. Thus $g' = [g_1', h_1'] \cdots [g_n', h_n'] \in G''$.

PROPOSITION 3·8. *For any sufficiently large prime p there is a group $\tilde{M}$ with the properties* (i)–(iv) *of Proposition* 3·6 *and the following properties*:

(v) *The group $\tilde{M}$ is perfect, that is $\tilde{M}' = \tilde{M}$.*

(vi) *Any proper normal subgroup of $\tilde{M}$ is contained in Z.*

(vii) *Any non-trivial proper subgroup S is infinite cyclic and either is contained in Z or $Z \subseteq S$ or SZ is cyclic with $(|SZ/S|, p) = 1$.*

Proof. We begin by denoting by M_1 a central extension of M by the subgroups $Z_1 \cong \mathbb{Z}$ having the properties (i)–(iv) of Proposition 3·6. Then $M_1' Z_1/Z_1$ is a normal subgroup of the simple group $M_1'/Z_1 \cong M$. Hence there are two cases: $M_1' \subseteq Z_1$ or $M_1' Z_1' = M_1$. The first case is impossible since $M_1/Z_1 \cong M$ is not abelian. As a subgroup of M_1, the group $\tilde{M} \overset{\text{def}}{=} M_1'$ is torsion free. By Lemma 3·7, the group $\tilde{M}$ is perfect. We set $Z = M_1' \cap Z_1$. Then $\tilde{M}/Z \cong M_1' Z_1/Z_1 = M_1/Z_1 \cong M$. This implies $Z \neq \{0\}$ since $\tilde{M}$ is torsion free. As Z is a non-zero subgroup of an infinite cyclic group, it is itself isomorphic to $\mathbb{Z}$. Since Z_1 is the centre of M_1, then $Z = Z_1 \cap M_1'$ is the centre of $\tilde{M} = M_1'$. Thus $\tilde{M}$ and $\tilde{Z}$ have the properties recorded in (i)–(iv) of Proposition 3·6, and (v) is satisfied.

(vi) Let H denote a normal subgroup of $\tilde{M}$. Then HZ/Z is a normal subgroup of $\tilde{M}/Z$. But $M \cong \tilde{M}/Z$ is simple. Thus either $H \subseteq Z$ or $HZ = \tilde{M}$. In the latter case $\tilde{M}/H = HZ/H \cong Z/(H \cap Z)$ is abelian. Hence H contains the commutator subgroup $\tilde{M}'$, which is $\tilde{M}$ by (v). Thus $H = \tilde{M}$.

(vii) Let H be a non-trivial proper subgroup of $\tilde{M}$. By (ii) HZ/Z is trivial or cyclic of order p or equals $\tilde{M}/Z$. In the first case $H \subseteq Z$ and H is infinite cyclic.

In the second case, HZ is an abelian group of at most two generators in a torsion free group, hence is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$. The second case does not allow a finite cyclic quotient by factoring a cyclic group. Hence H is cyclic.

If $HZ = \tilde{M}$, then H is normal in $\tilde{M}$ and the quotient group $\tilde{M}/H = HZ/Z$ is isomorphic to $Z/(H \cap Z)$. The possibility $H \cap Z = \{1\}$ is ruled out since this would imply $\tilde{M} \cong \mathbb{Z} \times M$ which we have already ruled out as $\tilde{M}$ is torsion free. Thus $H \cap Z \neq \{1\}$ and so $H \cap Z$ has finite index in $Z \cong \mathbb{Z}$, and thus H has finite index in $\tilde{M}$. Since H is normal in $\tilde{M}$, by (vi) we have $H = \tilde{M}$.

4. *Monothetic and p-adic monsters*

Now we consider our modified monster $\tilde{M}$ of Proposition 3·8 as a discrete, hence locally compact topological group and apply Lemma 2·2 with $\Gamma = \tilde{M}$. If we assume that $\tilde{M}$ is as in Proposition 3·8 then the centre Z is infinite cyclic. We select any monothetic compact group C as well as an appropriate morphism $\alpha : Z \to C$ with dense image. In the category of locally compact abelian groups, α is an epimorphism, and if C is not finite, then α is a monomorphism. We recall that $\hat{\alpha} : \hat{C} \to \hat{\mathbb{Z}} \cong \mathbb{T}, \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a monomorphism (with a dense image unless $\hat{C}$ is finite) and that every subgroup of $\mathbb{R}/\mathbb{Z}$ via duality gives rise to a monothetic group and a morphism $\alpha : Z \to C$. Special cases of monothetic groups are: (i) any compact connected group of weight not exceeding that of the continuum, (ii) the additive group of p-adic integers, (iii) the universal monothetic group $C = \hat{\mathbb{T}}_d$, where $\mathbb{T}_d$ is the circle group with its discrete topology and $\alpha : Z \to C$ is the adjoint of the identity morphism $\mathrm{id} : \mathbb{T}_p \to \mathbb{T}$.

