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Free compact groups IV: Splitting the component and the structure of the commutator group

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Abstract

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Introduction

The forgetful functor from the category of compact groups into the category of pointed spaces and base-point preserving maps has a left adjoint F. The image FX of a pointed space X under this functor is called the *free compact group FX over X*. The structure theory of compact groups allows the development of a structure theory of free compact groups. The program was introduced in [4], and its technical details were described in [7–10]. The theory is still incomplete today. In this note we present further progress.[1] In the first section we prove that FX splits semidirectly over its identity component $(FX)_0$ with the free profinite group over X as quotient. In the second section we give a complete description of the projective cover PFX of FX in case that X is connected. This allows us the calculation of the weight of FX and thus the proof of a conjecture expressed in [4]. Both sections illustrate the fact that the methods for the structure theory of FX come from transformation group and compact group theory while the definition of FX falls in the domain of universal algebra or category theory.

[1] In the preprint distributed and the lecture delivered at the conference, an overview of the theory of free compact groups up to the current state was presented.

1. Splitting the component

Let G_0 denote the identity component of the topological group G. *For any compact group G the quotient morphism $p: G \to G/G_0$ has a continuous cross-section* $\sigma: G/G_0 \to G$, that is, a continuous map satisfying $p \circ \sigma = \mathrm{id}_{G/G_0}$. This is a consequence of Mostert's Cross Section Theorem (see for instance [11, Appendix II, 1.12, p. 317]). In general, G_0 is not a semidirect factor. However, Lee showed that G always contains a compact zero-dimensional subgroup D such that $G = G_0 D$ while, in general, $G_0 \cap D$ is not singleton. (See [12].) If G, however, is any free compact group, then G_0 is a semidirect factor. Firstly, the assignment $G \mapsto G/G_0$ is left adjoint to the inclusion functor KZG $\to$ KG of the category of compact 0-dimensional groups into the category of compact groups. The group $FX/(FX)_0$ is called the *free compact zero-dimensional group* or the *free profinite group* and is denoted $F_z X$. (For free profinite groups there is a rather detailed theory due to Mel'nikov [13–15, 17].) If X is connected, then $F_z X = \{1\}$.

Theorem 1. *The identity component $(FX)_0$ of any free compact group FX is a semidirect factor. More specifically, there are closed totally disconnected subgroups D of FX with $FX = (FX)_0 D$, and every such contains a totally disconnected closed subgroup $T \cong F_z X$ of FX such that the function $(c, t) \mapsto ct$, $(FX)_0 \rtimes T \to FX$ is an isomorphism of compact groups.*

Proof. By Lee's Theorem, FX contains a compact zero-dimensional subgroup D such that $FX = (FX)_0 D$. The surjective morphism $\pi = (d \mapsto d(FX)_0)$, $D \to F_z X$ in the category of compact zero-dimensional groups has a topological cross-section $\sigma: F_z X \to D$ by Mostert's Cross Section Theorem. Thus the canonical map $\varepsilon: X \to F_z X$ gives a map $\sigma \circ \varepsilon: X \to D$ which, by the universal property of $F_z X$, factors through a morphism $\lambda: F_z X \to D$ satisfying $\pi \circ \lambda = \mathrm{id}_{F_z X}$. Then $\lambda(F_z X)$ is the required group T. $\square$

We have stated in the introduction of [8] the equivalent assertion that the exact sequence

$$0 \to (FX)_0 \to FX \to F_z X \to 0 \tag{1}$$

splits in KG. However, the proof of this claim was omitted and the theorem never even reappeared in the paper.

Here the extension problem of $(FX)_0$ by $F_z X$ is comparatively simple. Not all problems are settled in this context. In particular, we do not have a classification of all complementary subgroups T. Notably we do not know whether they are conjugate. At any rate, we are still left with the determination of $(FX)_0$.

2. The projective cover of a free compact connected group

In this section we present new results on the commutator subgroup of a free compact connected group. Our principal tools are the following remarks, applying to compact connected groups in general.

