Math. Proc. Camb. Phil. Soc. (1991), **110**, 293
Printed in Great Britain

Compact groups and products of the unit interval

BY JOAN CLEARY
The University of Wollongong, Wollongong, N.S.W. 2500, *Australia*
AND SIDNEY A. MORRIS
The University of New England, Armidale, N.S.W. 2351, *Australia*

(*Received* 9 *November* 1990)

1. *Introduction*

It is proved that if G is a compact connected Hausdorff group of uncountable weight, $w(G)$, then G contains a homeomorphic copy of $[0,1]^{w(G)}$. From this it is deduced that such a group, G, contains a homeomorphic copy of every compact Hausdorff group with weight $w(G)$ or less. It is also deduced that every infinite compact Hausdorff group G contains a Cantor cube of weight $w(G)$, and hence has $[0,1]^{w(G)}$ as a quotient space.

1·1. *Definitions and Notation*

(*a*) A finite subset $\{x_1, x_2, \ldots, x_k\}$ of a group G is said to be *independent* if it does not contain the identity, 1, of G and $x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k} = 1$, for integers $n_1, \ldots, n_k$, implies that $x_1^{n_1} = x_2^{n_2} = \ldots = x_k^{n_k} = 1$. An infinite subset X of a group G is said to be independent if each finite subset of X is independent. If G is any group, then it has a maximal independent set and every maximal independent set has the same cardinality. The cardinality of such a set is called the rank of G. (See [**4**], §16 and [**5**], §A·11.)

(*b*) If m is any cardinal number, then $\mathbb{D}_2^m$ denotes the product of m copies of the discrete 2-point space, $\mathbb{D}_2$, with the product topology, and is called a Cantor cube. Of course, $\mathbb{D}_2^{\aleph_0}$ is the Cantor space.

(*c*) If X is a topological space, then we denote $\max\{m : m$ is a cardinal and X has a subspace homeomorphic to $[0,1]^m\}$ by $\delta(X)$. In ([**10**], §4·7) it is noted that if G is a locally compact finite-dimensional Hausdorff group, then $\delta(G) = \dim(G)$.

(*d*) Let X be a topological space. We note $\min\{\mathrm{card}\,\mathscr{B} : \mathscr{B}$ is an open basis for $X\}$ by $w(X)$; this is said to be the *weight* of X.

(*e*) We denote the closed unit interval $[0,1]$ by $\mathbb{I}$.

(*f*) If G is a locally compact Hausdorff abelian group, then $\hat{G}$ denotes the locally compact Hausdorff abelian group known as the dual group of G [**5**].

2. *Subspaces of Compact Groups*

In this section we establish the fact that every connected (locally) compact Hausdorff group of uncountable weight, say m, contains a subspace homeomorphic to the cube $\mathbb{I}^m$.

PROPOSITION 2·1 ([**5**], §24·27). *Let G be a non-trivial discrete torsion-free abelian group of rank m, for some cardinal number m. Then $\hat{G}$ has a subspace homeomorphic to $\mathbb{I}^m$.* ∎

We now prove the abelian version of our main result.

PROPOSITION 2·2. *Let K be an infinite compact connected Hausdorff abelian group with $w(K) > \aleph_0$. Then K has a subspace homeomorphic to $\mathbb{I}^{w(K)}$.*

Proof. As K is compact and connected, $\hat{K}$ is a discrete torsion-free abelian group ([5], §23·17 and 24·25). By ([5], §24·14) we have $w(K) = w(\hat{K})$. But $w(\hat{K}) = \text{card}\,(\hat{K})$, as $\hat{K}$ is discrete. So card $\hat{K} > \aleph_0$, and thus card $(\hat{K}) = \text{rank}\,(\hat{K})$. The result then follows from Proposition 2·1. ∎

The following result shows that any compact connected Hausdorff group can be 'sandwiched' between two large products, where the structure of these products is known. Note that our definition of simple connectedness includes path-connectedness.