LEMMA 4·1. *Suppose that A is a compact monothetic group with an open subgroup B such that $A/B \cong \mathbb{Z}(p)$. Then $\hat{A}$ may be identified with a subgroup of $\mathbb{T}_d$ in such a way that $B^\perp \subseteq \hat{A}$ is the unique subgroup $1/p\,\mathbb{Z}/\mathbb{Z}$ of order p in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and that $\hat{A}/B^\perp$ is isomorphic to the character group $\hat{B}$ of B.*

Proof. These statements contain standard information on monothetic compact groups and duality.

Definition 4.2. We shall say that a group of the type of A in Lemma 4·1 is a monothetic p-extension of B.

Now we set $\mathcal{M} = \tilde{M}[\alpha]$ (see Notation 2·3).

THEOREM 4·3. *Let $\alpha : Z \to C$ be any morphism of the discrete infinite cyclic centre Z of $\tilde{M}$ into a compact group with $C = \overline{\alpha(Z)}$. Then the locally compact group $\mathcal{M} = \tilde{M}[\alpha]$ has the following properties:*

- (i) *The centre $\mathcal{Z}$ of $\mathcal{M}$ is open and isomorphic to C.*
- (ii) *The quotient group $\mathcal{M}/\mathcal{Z}$ is the Ivanov–Ol'shanskiĭ monster.*
- (iii) *Every element of $\mathcal{M}$ is contained in a monothetic subgroup A which contains $\mathcal{Z}$ and either equals $\mathcal{Z}$ or else is a monothetic p-extension of $\mathcal{Z}$.*
- (iv) *Any closed proper normal subgroup $\mathcal{H}$ is contained in $\mathcal{Z}$. In particular, the (algebraic) commutator subgroup is dense in $\mathcal{M}$.*
- (v) *If $\mathcal{H}$ is a proper closed subgroup of $\mathcal{M}$, then either $\mathcal{H} \subseteq \mathcal{Z}$ or $\mathcal{H}\mathcal{Z}$ is a monothetic p-extension of $\mathcal{Z}$.*

Proof. By Lemma 2·2 we know that $\mathcal{Z}$ is central in $\mathcal{M}$ and $\mathcal{M}/\mathcal{Z} \cong \tilde{M}/Z$. The group $\tilde{M}$ and thus the factor group $\tilde{M}/Z$ is discrete. Hence $\mathcal{Z}$ is open. By Propositions 3·6 and 3·8 we have $\tilde{M}/Z \cong M$, and by Lemma 3·4(iv), the Ivanov–Ol'shanskiĭ monster M is simple. In particular, the centre of M is trivial and thus the centre of $\mathcal{M}$ is contained in $\mathcal{Z}$ and therefore equals $\mathcal{Z}$. Thus (i) and (ii) are proved.

(iii) Let $\tilde{m} \in \mathscr{M} = C \times M/\Delta, \tilde{m} = (c, m)\Delta$. By Proposition 3·8(vii) there is a cyclic subgroup $T \subseteq \tilde{M}$ containing Z and m. We set $A = \overline{(\{0\} \times T)\Delta/\Delta}$ and note that in view of $Z \subseteq T$ we have $A = \overline{\alpha(Z) \times T}/\Delta = C \times T/\Delta$. By definition, A is monothetic and contains $\tilde{m}$ as well as $\mathscr{Z}$. By (ii) and 3·3(i) we either have $A = \mathscr{Z}$ or else $A/\mathscr{Z} \cong \mathbb{Z}(p)$.

(iv) If $\mathscr{H} \subseteq \mathscr{Z}$ we are finished. Since $\mathscr{H}\mathscr{Z}/\mathscr{Z}$ is a normal subgroup of the simple group $\mathscr{M}/\mathscr{Z} \cong M$ the remaining case is $\mathscr{H}\mathscr{Z} = \mathscr{M}$. Then $\mathscr{M}/\mathscr{H} = \mathscr{H}\mathscr{Z}/\mathscr{H} \cong \mathscr{Z}/(\mathscr{Z} \cap \mathscr{H})$ is abelian, whence $\mathscr{M}' \subseteq \mathscr{H}$. Now let $M^* = (\{0\} \times \tilde{M})\Delta/\Delta$ denote the algebraic copy of $\tilde{M}$ in $\mathscr{M}$ according to 2·2(ii). Then $(M^*)' = M^*$ by 3·8(v). Hence $M^* = (M^*)' \subseteq \mathscr{M}' \subseteq \mathscr{H}$. Since M^* is dense in $\mathscr{M}$ by 2·2(iv), we obtain $\mathscr{H} = \mathscr{M}$ contradicting the assumption that $\mathscr{H}$ is a proper subgroup.