Let $\mathscr{L}$ denote the class of all simple compact Lie algebras (i.e. Lie-simple Lie algebras whose Cartan–Killing form is negative definite). The following result is a part of the structure theory of compact connected groups (see for instance [1, 2, 5]).

Proposition 2. *For any compact connected group G, for each $\mathfrak{s} \in \mathscr{L}$ there is a characteristic largest semisimple subgroup $S_{\mathfrak{s}}G$ such that every simple homomorphic image of $S_{\mathfrak{s}}G$ has $\mathfrak{s}$ as Lie algebra and the commutator subgroup G' is the quotient of $\prod_{\mathfrak{s} \in \mathscr{L}} S_{\mathfrak{s}}G$ modulo a zero-dimensional subgroup of the center $\prod_{\mathfrak{s} \in \mathscr{L}} ZS_{\mathfrak{s}}G$.* $\square$

We shall call $S_{\mathfrak{s}}G$ *the $\mathfrak{s}$-component* of G. Let $C_{\mathfrak{s}}X$ denote the closed subgroup generated by all $S_{\mathfrak{s}'}G$ with $\mathfrak{s}' \neq \mathfrak{s}$. The simply connected, respectively centerfree (adjoint) compact Lie group with Lie algebra $\mathfrak{s}$ will be denoted $L_{\mathfrak{s}}$, respectively, $K_{\mathfrak{s}}$.

We notice the following:

Proposition 3. (i) *For any compact connected group G and any $\mathfrak{s} \in \mathscr{L}$ we have $G' = C_{\mathfrak{s}}G \cdot S_{\mathfrak{s}}G$ and $G = Z_0G \cdot C_{\mathfrak{s}}G \cdot S_{\mathfrak{s}}G$.*

(ii) The Sandwich Theorem. *There is a unique cardinal $\aleph(\mathfrak{s}, G)$ and there are surjective morphisms $L_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)} \to S_{\mathfrak{s}}G \to K_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)}$ whose kernels are totally disconnected central subgroups, the first one of $Z(L_{\mathfrak{s}})^{\aleph(\mathfrak{s},G)}$, the second one of $ZS_{\mathfrak{s}}G$.*

(iii) *For each $\mathfrak{s} \in \mathscr{L}$ one has an isomorphism*

$$gZS_{\mathfrak{s}}G \mapsto gZG \cdot C_{\mathfrak{s}}G, \qquad S_{\mathfrak{s}}G/ZS_{\mathfrak{s}}G \to G/(ZG \cdot C_{\mathfrak{s}}G). \qquad \square \tag{2}$$

The situation is best illustrated in the following diagram:

$$\begin{array}{c} \left.\begin{array}{r} K_{\mathfrak{s}}^{w(X)^{\aleph_0}} \left\{ \begin{array}{c} L_{\mathfrak{s}}^{w(X)^{\aleph_0}} \\ | \\ ZL_{\mathfrak{s}}^{w(X)^{\aleph_0}} \end{array} \right. \\ | \\ D \end{array} \right\} F_{\mathfrak{s}}X \\ | \\ \{1\} \end{array}$$

The Sandwich Theorem tells us in essence the structure of the $\mathfrak{s}$-component $S_{\mathfrak{s}}G$ of G; it depends only on the cardinal $\aleph(\mathfrak{s}, G)$ and a closed subgroup D of $Z(L_{\mathfrak{s}})^{\aleph(\mathfrak{s},G)}$.

An ideal of an ideal in a Lie algebra need not be an ideal, but this is the case for compact Lie algebras. A consequence of this fact is the first item of the following theorem. For a proof of the other statements see for instance [2, Volume II, pp. 81ff.].

Proposition 4. (i) *If G is a connected compact group, N a connected closed normal* *subgroup of G and M a connected closed normal subgroup of N, then M is normal in G. The class of all compact connected groups together with morphisms with normal image form a category* KGN.