THEOREM 2·3. *Let G be a compact connected Hausdorff group and A the connected component at the identity of the centre of G. Then there exist a family of compact simply connected simple Lie groups $\{L_\lambda : \lambda \in \Lambda\}$ and a totally disconnected closed subgroup K of the centre of $A \times \prod_{\lambda\in\Lambda} L_\lambda$, such that if*

(i) *C_λ is the centre of L_λ, for each $\lambda \in \Lambda$,*

(ii) *ϕ is the quotient homomorphism $A \times \prod_{\lambda\in\Lambda} L_\lambda \to G$ which factors out K, and*

(iii) *ψ is the quotient homomorphism of G onto $\prod_{\lambda\in\Lambda}(L_\lambda/C_\lambda)$ which factors out $\phi(A \times \prod_{\lambda\in\Lambda} C_\lambda)$, the image in G of the centre of $A \times \prod_{\lambda\in\Lambda} L_\lambda$, then G is 'sandwiched' between the two groups $A \times \prod_{\lambda\in\Lambda} L_\lambda$ and $\prod_{\lambda\in\Lambda}(L_\lambda/C_\lambda)$ as follows:*

$$A \times \prod_{\lambda\in\Lambda} L_\lambda \xrightarrow{\phi} G \xrightarrow{\psi} \prod_{\lambda\in\Lambda} (L_\lambda/C_\lambda),$$

and $\psi \circ \phi$ is the quotient homomorphism which factors out $A \times \prod_{\lambda\in\Lambda} C_\lambda$.

Proof. This is an immediate consequence of ([13], §6.5.6), which states that

$$G = \left(A \times \prod_{\lambda\in\Lambda} L_\lambda\right)\Big/ K,$$

where K is a totally disconnected closed subgroup of the centre of $A \times \prod_{\lambda\in\Lambda} L_\lambda$. ∎

Note that if A is any subspace of a topological space, G, then $w(A) \leqslant w(G)$. Note also that for any $m \geqslant 1$, we have $w(\mathbb{I}^m) = \max\{m, \aleph_0\}$. From these observations and the definition of $\delta(G)$, we see that $\delta(G) \leqslant w(G)$.

We now prove the main result which says that every compact connected Hausdorff group, G, of uncountable weight contains a subspace homeomorphic to $\mathbb{I}^{w(G)}$; that is, $\delta(G) = w(G)$.

THEOREM 2·4. *Let G be an infinite compact connected Hausdorff group with $w(G) > \aleph_0$. Then $\delta(G) = w(G)$.*

Proof. From Theorem 2·3, we have the following open continuous surjective homomorphisms:

$$A \times \prod_{\lambda\in\Lambda} L_\lambda \xrightarrow{\phi} G \xrightarrow{\psi} \prod_{\lambda\in\Lambda} (L_\lambda/C_\lambda)$$

from which we obtain the inequality

$$w\left(A \times \prod_{\lambda \in \Lambda} L_\lambda\right) \geqslant w(G) \geqslant w\left(\prod_{\lambda \in \Lambda} (L_\lambda / C_\lambda)\right).$$

As each L_λ is a non-trivial connected Lie group and, in particular, a non-discrete metrizable group, $w(L_\lambda\} = \aleph_0$. For the same reason, $w(L_\lambda/C_\lambda) = \aleph_0$. So we have

$$\max\{w(A), \aleph_0, \operatorname{card}\Lambda\} \geqslant w(G) \geqslant \max\{\operatorname{card}\Lambda, \aleph_0\}.$$

As A is a topological subgroup of $G, w(A) \leqslant w(G)$. Therefore, we have

$$w(G) = \max\{w(A), \aleph_0, \operatorname{card}\Lambda\}.$$

As $w(G) > \aleph_0$, there are two possibilities for $w(G)$ – either $w(G) = w(A)$ or $w(G) = \operatorname{card}\Lambda$. We examine each possibility.