(v) Assume that $\mathscr{H} \nsubseteq \mathscr{Z}$. Then $\mathscr{H}\mathscr{Z}/\mathscr{Z}$ is of order p or equals $\mathscr{M}/\mathscr{Z} \cong M$ by 3·3(i). Assume the former. Then $\mathscr{H}\mathscr{Z} \cap M^*$ is a proper subgroup of M^* containing Z^*. Hence 3·8(vii) applies and shows that $\mathscr{H}\mathscr{Z} \cap M^*$ is cyclic. Since $\mathscr{H}\mathscr{Z}$ is open in $\mathscr{M}$ and M^* is dense in $\mathscr{M}$ it follows that $\mathscr{H}\mathscr{Z}$ is monothetic. Then $\mathscr{H}\mathscr{Z}$ is a monothetic p-extension. Now suppose that $\mathscr{H}\mathscr{Z} = \mathscr{M}$. Then, since $\mathscr{Z}$ is central in $\mathscr{M}$, the subgroup $\mathscr{H}$ is normal in $\mathscr{M}$. Then section (iv) applies and shows that $\mathscr{H} \subseteq \mathscr{Z}$, which implies $\mathscr{H}\mathscr{Z} = \mathscr{Z} \neq \mathscr{M}$, a contradiction. ∎

Let us make some observations on monothetic compact groups which are relevant in our context. We understand monothetic p-extensions in the same measure as we understand the subgroups $\hat{A}$ of $\mathbb{R}/\mathbb{Z}$ containing $1/p\,\mathbb{Z}/\mathbb{Z}$. We recall that $\mathbb{R}/\mathbb{Z}$ has the (fully characteristic) torsion group which, in turn, contains the (fully characteristic) p-Sylow subgroup $\mathbb{Z}_{p^\infty} = 1/p^\infty\,\mathbb{Z}/\mathbb{Z}$ (with $1/p^\infty\,\mathbb{Z} = \{m/p^n : m, n \in \mathbb{Z}\}$) which contains the (fully characteristic) subgroup $1/p\,\mathbb{Z}/\mathbb{Z}$. The group $\mathbb{T}_d$ itself is a direct sum of the (uniquely determined) direct summands $\mathbb{Z}_{q^\infty}$, q ranging through the set of all primes, and a (not uniquely determined) direct summand isomorphic to $\mathbb{R}_d$.

The closed subgroups K of any monothetic group A are fully classified by the subgroups $K^\perp$ of $\hat{A}$. The character group of K may be identified with $\hat{A}/K^\perp$. We note in passing that in general we find closed subgroups K which are not monothetic: take $\hat{A} = \mathbb{T}_d = \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{x\in X} \mathbb{Q}\cdot e_x$ with a free family $\{e_x : x \in X\}$ of continuum cardinality, and set $K^\perp = \{0\} \oplus \bigoplus_{x \in X} \mathbb{Z}\cdot e_x$. Then $\hat{K} \cong \tilde{A}/K^\perp \cong \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})^{(X)}$, and thus $K \cong (\prod_{p\,\text{prime}} \Delta_p)^X$ which is vastly non-monothetic.

Also, the question whether a closed subgroup K of a monothetic group A meets the subgroup $\alpha(Z)$ of A is clarified via duality: if $K \subseteq A$ we set $P = \alpha^{-1}(K) \subseteq Z$. Then $\alpha(P) = \alpha(Z) \cap K$. Thus we need to find P. Either P is infinite cyclic or P is zero. The adjoint of the inclusion $j: P \to \mathbb{Z}$ is a quotient morphism $\hat{j}: \mathbb{T} \to \hat{P}$ which either has finite kernel or is constant (with $\hat{P}$ a singleton). An inspection of the dual situation reveals that $\ker \hat{j} = \hat{\alpha}(K^\perp)$. We recall that α has a dense image so that $\hat{\alpha}$ is injective, whence $\alpha(K^\perp)$ is finite if and only if $K^\perp$ is finite. Therefore:

Remark. If K is a closed subgroup of the monothetic group A with injective defining morphism $\alpha: Z \to A$, then the intersection $\alpha(Z) \cap K$ is non-trivial (that is, infinite cyclic) if and only if $K^\perp$ is finite if and only if A/K is finite.