(ii) *Every group $A \times \prod_{j\in J} X_j$ with a compact abelian group A with $\hat{A}$ a rational vector space and with simply connected compact Lie groups S_j is a projective in* KGN.

(iii) *There is a self-functor P of* KGN *given by*

$$PG = PZ_0 G \times \prod_{\mathfrak{s}\in\mathscr{L}} L_{\mathfrak{s}}^{\aleph(\mathfrak{s},G)}$$

and a natural transformation $\tau_G : PG \to G$ with a totally disconnected kernel. □

We shall call *PG the projective cover* of G. This notation is consistent with that which we introduced for abelian groups in [7] and [8] inasmuch as it reduces standard projectivity if G is abelian. Projectivity in the category KGN is discussed in [2, pp. 84ff.].

Here repeatedly we have presented given compact connected groups as factor groups of known compact connected groups with a product structure modulo totally disconnected subgroups. It is therefore noteworthy to observe that such quotient maps do not decrease weight (see [10]) and that the weight of products is easily computed.

Proposition 5. (i) *Let G be a compact group and D a totally disconnected normal subgroup contained in G_0. Then $w(G) = w(G/D)$.*

(ii) *If $\{G_j \mid j \in J\}$ is a family of infinite compact separable metric groups and H their product, then $w(H) = \max\{\aleph_0, \operatorname{card}(J)\}$.* □

A particular consequence of Propositions 3 through 5 is that we have

$$w(S_{\mathfrak{s}}G) = \max\{\aleph_0, \aleph(\mathfrak{s}, G)\}, \tag{3}$$

which implies that

$$\aleph(\mathfrak{s}, G) = w(S_{\mathfrak{s}}G) \quad \text{whenever } \aleph(\mathfrak{s}, G) \text{ is infinite,}$$

and this is certainly true if $w(S_{\mathfrak{s}}G)$ is uncountable.

In the following we shall completely describe the projective over *PFX* of a free compact group for connected X. First we shall show as much as we presently can about the $\mathfrak{s}$-components of *FX*. We need some preliminary information.

Definition 6. (i) A basepoint preserving function $f : X \to G$ from a pointed space into a topological group is said to be *essential* if it is basepoint preserving and G is topologically generated by $f(X)$, that is, G is the smallest closed subgroup containing $f(X)$.

(ii) For any compact group G, and any compact pointed space X, the *essential*

G-free compact group $F_G X$ is a compact group together with a natural map $e_X : X \to F_G X$ such that for every essential function $f : X \to G$ mapping the base-point of X to the identity of G there is a unique continuous homomorphism $f' : F_G X \to G$ such that $f = f' \circ e_X$.

(iii) Let X be a pointed compact space and G any compact group. We denote by $E(X, G)$ the set of all essential functions $f : X \to G$. The automorphism group Aut G acts on the set $E(X, G)$ on the left by

$$(\alpha, f) \mapsto \alpha \circ f, \qquad \operatorname{Aut} G \times E(X, G) \to E(X, G).$$

We shall denote the orbit space $E(X, G)/\operatorname{Aut} G$ by $A(X, G)$.

If X is a compact pointed space with at least five points, then

$$\operatorname{card} A(X, G) = w(X)^{\aleph_0}. \tag{4}$$

A compact group G will be called *homomorphically simple* if each endomorphism of G is either constant or an automorphism.

Lemma 7. *Let G be a compact connected homomorphically simple compact Lie group and X an compact pointed space with at least three points. Then X is embedded into $F_G X$ and there is a natural isomorphism $\psi_X : G^{A(G,X)} \to F_G X$.* □

Since there is a subdirect product representation of $F_G X$ in $G^{E(G,X)}$ it is possible to describe the natural function $X \to F_G X$ rather explicitly through an evaluation function. (The details are given in [9].)