(i) Assume $w(G) = w(A)$. As A is a compact connected Hausdorff abelian group, and $w(A) > \aleph_0$, from Proposition 2·2, A contains a subspace homeomorphic to $\mathbb{I}^{w(A)} = \mathbb{I}^{w(G)}$. Thus $\delta(G) \geqslant w(G)$ and so $\delta(G) = w(G)$, as required.

(ii) Assume $w(G) = \operatorname{card}\Lambda$. Let $\phi_r : \prod_{t \in I} L_\lambda \to G$ be the restriction of ϕ to $\{1\} \times \prod_{\lambda \in \Lambda} L_\lambda$, where 1 is the identity element of the group A. As L_λ is a simple Lie group, each C_λ, being a closed normal subgroup of L_λ, must be discrete (and finite). It follows from ([**11**], proposition 24) that the quotient map, ρ_λ, from L_λ onto L_λ/C_λ is a local isomorphism, for each $\lambda \in \Lambda$. Thus there exists an open neighbourhood, N_λ, of the identity in L_λ, so that N_λ maps homeomorphically onto its image in L_λ/C_λ. Hence, the composite map $\phi \circ \phi_r = \prod_{\lambda \in \Lambda} \rho_\lambda$

$$\prod_{\lambda \in \Lambda} L_\lambda \xrightarrow{\phi_r} G \xrightarrow{\phi} \prod_{\lambda \in \Lambda} (L_\lambda/C_\lambda),$$

when restricted to $\prod_{\lambda \in \Lambda} N_\lambda$, is one-to-one. As L_λ is a connected Lie group, it is locally Euclidean, and so N_λ contains a homeomorphic copy of $\mathbb{I}$. Therefore $\prod_{\lambda \in \Lambda} N_{\lambda \in \Lambda}$ contains a homeomorphic copy of $\mathbb{I}^{\operatorname{card}\Lambda}$. As $\psi \circ \phi_r$ is a one-to-one continuous map of the compact space $\mathbb{I}^{\operatorname{card}\Lambda}$ onto its image, it follows that $\psi(\phi_r(\mathbb{I}^{\operatorname{card}\Lambda}))$ is homeomorphic to $\mathbb{I}^{\operatorname{card}\Lambda}$. Therefore $\phi_r(\mathbb{I}^{\operatorname{card}\Lambda})$ is also homeomorphic to $\mathbb{I}^{\operatorname{card}\Lambda}$. So G contains a homeomorphic copy of $\mathbb{I}^{\operatorname{card}\Lambda}$. But $\mathbb{I}^{\operatorname{card}\Lambda} = \mathbb{I}^{w(G)}$, and so $\delta(G) \geqslant w(G)$. So once again we have $\delta(G) = w(G)$, as required.

Remark 2·5. Theorem 2·4 is the best possible in the following sense: there exists a group G with $w(G) = \aleph_0$, but $\delta(G) = 1$. The compact group $\mathbb{T}$ has this property, as $\mathbb{T}$ contains $\mathbb{I}$, but $\mathbb{T}$ does not contain a copy of $\mathbb{I}^2$. This is so since $\dim(\mathbb{T}) = 1$ and $\dim(\mathbb{I}^2) = 2$. Further, observe that for each positive integer $n, w(\mathbb{T}^n) = \aleph_0$ and $\delta(\mathbb{T}^n) = n$. ∎

As a corollary to Theorem 2·4, we have the following surprising result.