In particular, if $A = \Delta_p$ is the p-adic group and thus the character group of $\mathbb{Z}_{p^\infty}$, and if $\alpha = \alpha_p: \mathbb{Z} \to \Delta_p$ is the embedding of $\mathbb{Z}$, then im α meets every non-trivial closed subgroup non-trivially. The p-adic groups are the only infinite monothetic groups with this property. ∎

If A is a monothetic p-extension of B we note that, as a consequence of some of our observations, if $\hat{A}/B^{\perp}$ is a p-group, then $\hat{A}$ is itself a p-group. As a special consequence we have:

LEMMA 4·4. *A monothetic p-extension of a p-adic group (i.e. a group isomorphic to Δ_p) is a p-adic group.*

We have made reference to the additive group Δ_p of p-adic integers. We let $\alpha_p : \mathbb{Z} \to \Delta_p$ denote the standard morphism. Depending on one's preference of representing Δ_p, the function α_p is the inclusion if Δ_p is the completion with respect to the p-adic metric. We note that we may write the elements of Δ_p in their p-adic expansion $\sum_{n=0}^{\infty} a_n p^n$ with $a_n \in \{0, \ldots, p-1\}$; the elements of $\mathbb{Z}$ then are exactly the ones whose expansion is polynomial. If, on the other hand, we use the inverse limit representation $\Delta_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$, then $\alpha_p(z) = (z + p^n\mathbb{Z})_{n\in\mathbb{N}}$. If, finally, one wants to consider Δ_p as the character group of $\mathbb{Z}^{p\infty}$, then α_p: $\mathbb{Z} \to \Delta_p$ is the adjoint morphism of the inclusion $\mathbb{Z}_{p^\infty} \to \mathbb{T}$.

Now we specialize Theorem 4·3 to the case that C is the additive group Δ_p of p-adic integers.

Recalling that Z denotes the infinite cyclic centre of $\tilde{M}$, we let $j : Z \to \mathbb{Z}$ denote one of the two possible isomorphisms and set

$$\alpha : Z \to \Delta_p, \quad \alpha = \alpha_p \circ j : Z \xrightarrow{j} \mathbb{Z} \xrightarrow{\alpha_p} \Delta_p.$$

We obtain $\mathscr{M} = \tilde{M}[\alpha]$, a locally compact monster group whose bizarre properties, of course, result, in the final evaluation, from those of the Ivanov–Ol'shanskiĭ monster.

THEOREM 4·5. *There is a locally compact group $\mathscr{M}$ with the following properties:*

(i) *The centre $\mathscr{Z}$ of $\mathscr{M}$ is open and isomorphic to Δ_p.*
(ii) *The quotient group $\mathscr{M}/\mathscr{Z}$ is the Ivanov–Ol'shanskiĭ monster.*
(iii) *Every element of $\mathscr{M}$ is contained in a subgroup which contains $\mathscr{Z}$ and is isomorphic to Δ_p. In particular, $\mathscr{M}$ is torsion free.*
(iv) *Any closed proper normal subgroup $\mathscr{H}$ is contained in $\mathscr{Z}$. In particular, the (algebraic) commutator subgroup is dense in $\mathscr{M}$.*
(v) *If $\mathscr{H}$ is a proper closed subgroup of $\mathscr{M}$, then $\mathscr{H} \cong \Delta_p$. Either $\mathscr{H} \subseteq \mathscr{Z}$ or $\mathscr{Z} \subset \mathscr{H}$ in which case the index is p.*

Proof. By construction, $\mathscr{Z} \cong \Delta_p$ is p-adic. Hence every monothetic p-extension of $\mathscr{Z}$ is p-adic by Lemma 4·4. Assertions (i), (ii), (iii) and (iv) now follow from the corresponding assertions in Theorem 4·3.

(v) From 4·3 (v) we know that either $\mathscr{H} \subseteq \mathscr{Z}$ or else that $\mathscr{H}\mathscr{Z}$ is a monothetic p-extension of $\mathscr{Z} \cong \Delta_p$. Suppose $\mathscr{H} \nsubseteq \mathscr{Z}$. By Lemma 4·4 we have $\mathscr{H}\mathscr{Z} \cong \Delta_p$. But the closed subgroups of Δ_p are totally ordered under inclusion. Hence we must have $\mathscr{Z} \subset \mathscr{H}$. ∎

Thus we have achieved our aim of constructing a non-abelian topological group in which every proper non-trivial closed subgroup is open and isomorphic (to Δ_p) so this one example suffices to show that the conditions in [2] and [3] cannot be weakened.

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