Theorem 8 (The Sandwich Theorem for free compact groups). *Let X denote a compact connected pointed nonsingleton Hausdorff space. Then for each $\mathfrak{s} \in \mathscr{L}$ we have surjective morphisms with totally disconnected kernels*

$$L_{\mathfrak{s}}^{w(X)^{\aleph_0}} \to S_{\mathfrak{s}} FX \to K_{\mathfrak{s}}^{w(X)^{\aleph_0}} \tag{5}$$

and there are exact sequences

$$0 \to ZS_{\mathfrak{s}} FX \to S_{\mathfrak{s}} FX \to K_{\mathfrak{s}}^{w(X)^{\aleph_0}} \to 0, \tag{6}$$

$$0 \to D \to L_{\mathfrak{s}}^{w(X)^{\aleph_0}} \to S_{\mathfrak{s}} FX \to 0 \tag{7}$$

with a zero-dimensional central group $D \subseteq ZL_{\mathfrak{s}}^{w(X)^{\aleph_0}}$ which satisfies

$$(ZL_{\mathfrak{s}}^{w(X)^{\aleph_0}})/D \cong ZS_{\mathfrak{s}} FX.$$

In particular, $\aleph(\mathfrak{s}, FX) = w(X)^{\aleph_0}$.

Proof. Clearly, we have to show (6) and (7). By Lemma 7, we know $F_{K_{\mathfrak{s}}} X \cong K_{\mathfrak{s}}^{A(X, K_{\mathfrak{s}})}$. Since X is Hausdorff, nonsingleton and connected, (4) shows $A(X, L_{\mathfrak{s}}) = w(X)^{\aleph_0}$. Hence there is an isomorphism

$$\psi_X : K_{\mathfrak{s}}^{w(X)^{\aleph_0}} \to F_{K_{\mathfrak{s}}} X. \tag{8}$$

We recall now that it is no loss of generality to assume that X is contained in FX. The natural continuous map $e_X: X \to F_{K_{\mathfrak{g}}}X$ of Definition 6(ii) and the universal property of FX guarantees a morphism $f': FX \to F_{K_{\mathfrak{g}}}X$ with $f'|_X = e_X$. Since $e_X(X)$ generates $F_{K_{\mathfrak{g}}}X$ as a compact group, this f' is surjective. For the purposes of the proof, we choose the abbreviation $F_{\mathfrak{g}}X = S_{\mathfrak{g}}(FX)$. The quotient morphism $FX \to FX/ZFX \cdot C_{\mathfrak{g}}FX$ and the inverse of the isomorphism (2) give us a surjective morphism

$$p_{\mathfrak{g}}: FX \to F_{\mathfrak{g}}X/ZF_{\mathfrak{g}}X.$$

Since the morphisms $F_{K_{\mathfrak{g}}}X \to K_{\mathfrak{g}}$ separate the points by (8), the morphism f' annihilates $ZFX \cdot C_{\mathfrak{g}}FX = Z_0FX \cdot C_{\mathfrak{g}}FX \cdot ZF_{\mathfrak{g}}X$ and thus induces a surjective morphism

$$f: F_{\mathfrak{g}}X/ZF_{\mathfrak{g}}X \to F_{K_{\mathfrak{g}}}X, \qquad f(gZF_{\mathfrak{g}}X) = f'(g),$$

satisfying $f' = f \circ p_{\mathfrak{g}}$ and thus

$$fp_{\mathfrak{g}}|_X = e_X.$$

By the Sandwich Theorem 3(ii), $F_{\mathfrak{g}}X/ZF_{\mathfrak{g}}X$ is some power of $K_{\mathfrak{g}}$. Hence $p_{\mathfrak{g}}|_X: X \to F_{\mathfrak{g}}X/ZF_{\mathfrak{g}}X$ is essential into a power of $K_{\mathfrak{g}}$. Thus, by the universal property of $F_{K_{\mathfrak{g}}}X$, there is a surjective morphism $g: F_{K_{\mathfrak{g}}}X \to F_{\mathfrak{g}}X/ZF_{\mathfrak{g}}'X$ with $p_{\mathfrak{g}}|_X = g \circ e_X$. In particular, $fge_X = fp_{\mathfrak{g}}|_X = e_X$. Thus the two functions $fg \circ e_X$, $\mathrm{id}_{F_{K_{\mathfrak{g}}}X} \circ e_X: X \to F_{K_{\mathfrak{g}}}X$ agree, and the uniqueness in the universal property of $F_{K_{\mathfrak{g}}}X$ shows $fg = \mathrm{id}_{F_{K_{\mathfrak{g}}}X}$. The surjectivity of g establishes that g is an isomorphism with $f = g^{-1}$. Thus (6) is proved. At this point we also know that $\aleph(\mathfrak{g}, FX) = w(X)^{\aleph_0}$.