THEOREM 2·6. *Let G be an infinite compact connected Hausdorff group with weight $w(G) > \aleph_0$. Then G contains a homeomorphic copy of every compact Hausdorff group of weight $w(G)$ or less.*

Proof. Let H be a compact Hausdorff group. That G contains a homeomorphic copy of H is clearly true when H is finite. So without loss of generality, assume that H is an infinite compact Hausdorff group of weight $w(H) \leqslant w(G)$. By the proof of theorem 1 of [2] we see that H is topologically isomorphic to a subgroup of $S^{w(H)}$, where S is the product $\prod_{n=1}^{\infty} \mathrm{SU}(n)$ of special unitary groups. As each $\mathrm{SU}(n)$ is a subspace of $\mathbb{I}^{\aleph_0}$, so too is S. Therefore H is homeomorphic to a subspace of $(\mathbb{I}^{\aleph_0})^{w(H)} = \mathbb{I}^{w(H)}$, which is in turn a subspace of $\mathbb{I}^{w(G)}$. By Theorem 2·4 this implies that H is homeomorphic to a subspace of G.

Remark 2·7. (i) Note that the condition that G be connected cannot be dropped in Theorems 2·4 and 2·6. For instance, if G is a totally disconnected group, then it contains no connected subspaces. (ii) Also, the condition that the weight of G be strictly greater than $\aleph_0$ cannot be dropped, because if $G = \mathbb{T}$, then $w(G) = \aleph_0$, but G does not contain even $\mathbb{T}^2$, which has weight $\aleph_0$. ▌

We now generalize Theorem 2·4 to connected locally compact groups.

THEOREM 2·9. *Let G be a connected locally compact Hausdorff group with weight $w(G) > \aleph_0$. Then $\delta(G) = w(G)$.*

Proof. Using the Iwasawa structure theorem, ([11], p. 118), G is homeomorphic $\mathbb{R}^n \times K$ where K is a connected compact subgroup of G. Note that $\mathbb{R}^n$ contains $\mathbb{I}^n$ but not $\mathbb{I}^{n+1}$, and so it is clear that $\delta(G) = n + \delta(K)$.

As $w(G) = \max\{\aleph_0, w(K)\}$ is strictly greater than $\aleph_0$, $w(G)$ must be $w(K)$ which is strictly greater than $\aleph_0$. So from Theorem 2·4, $\delta(K) = w(K)$, and hence $\delta(G) = w(G)$ as required. ▌

3. *Consequences of the main result*

In this section, we show that an infinite compact Hausdorff group, G, contains a Cantor cube, and more particularly, the product $\mathbb{D}_2^{w(G)}$. From this it is easy to deduce that G maps onto $\mathbb{I}^{w(G)}$.

THEOREM 3·1. *Let G be an infinite compact Hausdorff group. Then G contains a subspace homeomorphic to $\mathbb{D}_2^{w(G)}$.*

Proof. From a result of Mostert [12] it can be deduced that G is homeomorphic to $G_0 \times G/G_0$, where G_0 is the connected component of the identity of G. So $w(G) = \max\{w(G_0), w(G/G_0)\}$. But G/G_0 is a compact totally disconnected group and so, by ([5], §9·15 and [6], 28·58), is $\aleph$ homeomorphic to $\mathbb{D}_2^{w(G/G_0)}$.

As G_0 is an infinite compact connected Hausdorff group, from Theorem 2·4 it contains a subspace homeomorphic to $\mathbb{I}^{w(G_0)}$. If $w(G_0) > \aleph_0$, then, as $\mathbb{I}$ contains $\mathbb{D}_2$, $\mathbb{I}^{w(G_0)}$ contains $\mathbb{D}_2^{w(G_0)}$. If $w(G_0) = \aleph_0$, then G_0 is metrizable, and so contains a subspace homeomorphic to $\mathbb{I}$, which in turn contains the Cantor space $\mathbb{D}_2^{\aleph_0} = \mathbb{D}_2^{w(G_0)}$. So G_0 contains a subspace homeomorphic to $\mathbb{D}_2^{w(G_0)}$.

Putting these results together, we see that G contains a subspace homeomorphic to $\mathbb{D}_2^{w(G_0)} \times \mathbb{D}_2^{w(G/G_0)} = \mathbb{D}_2^{\max\{w(G_0), w(G/G_0)\}} = \mathbb{D}_2^{w(G)}$. ▌

Our final result shows that each compact Hausdorff group, G, has $\mathbb{I}^{w(G)}$ as a quotient space.