The Sandwich Theorem 3(ii) tells us that $F_{\mathfrak{g}}X$ is the quotient modulo a central subgroup of the power $L_{\mathfrak{g}}^{\aleph(\mathfrak{g}, FX)}$. Then (7) follows. □

We have shown in [10] that one can canonically associate with any (locally) compact group G a cardinal invariant, the generating rank of G. Before we define this cardinal, we agree to say that a subset X of a topological group G is *suitable* if it is discrete, if it generates G topologically and if it is closed in $G \setminus \{1\}$. Equivalently, the suitable subsets are those discrete subsets whose only possible accumulation point is the identity and which generate the group topologically. Now the *generating rank of* G is defined by

$$s(G) = \min\{\aleph: \text{there is a suitable subset } X \text{ of } G \text{ such that card } X = \aleph\}.$$

In order that this invariant becomes a meaningful tool we proved in [10] the following result:

Every locally compact group has a suitable subset.

The weight and the generating rank of a locally compact group are linked (see [10]). A calculation of weights (see [9, 10]) and generating ranks with the aid of the Sandwich Theorem yields the following conclusion:

Corollary 9. *For a compact connected nonsingleton pointed Hausdorff space X we have $w(S_{\mathfrak{s}}FX) = w(X)^{\aleph_0}$ and*

$$s(S_{\mathfrak{s}}FX) = \begin{cases} 2 & \text{if } w(X) \le c, \\ w(X)^{\aleph_0} & \text{if } w(X) > c. \end{cases}$$

As a consequence, $w(F'X) = w(X)^{\aleph_0}$ and

$$s(F'X) = \begin{cases} 2 & \text{if } w(X) \le c, \\ w(X)^{\aleph_0} & \text{if } w(X) > c. \end{cases} \qquad \square$$

The following result completely describes the structure of the projective cover PFX.

Theorem 10. *Let X denote a nonsingleton compact connected pointed space and PFX the projective cover of FX. Then*

$$PFX \cong \left(\hat{\mathbb{Q}} \times \prod_{\mathfrak{s} \in \mathcal{L}} L_{\mathfrak{s}} \right)^{w(X)^{\aleph_0}}. \tag{9}$$

Moreover, $w(FX) = w(PX) = w(X)^{\aleph_0}$ and

$$s(FX) = \begin{cases} 2 & \text{if } w(X) \le c, \\ w(X)^{\aleph_0} & \text{if } w(X) > c. \end{cases}$$

Proof. In view of the Characteristic Sequence discussed in [8] linking a free compact *abelian* group with its projective cover, the results on the generating degree of [9], and Proposition 4(iii) above, it suffices to recall from Theorem 6 that $\aleph(\mathfrak{s}, FX) = w(X)^{\aleph_0}$ for all $\mathfrak{s} \in \mathcal{L}$. $\square$

For the structure of the commutator group $F'X$ of a free compact group FX for connected X we deduce at once the following information:

Corollary 11. *If X is compact connected then $F'X$ is a quotient of*

$$\prod_{\mathfrak{s} \in \mathcal{L}} L_{\mathfrak{s}}^{w(X)^{\aleph_0}} \cong \left(\prod_{\mathfrak{s} \in \mathcal{L}} L_{\mathfrak{s}} \right)^{w(X)^{\aleph_0}}$$

modulo a central compact (totally disconnected) subgroup. $\square$

Unfortunately, we still lack the precise insight into what subgroup we have to factor.

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