THEOREM 3·2 ([9], chapter 24, §1·45). *If G is an infinite compact Hausdorff group, then there is a continuous map from G onto $\mathbb{I}^{w(G)}$.*

Proof. Note that G is a normal space. From Theorem 3·1, G contains a subspace homeomorphic to $\mathbb{D}_2^{w(G)}$, which is a closed subset of G.

The Cantor space $\mathbb{D}_2^{\aleph_0}$ can be continuously mapped onto $\mathbb{I}$, from ([7], §15·3). Hence $(\mathbb{D}_2^{\aleph_0})^{w(G)} = \mathbb{D}_2^{w(G)}$ maps continuously onto $\mathbb{I}^{w(G)}$. Finally, using Tietze's theorem, there exists a continuous map from G onto $\mathbb{I}^{w(G)}$.

Remark 3·3. We have shown in the proof of Theorem 3·1 that if X is a compact Hausdorff space, then (i) implies (ii), where

(i) X has $\mathbb{D}_2^m$ as a subspace;

(ii) there is a continuous mapping of X onto $\mathbb{I}^m$.

Juhász, ([**8**], theorem 3·18), shows that condition (ii) is equivalent to various other conditions. One might ask if (ii) implies (i). Juhász's result suggests that this is likely. However, it is false. To see this, we observe that the Stone-Čech compactification $\beta\mathbb{N}$ of the discrete space of natural numbers $\mathbb{N}$ has property (ii) but fails to have property (i). As $\mathbb{I}^c$ is separable ([**1**], §4·19), there exists a continuous map ϕ of $\mathbb{N}$ onto a dense subspace of $\mathbb{I}^c$. Therefore ϕ extends to a continuous map of $\beta\mathbb{N}$ onto $\mathbb{I}^c$. So (ii) is true for $m = c$. To see that (i) is false it suffices to observe that every infinite closed subspace of $\beta\mathbb{N}$ has cardinality 2^c ([**14**], theorem 3·3). So $\beta\mathbb{N}$ does not contain the Cantor space $\mathbb{D}_2^{\aleph_0}$, as card $\mathbb{D}_2^{\aleph_0} = c$. Hence $\beta\mathbb{N}$ does not contain the Cantor cube $\mathbb{D}_2^c$.

REFERENCES

[1] R. E. Chandler. *Hausdorff Compactifications* (Marcel Dekker, 1976).

[2] Joan Cleary and Sidney A. Morris. Trinity – a tale of three cardinals. *Proc. Centre Math. Anal. Austral. Nat. Univ.* **14** (1986), 117–127.

[3] Ryszard Engelking. *General Topology* (Polish Scientific Publishers, 1977).

[4] László Fuchs. *Infinite Abelian Groups* (Academic Press, 1970).

[5] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis I* (Springer-Verlag, 1963).

[6] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis II* (Springer-Verlag, 1970).

[7] G. Jameson. *Topology and Normed Spaces* (Chapman and Hall, 1974).

[8] I. Juhász. *Cardinal Functions in Topology – Ten Years Later* (Mathematisch Centrum, Amsterdam, 1983).

[9] K. Kunen and J. E. Vaughan (eds). *Handbook of Set-theoretical Topology* (North-Holland, 1984).

[10] D. Montgomery and L. Zippin. *Topological Transformation Groups* (Interscience, 1965).

[11] Sidney A. Morris. *Pontryagin Duality and the Structure of Locally Compact Abelian Groups* (Cambridge University Press, 1977).

[12] Paul. S. Mostert. Sections in principal fibre spaces. *Duke Math. J.* **23** (1956), 57–71.

[13] John F. Price. *Lie Groups and Compact Groups* (Cambridge University Press, 1977).

[14] Russell C. Walker. *The Stone-Čech Compactification* (Springer-Verlag, 1974).